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Convergence of Compressible Euler-Maxwell Equations to Compressible Euler-Poisson Equations***

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Abstract In this paper, the convergence of time-dependent Euler-Maxwell equations to compressible Euler-Poisson equations in a torus via the non-relativistic limit is studied. The local existence of smooth solutions to both systems is proved by using energy estimates for first order symmetrizable hyperbolic systems. For well prepared initial data the convergence of solutions is rigorously justified by an analysis of asymptotic expansions up to any order. The authors perform also an initial layer analysis for general initial data and prove the convergence of asymptotic expansions up to first order.

Keywords Euler-Maxwell equations, Compressible Euler-Poisson equations, Non-relativistic limit, Asymptotic expansion and convergence
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1 Introduction

Let n and u be the density and velocity vector of the electric particles in a plasma, E and B be respectively the electric field and magnetic field. They are functions of a three-dimensional position vector $x \in \mathbb{T}^3$ and of the time $t > 0$, where $\mathbb{T} = \mathbb{R}/2\pi$ is the torus. The fields E and B are coupled to the electron density through the Maxwell equations and act on electrons via the Lorentz force. Let $c = (\epsilon_0 \nu_0)^{-\frac{1}{2}}$ be the speed of light, where ϵ_0 and ν_0 are the vacuum permittivity and permeability. The dynamics of the compressible electrons for plasma physics in a uniform background of non-moving ions with fixed density $b(x)$ obey the (scaled) one-fluid Euler-Maxwell system (see [1, 4, 13])

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{1.1}$$

$$\partial_t (nu) + \operatorname{div}(nu \otimes u) + \nabla p(n) = -n\Big(E + \frac{1}{c} u \times B\Big), \tag{1.2}$$

$$\partial_t E - c\nabla \times B = nu, \quad \partial_t B + c\nabla \times E = 0, \tag{1.3}$$

$$\operatorname{div} E = b(x) - n, \quad \operatorname{div} B = 0 \tag{1.4}$$

for $x \in \mathbb{T}^3$ and $t > 0$, subject to initial conditions

$$t = 0: \quad (n, u, E, B) = (n_0^c, u_0^c, E_0^c, B_0^c) \tag{1.5}$$

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for $x \in \mathbb{T}^3$. In the above equations, $p = p(n)$ is the pressure, assumed to be smooth and strictly increasing for $n > 0$, $j = nu$ is the current density and $E + \frac{1}{c} u \times B$ represents the Lorentz force. Equations (1.1)–(1.2) are the mass and momentum balances for the electrons, respectively, while (1.3)–(1.4) are the classical Maxwell equations. It is a well-known fact that equations (1.4) are redundant with equations (1.3), as soon as they are satisfied by the initial conditions. However, we keep them in the system because this redundancy may be lost in the asymptotic limit.

There have been a lot of studies on the Euler-Poisson equations and their asymptotic analysis contrarily to the study on the Euler-Maxwell equations. See [2, 3, 6, 10, 11, 15, 18] and the references therein. In particular, the convergence of compressible Euler-Poisson equations to incompressible Euler equations is shown independently in [12, 17]. The first mathematical study of the Euler-Maxwell equations with extra relaxation terms is due to Chen et al [5], where a global existence result to weak solutions in one-dimensional case is established by the fractional step Godunov scheme together with a compensated compactness argument. The paper [5] exhibits also some applications of the model (1.1)–(1.4) in semiconductor theory. Since then few progress have been made on the Euler-Maxwell equations.

In this paper we are interested in the non-relativistic limit $c \to \infty$ in the problem (1.1)–(1.5) for the Euler-Maxwell equations. In the case that the problems are confined in a torus, we prove the existence of smooth solutions to the problem (1.1)–(1.5) and their convergences to the solutions of the compressible Euler-Poisson equations in a time interval independent of c. For this propose, we use the method of asymptotic expansions constructed by solving the compressible Euler-Poisson equations and a linear curl-div system. The convergence of the expansions is achieved through the energy estimates for error equations derived from the asymptotic expansions and the Euler-Maxwell equations. Here we have to deal with some coupling and singular terms. For the variables n and u, we adapt the techniques of Majda [9] for symmetrizable hyperbolic equations. For the fields E and B we observe that from the Maxwell equations (1.3) E and B satisfy the relation

$$\frac{d}{dt} \int_{\mathbb{T}^3} (E^2 + B^2) dx = 2 \int_{\mathbb{T}^3} nuE dx.$$

Together with the Euler equations this yields uniform energy estimates for E and B with respect to c.

This paper is organized as follows. In the next section, we derive formal asymptotic expansions of the problem (1.1)–(1.5). The existence of the expansions is proved in Section 3. Section 4 is devoted to justify the asymptotic expansions up to any order under the condition that the initial expansions are well prepared which exclude the formation of initial layers. In the last section, we perform an initial layer analysis of the problem (1.1)–(1.5) for general initial data. The constructed initial layers do not decay to zero and are even non-local with respect to fast variables. Due to the special structure of the systems, we justify the convergence of the expansions up to first order for general initial data. Finally, the proof of Lemma 4.1 used in Sections 4 and 5 is given in Appendix.

2 Formal Asymptotic Expansions

For smooth solutions of the Euler-Maxwell system (1.1)–(1.5), the second equation (1.2) is equivalent to

$$\partial_t u + (u \cdot \nabla)u + \nabla h(n) = -\Big(E + \frac{1}{c} u \times B\Big),$$

where the enthalpy $h(n)$ is defined by

$$h(n) = \int_1^n \frac{p'(s)}{s} ds.$$

Thus, regarding c as a singular perturbation parameter, we can rewrite the problem (1.1)–(1.5) as

$$\partial_t n + \mathrm{div}(nu) = 0, \tag{2.1}$$

$$\partial_t u + (u \cdot \nabla)u + \nabla h(n) = -\Big(E + \frac{1}{c} u \times B\Big), \tag{2.2}$$

$$\partial_t E - c\nabla \times B = nu, \quad \partial_t B + c\nabla \times E = 0, \tag{2.3}$$

$$\mathrm{div} E = b(x) - n, \quad \mathrm{div} B = 0, \tag{2.4}$$

$$t = 0: \quad (n, u, E, B) = (n_0^c, u_0^c, E_0^c, B_0^c). \tag{2.5}$$

Denote by (n^c, u^c, E^c, B^c) the classical solutions to the problem (2.1)–(2.5). In this section, we are going to study the formal expansions of (n^c, u^c, E^c, B^c) as $c \to \infty$. To this end, we assume that the initial data $(n_0^c, u_0^c, E_0^c, B_0^c)$ have the asymptotic expansion with respect to the speed of light c:

$$(n_0^c, u_0^c, E_0^c, B_0^c) = \sum_{j=0}^{m} c^{-j}(n_j, u_j, E_j, B_j) + O(c^{-m}),$$

with $(E_j, B_j)_{0\le j\le m}$ being determined by $(n_j, u_j)_{0\le j\le m}$ and $b(x)$ (see Remark 2.1).

Take the following ansatz:

$$(n^c, u^c, E^c, B^c) = \sum_{j\ge 0} c^{-j}(n^j, u^j, E^j, B^j), \tag{2.6}$$

in terms of c for the solutions to the problem (2.1)–(2.5). Substituting the expansions (2.6) into the system (2.1)–(2.5), we obtain

(1) The leading profiles (n^0, u^0, E^0, B^0) satisfy the following problem:

$$\begin{cases} \partial_t n^0 + \mathrm{div}(n^0 u^0) = 0, \\ \partial_t u^0 + (u^0 \cdot \nabla)u^0 + \nabla h(n^0) = -E^0, \\ \mathrm{div} E^0 = b(x) - n^0, \quad \nabla \times E^0 = 0, \\ \mathrm{div}\ B^0 = 0, \quad \nabla \times B^0 = 0, \\ t = 0\ : \quad (n^0, u^0) = (n_0, u_0). \end{cases} \tag{2.7}$$

This is the so-called Euler-Poisson system in plasma physics because the equation $\nabla \times E^0 = 0$ implies that the electric field is the gradient of some potential function.

(2) For any $j \ge 1$, the profiles (n^j, u^j, E^j, B^j) satisfy the following problem for linearized

equations:

$$
\begin{cases}
\partial_t n^j + \sum_{k=0}^{j} \operatorname{div}(n^k u^{j-k}) = 0, \\
\partial_t u^j + \sum_{k=0}^{j} (u^k \cdot \nabla) u^{j-k} + \nabla(h'(n^0)n^j + h^{j-1}((n^k)_{k\le j-1}) = -E^j - \sum_{k=0}^{j-1} u^k \times B^{j-1-k}, \\
\partial_t E^{j-1} - \nabla \times B^j = \sum_{k=0}^{j-1} n^k u^{j-1-k}, \quad \partial_t B^{j-1} + \nabla \times E^j = 0, \\
\operatorname{div} E^j = -n^j, \quad \operatorname{div} B^j = 0, \\
t = 0: \quad (n^j, u^j) = (n_j, u_j),
\end{cases}
\tag{2.8}
$$

where $h^0 = 0$ and for $j \ge 2$, $h^{j-1}((n^k)_{k\le j-1})$ is defined by

$$
h\Big(n^0 + \sum_{j\ge 1} c^{-j} n^j\Big) = h(n^0) + h'(n^0) \sum_{j\ge 1} c^{-j} n^j + \sum_{j\ge 2} c^{-j} h^{j-1}((n^k)_{k\le j-1}).
$$

The fact that h^{j-1} depends only on $(n^k)_{k\le j-1}$ can be obtained from the following relation:

$$
h^{j-1}((n^k)_{k\le j-1}) = \frac{1}{j!} \frac{d^j}{d\lambda^j} h\Big(n^0 + \sum_{i\ge 1} \lambda^i n^i\Big)\Big|_{\lambda=0} - h'(n^0) n^j, \quad \text{with} \ \ \lambda = c^{-1}.
$$

Remark 2.1 From $(2.7)_{3,4}$ and $(2.8)_{3,4}$, we see that each order profile of (B^c, E^c) is given by profiles of (n^c, u^c) explicitly, and hence each order profile of the initial data (B^c, E^c) given in (2.5) should be determined by (n_0^c, u_0^c) and $b(x)$ completely. Thus, in Section 4, certain well-prepared initial value conditions are imposed for $(E^c, B^c)(t=0)$, i.e, $(E_j, B_j)(x) = (E^j, B^j)(x, 0)$, $j = 1, \cdots, m$ (refer to Remark 3.1 in Section 3 and the assumption (4.11) given in Section 4), which avoid the presence of initial layers. Of course, it is an interesting problem to study the case of general initial data where an initial layer will occur (see Section 5).

Remark 2.2 The system (2.8) for $j \ge 2$ is new in plasma physics, where the magnetic field B^j satisfies the linear curl-div equation

$$
-\nabla \times B^j = f^j((n^k, u^k, E^k)_{0\le k\le j-1}), \quad \operatorname{div} B^j = 0,
$$

but the electric field is a rotational one with the rotation $\nabla \times E^j = -\partial_t B^{j-1}$ for $j \ge 2$.

Remark 2.3 For the fixed integer $m \ge 1$, $(2.8)_4$ for $j \le m-1$ can be obtained from $(2.8)_3$ by taking div of equations $(2.8)_3$, but when $j = m$, it is no longer redundant for solving E^m and B^m. This is the reason why we keep it in the system (2.1)–(2.5).

3 Determination of Formal Expansions

3.1 Preliminary

From the equations (2.8), we know that once (n^0, u^0, E^0, B^0) are solved from the problem

(2.7), (n^1, u^1, E^1, B^1) are solutions to the following problem for a linearized equations:

$$\begin{cases} \partial_t n^1 + \mathrm{div}(n^0 u^1 + n^1 u^0) = 0, \\ \partial_t u^1 + (u^0 \cdot \nabla) u^1 + (u^1 \cdot \nabla) u^0 + \nabla(h'(n^0) n^1) = -E^1 - u^0 \times B^0, \\ \mathrm{div} E^1 = -n^1, \quad \nabla \times E^1 = -\partial_t B^0, \\ \mathrm{div} B^1 = 0, \quad \nabla \times B^1 = \partial_t E^0 - n^0 u^0, \\ t = 0: \quad (n^1, u^1) = (n_1, u_1). \end{cases} \tag{3.1}$$

Inductively, suppose that $(n^k, u^k, E^k, B^k)_{k \le j-1}$ are solved already for some $j \ge 2$, from (2.8) we know that (n^j, u^j, E^j) satisfy the following linear problem

$$\begin{cases} \partial_t n^j + \mathrm{div}(n^0 u^j + n^j u^0) = -\displaystyle\sum_{k=1}^{j-1} \mathrm{div}(n^k u^{j-k}), \\ \partial_t u^j + (u^0 \cdot \nabla) u^j + (u^j \cdot \nabla) u^0 + \nabla(h'(n^0) n^j) \\ \qquad = -E^j - \nabla(h^{j-1}((n^k)_{k \le j-1})) - \displaystyle\sum_{k=1}^{j-1} (u^k \cdot \nabla) u^{j-k} - \sum_{k=0}^{j-1} u^k \times B^{j-1-k}, \\ \mathrm{div} E^j = -n^j, \quad \nabla \times E^j = -\partial_t B^{j-1}, \\ t = 0: \quad (n^j, u^j) = (n_j, u_j) \end{cases} \tag{3.2}$$

and B^j satisfies the linear curl-div equations

$$-\nabla \times B^j = -\partial_t E^{j-1} + \sum_{k=0}^{j-1} n^k u^{j-1-k}, \quad \mathrm{div} B^j = 0. \tag{3.3}$$

Thus, in order to determine the profiles of (n^c, u^c, E^c, B^c) we require to solve the nonlinear problem (2.7) for (n^0, u^0, E^0, B^0), the linear system (3.2) and the linear curl-div equations (3.3).

Remark 3.1 It follows from (3.1)–(3.3) and Remark 2.1 that for any fixed $j \in \mathbb{N}$, $(E^j, B^j)(x, 0)$ will be determined by $b(x)$ and $(n_k, u_k)_{0 \le k \le j}$.

For convenience, for a given scalar or vector function $v(x,t)$, denote the mean value of $v(x,t)$ in $\mathbb{T}^3$ with respect to x by

$$\mathbf{m}(v) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} v(x, \cdot) dx.$$

In the following, we look for the profiles (n^j, u^j, E^j, B^j) satisfying $\mathbf{m}(E^j) = \mathbf{m}(B^j) = 0$. The main assumptions on the data are as follows:

(H) the ion density b and the initial data $(n_j, u_j)_{j \ge 0}$, satisfy the following conditions:

$$b, n_j, u_j \in H^s(\mathbb{T}^3) \quad \text{for } s \ge N+2,\ N \ge j \quad \text{and} \quad n_0 \ge \delta, \quad \mathbf{m}(n_0 - b) = \mathbf{m}(n_j) = 0 \quad \text{for } j \ge 1$$

for some integers s, $N \ge 1$ and constant $\delta > 0$.

3.2 Existence and uniqueness of solutions (n^0, u^0, E^0, B^0)

Let us begin with the following result.

Lemma 3.1 *Let $s \geq 0$ be an integer. Assume that $f(\cdot,t) \in (H^s(\mathbb{T}^3))^3$, $g(\cdot,t) \in H^s(\mathbb{T}^3)$ with $\mathrm{div} f(\cdot,t) = 0$ and $\mathbf{m}(g) = 0$ for $t \geq 0$. Then there exists a unique classical solution $z(\cdot,t) \in (H^{s+1}(\mathbb{T}^3))^3$ to the following linear curl-div equations*

$$\nabla \times z = f, \tag{3.4}$$

$$\mathrm{div}\, z = g \tag{3.5}$$

for $x \in \mathbb{T}^3$ and $t > 0$, with $\mathbf{m}(z) = 0$. Moreover, for all $t \geq 0$, the solution z satisfies the following estimate

$$\|z(\cdot,t)\|_{H^{s+1}(\mathbb{T}^3)} \leq C_1(\|f(\cdot,t)\|_{H^s(\mathbb{T}^3)} + \|g(\cdot,t)\|_{H^s(\mathbb{T}^3)}) \tag{3.6}$$

for some positive constant C_1 independent of t.

Proof Uniqueness

Let z_1, z_2 be any two solutions of the equations. Then $\widetilde{z} = z_1 - z_2$ solves

$$\nabla \times \widetilde{z} = 0, \tag{3.7}$$

$$\mathrm{div}\, \widetilde{z} = 0, \tag{3.8}$$

$$\mathbf{m}(\widetilde{z}) = 0. \tag{3.9}$$

Acting $\nabla\times$ on (3.7) and using (3.8) and the vector analysis formula

$$\nabla \times (\nabla \times \widetilde{z}) = \nabla \mathrm{div} \widetilde{z} - \Delta \widetilde{z},$$

we get

$$-\Delta \widetilde{z} = 0,$$

which, together with (3.9), yields $\widetilde{z} = 0$. This proves the uniqueness.

Existence

The proof of existence is constructive. Since the system (3.4)–(3.5) is linear, it suffices to prove the existence of smooth solutions of the following two linear systems:

$$\nabla \times z_1 = f, \quad \mathrm{div}\, z_1 = 0, \tag{3.10}$$

$$\nabla \times z_2 = 0, \quad \mathrm{div}\, z_2 = g \tag{3.11}$$

for $x \in \mathbb{T}^3$ and $t > 0$, with $\mathbf{m}(z_1) = \mathbf{m}(z_2) = 0$.

First, let p be the unique periodic solution of the Poisson equation

$$-\Delta p = f, \quad x \in \mathbb{T}^3,\ t \geq 0, \quad \mathbf{m}(p) = 0.$$

Then it follows from the Poisson equation that $\mathrm{div}\, p = r(t)$ for some $r(t)$ independent of x due to $\mathrm{div} f = 0$ and the periodicity of the vector potential p, and hence $z_1 = \nabla \times p - \mathbf{m}(\nabla \times p)$ solves the linear system (3.10).

Next, let q solve the system

$$-\Delta q = g, \quad x \in \mathbb{T}^3,\ t \geq 0, \quad \mathbf{m}(q) = 0.$$

Then $z_2 = -\nabla q - \mathbf{m}(-\nabla q)$ solves the system (3.11).

Finally, noting that

$$\|\nabla z\|^2_{L^2(\mathbb{T}^3)} = \|\nabla \times z\|^2_{L^2(\mathbb{T}^3)} + \|\mathrm{div}\, z\|^2_{L^2(\mathbb{T}^3)},$$

we can easily establish the estimate (3.6) by using (3.4)–(3.5) and the Poincare's inequality due to $\mathbf{m}(z) = 0$. Notice that C_1 does not depend upon $f(x,t)$ and $g(x,t)$. Hence C_1 is independent of t.

The proof of Lemma 3.1 is complete.

Now we turn to the existence and uniqueness of smooth solutions to the nonlinear problem (2.7). Using Lemma 3.1, we get

$$B^0 = 0,$$

due to $\nabla \times B^0 = 0, \mathrm{div} B^0 = 0$ and $\mathbf{m}(B^0) = 0$.

Since $\nabla \times E^0 = 0$ with $\mathbf{m}(E^0) = 0$, we get

$$E^0 = -\nabla\psi^0 + \mathbf{m}(\nabla\psi^0), \tag{3.12}$$

and hence

$$-\Delta\psi^0 = b - n^0. \tag{3.13}$$

By Green's formulation, it follows from (3.13) and $(2.7)_1$ that

$$\nabla\psi^0(x,t) = -\nabla\Delta^{-1}(b(x) - n_0(x)) - \nabla\Delta^{-1}\mathrm{div}\int_0^t (n^0u^0)(x,s)ds. \tag{3.14}$$

Hence, together with (3.12), (3.14) and $B^0 = 0$, the system (2.7) can be reduced to the equivalent form of the following equations with a non-local source term

$$\partial_t n^0 + \mathrm{div}(n^0u^0) = 0, \tag{3.15}$$

$$\begin{aligned}\partial_t u^0 + (u^0\cdot\nabla)u^0 + \nabla h(n^0) = &-\Big(\nabla\Delta^{-1}(b(x) - n_0(x)) + \nabla\Delta^{-1}\mathrm{div}\int_0^t (n^0u^0)(x,s)ds\Big)\\ &+\mathbf{m}\Big(\nabla\Delta^{-1}(b(x) - n_0(x)) + \nabla\Delta^{-1}\mathrm{div}\int_0^t (n^0u^0)(x,s)ds\Big),\end{aligned} \tag{3.16}$$

$$t = 0: \quad (n^0, u^0) = (n_0, u_0). \tag{3.17}$$

Thus, to solve the system (2.7), it suffices to solve the system (3.15)–(3.17). Since the non-local source term $\nabla\Delta^{-1}\mathrm{div}\int_0^t (n^0u^0)(x,s)ds$ is a sum of products of Riesz transforms of $\int_0^t (n^0u^0)(x,s)ds$, we have, by the L^2 boundedness of the Riesz transformation (see [16]),

$$\Big\|\nabla\Delta^{-1}\mathrm{div}\int_0^t (n^0u^0)(\,\cdot\,,s)ds\Big\|_{H^s(\mathbb{T}^3)} \le C_2\Big\|\int_0^t (n^0u^0)(\,\cdot\,,s)ds\Big\|_{H^s(\mathbb{T}^3)}$$

for some constant $C_2 > 0$ independent of t.

Also, since $\nabla\Delta^{-1}$ is a linear bounded operator from $V = \{v \in L^2(\mathbb{T}^3) \mid \mathbf{m}(v) = 0\}$ into $H^1(\mathbb{T}^3)$, we have

$$\|\nabla\Delta^{-1}(b - n_0)\|_{H^{s+1}(\mathbb{T}^3)} \le C_3\|b - n_0\|_{H^s(\mathbb{T}^3)}$$

for some constant $C_3 > 0$. Noticing the above two crucial facts, using the standard iteration techniques of local existence theory for symmetrizable hyperbolic systems (see [9]), we have

Proposition 3.1 *Under the assumption* (H)*, the periodic problem* (3.15)–(3.17) *has a unique smooth solution* (n^0, u^0) *with* $n^0 \geq \frac{\delta}{2}$*, well-defined on* $\mathbb{T}^3 \times [0, T_0)$ *for some* $0 < T_0 \leq +\infty$ *depending only on* n_0 *and* u_0*. Hence the nonlinear periodic problem* (2.7) *admits a unique solution* (n^0, u^0, E^0, B^0) *satisfying*

$$n^0, u^0 \in \bigcap_{l=0}^{N} C^l([0, T_0), H^{s-l}(\mathbb{T}^3)), \quad E^0 \in \bigcap_{l=0}^{N+1} C^l([0, T_0), H^{s+1-l}(\mathbb{T}^3)), \quad \mathbf{m}(E^0) = B^0 = 0.$$

The regularity of (n^0, u^0) stated above is easily obtained from (3.15)–(3.17) while the regularity of E^0 is easily obtained from the Poisson equations (3.12)–(3.14), which improves the regularity of E^0 with respect to the time t.

3.3 Existence and uniqueness of solutions (n^j, u^j, E^j, B^j) for $j \geq 1$

Now, let us briefly describe the solvability of (n^j, u^j, E^j, B^j) for any $j \geq 1$ from the problems (3.1)–(3.3) provided that we have known $(n^k, u^k, E^k, B^k)_{k \leq j-1}$ already. Obviously, from (3.1) and $B^0 = 0$ we know that (n^1, u^1, E^1) satisfy the following linearized Euler-Poisson system

$$\begin{cases} \partial_t n^1 + \mathrm{div}(n^0 u^1 + n^1 u^0) = 0, \\ \partial_t u^1 + (u^0 \cdot \nabla) u^1 + (u^1 \cdot \nabla) u^0 + \nabla(h'(n^0) n^1) = -E^1, \\ \mathrm{div} E^1 = -n^1, \quad \nabla \times E^1 = 0, \\ t = 0: \quad (n^1, u^1) = (n_1, u_1) \end{cases} \tag{3.18}$$

and B^1 satisfies

$$-\nabla \times B^1 = -\partial_t E^0 + n^0 u^0, \quad \mathrm{div} B^1 = 0. \tag{3.19}$$

Similarly to Lemma 3.1 and Proposition 3.1, we have

Proposition 3.2 *Let* $T_1 \in (0, T_0)$ *and the assumption* (H) *hold. Then the periodic problem* (3.18)–(3.19) *or the periodic problem* (3.1) *has a unique smooth solution* (n^1, u^1, E^1, B^1)*, well-defined on* $[0, T_1]$*, satisfying*

$$n^1, u^1 \in \bigcap_{l=0}^{N} C^l([0, T_1], H^{s-l-1}(\mathbb{T}^3)),$$
$$E^1 \in \bigcap_{l=0}^{N+1} C^l([0, T_1], H^{s-l}(\mathbb{T}^3)), \quad B^1 \in \bigcap_{l=0}^{N} C^l([0, T_1], H^{s+1-l}(\mathbb{T}^3))$$

in the class $\mathbf{m}(E^1) = \mathbf{m}(B^1) = 0$.

The rest is to solve (3.2) and (3.3) with $j \geq 2$. For (3.3), we get from Lemma 3.1 that (3.3) has a unique smooth solution $B^j = B^j(E^{j-1}, (n^k, u^k)_{0 \leq k \leq j-1})$ satisfying

$$\|B^j\|_{C^{l-1}(0,T_1;H^{s+1-l}(\mathbb{T}^3))} \leq C_4(T_1)\Big(\|E^{j-1}\|_{C^l(0,T_1;H^{s-l}(\mathbb{T}^3))} + \Big\|\sum_{k=0}^{j-1} n^k u^{j-1-k}\Big\|_{C^{l-1}(0,T_1;H^{s-l}(\mathbb{T}^3))}\Big)$$

in the class $\mathbf{m}(B^j) = 0$ provided that we have known E^{j-1} and $(n^k, u^k)_{0 \leq k \leq j-1}$, where $C_4 > 0$ is a constant.

Notice that the system (3.2) with $j \geq 2$ is no longer the linearized Euler-Poisson system because, generally speaking, E^j is not the gradient of some potential function due to the fact

$B^{j-1} \neq 0$. But the existence of the smooth solution to (3.2) can be constructed by so-called Hodge decomposition and some techniques as same as those to establish the well-posed theory for the Euler-Poisson system. Let us outline some main ideas.

First, recall the Hodge decomposition as follows. For any vector field $E^j \in (L^2(\mathbb{T}^3))^3$, denote by $\mathcal{P}E^j$ and $\mathcal{Q}E^j$ respectively the divergence-free part and gradient part of E^j. Then $\mathcal{Q}E^j = \nabla\Delta^{-1}(\operatorname{div} E^j)$ and $\mathcal{P}E^j = E^j - \mathcal{Q}E^j$.

Thus, using the Hodge decomposition, we can rewrite (3.2) as

$$\begin{cases} \partial_t n^j + \operatorname{div}(n^0 u^j + n^j u^0) = -\displaystyle\sum_{k=1}^{j-1} \operatorname{div}(n^k u^{j-k}), \\ \partial_t u^j + (u^0 \cdot \nabla) u^j + (u^j \cdot \nabla) u^0 + \nabla(h'(n^0) n^j) \\ \qquad = -\mathcal{Q}E^j - \mathcal{P}E^j - \nabla(h^{j-1}(n^k)_{k \le j-1})) - \displaystyle\sum_{k=1}^{j-1} (u^k \cdot \nabla) u^{j-k} - \sum_{k=0}^{j-1} u^k \times B^{j-1-k}, \\ \operatorname{div} \mathcal{Q}E^j = -n^j, \quad \nabla \times \mathcal{Q}E^j = 0, \\ \operatorname{div} \mathcal{P}E^j = 0, \quad \nabla \times \mathcal{P}E^j = \partial_t B^{j-1}, \\ t = 0: \quad (n^j, u^j) = (n_j, u_j). \end{cases} \tag{3.20}$$

Since the equation of $\mathcal{P}E^j$ is independent of the others in the system (3.20), it follows from Lemma 3.1 that $(3.20)_4$ has a unique smooth solution $\mathcal{P}E^j = \mathcal{P}E^j(\partial_t B^{j-1})$ satisfying

$$\|\mathcal{P}E^j\|_{C^{l-1}([0,T_1];H^{s+1-l}(\mathbb{T}^3))} \le C_5(T_1) \|B^{j-1}\|_{C^l([0,T_1];H^{s-l}(\mathbb{T}^3))} \tag{3.21}$$

in the class $\mathbf{m}(\mathcal{P}E^j) = 0$ provided that $B^{j-1} \in C^l([0,T_1]; H^{s-l}(\mathbb{T}^3))$, where $C_5 > 0$ is a constant.

Only if $\mathcal{P}E^j$ is determined, (3.20) becomes a linear Euler-Poisson system for the unknown variables $(n^j, u^j, \mathcal{Q}E^j)$. Thus, as for Euler-Poisson system (3.1), it is easy to prove that the problem (3.20) has a unique smooth solution $(n^j, u^j, \mathcal{Q}E^j)$ satisfying

$$\begin{aligned} n^j, u^j &\in \bigcap_{l=0}^{N_2} C^l([0,T_1], H^{s-l-1}(\mathbb{T}^3)), \\ \mathcal{Q}E^j &\in \bigcap_{l=0}^{N_2-1} C^l([0,T_1], H^{s-l}(\mathbb{T}^3)) \cap C^{N_2}([0,T_1], H^{s-N_2-1}(\mathbb{T}^3)) \end{aligned}$$

in the class $\mathbf{m}(\mathcal{Q}E^j) = 0$ provided that we have, for all $k \le j-1$ and $N_1 \ge N_2$, that

$$\begin{aligned} &n^k, u^k \in \bigcap_{l=0}^{N_1} C^l([0,T_1], H^{s-l}(\mathbb{T}^3)), \\ &B^k \in \bigcap_{l=0}^{N_2} C^l([0,T_1], H^{s-1-l}(\mathbb{T}^3)), \quad \mathcal{P}E^j \in \bigcap_{l=0}^{N_2-1} C^l([0,T_1], H^{s-l}(\mathbb{T}^3)). \end{aligned}$$

In summary, we obtain

Proposition 3.3 *Let* $T_1 \in (0, T_0)$ *and the assumption* (H) *hold. Then the problem* (3.2)–(3.3) *with* $j \ge 2$ *or the problem* (2.8) *with* $j \ge 2$ *has a unique smooth solution* (n^j, u^j, E^j, B^j)

satisfying

$$\begin{cases} n^j, u^j \in \bigcap_{l=0}^{N-(j-2)} C^l([0,T_1], H^{s-l-j}(\mathbb{T}^3)), \\ E^j \in \bigcap_{l=0}^{N-(j-2)} C^l([0,T_1], H^{s-l-(j-1)}(\mathbb{T}^3)), \\ B^j \in \bigcap_{l=0}^{N-(j-1)} C^l([0,T_1], H^{s-l-(j-2)}(\mathbb{T}^3)), \quad \textit{if } j \textit{ is an odd number}; \end{cases}$$

$$\begin{cases} n^j, u^j \in \bigcap_{l=0}^{N-(j-1)} C^l([0,T_1], H^{s-l-j}(\mathbb{T}^3)), \\ E^j \in \bigcap_{l=0}^{N-(j-1)} C^l([0,T_1], H^{s-l-(j-1)}(\mathbb{T}^3)), \\ B^j \in \bigcap_{l=0}^{N-(j-2)} C^l([0,T_1], H^{s-l-(j-2)}(\mathbb{T}^3)), \quad \textit{if } j \textit{ is an even number} \end{cases}$$

in the class $\mathbf{m}(E^j) = \mathbf{m}(B^j) = 0$.

4 Justification of the Expansions

In this section, we rigorously justify the asymptotic expansions of solutions (n^c, u^c, E^c, B^c) to the periodic problem (2.1)–(2.5) developed in Section 2 under the assumption of well-prepared initial data. As a consequence, we obtain the existence of exact solutions (n^c, u^c, E^c, B^c) to (2.1)–(2.5) in a time interval independent of c, and the convergence of (n^c, u^c, E^c, B^c) to the solution (n^0, u^0, E^0, B^0) of the compressible Euler-Poisson equations (2.7) as the light speed c goes to infinity.

In the following, we denote by C various generic constants independent of the light speed c, which can be different from one line to another one.

4.1 Derivation of error equations

For any fixed integers $m \geq 1$ and $s_0 > \frac{5}{2}$, let the assumption (H) hold with $N = m$ and $s = m + s_0 + 3$. Set

$$(n^c_{a,m}, u^c_{a,m}, E^c_{a,m}, B^c_{a,m}) = \sum_{j=0}^{m} c^{-j}(n^j, u^j, E^j, B^j),$$

with (n^j, u^j, E^j, B^j) being given by Propositions 3.1–3.3. From the asymptotic analysis in Sections 2–3, we know that $(n^c_{a,m}, u^c_{a,m}, E^c_{a,m}, B^c_{a,m})$ satisfy the following problem:

$$\begin{cases} \partial_t n^c_{a,m} + \mathrm{div}(n^c_{a,m} u^c_{a,m}) = R^c_n, \\ \partial_t u^c_{a,m} + (u^c_{a,m} \cdot \nabla) u^c_{a,m} + \nabla h(n^c_{a,m}) = -E^c_{a,m} - c^{-1} u^c_{a,m} \times B^c_{a,m} + R^c_u, \\ \partial_t E^c_{a,m} - c \nabla \times B^c_{a,m} = n^c_{a,m} u^c_{a,m} + R^c_E, \quad \mathrm{div} E^c_{a,m} = b(x) - n^c_{a,m}, \\ \partial_t B^c_{a,m} + c \nabla \times E^c_{a,m} = R^c_B, \quad \mathrm{div} B^c_{a,m} = 0, \\ \mathbf{m}(E^c_{a,m}) = \mathbf{m}(B^c_{a,m}) = 0, \\ t = 0: \quad (n^c_{a,m}, u^c_{a,m}, E^c_{a,m}, B^c_{a,m}) = \sum_{j=0}^{m} c^{-j}(n_j, u_j, E^j(\,\cdot\,,0), B^j(\,\cdot\,,0)), \end{cases} \tag{4.1}$$

where the remainders R_n^c, R_u^c, R_E^c and R_B^c satisfy

$$\mathrm{div} R_E^c = -R_n^c, \quad \mathrm{div} R_B^c = 0 \tag{4.2}$$

and for any $0 \le s_1 \le s_0$,

$$\sup_{0\le t\le T_1} \|(R_n^c, R_u^c, R_E^c, R_B^c)(\,\cdot\,,t)\|_{H^{s_1}(\mathbb{T}^3)} \le Cc^{-m}. \tag{4.3}$$

Let (n^c, u^c, E^c, B^c) be the unknown solutions to the problem (2.1)–(2.5), and denote by

$$(N^c, U^c, F^c, G^c) = (n^c - n_{a,m}^c, u^c - u_{a,m}^c, E^c - E_{a,m}^c, B^c - B_{a,m}^c). \tag{4.4}$$

Obviously, (N^c, U^c, F^c, G^c) satisfy the following problem:

$$\begin{cases} \partial_t N^c + \mathrm{div}(N^c(U^c + u_{a,m}^c) + n_{a,m}^c U^c) = -R_n^c, \\ \partial_t U^c + [(U^c + u_{a,m}^c)\cdot\nabla]U^c + (U^c\cdot\nabla)u_{a,m}^c + F^c + \nabla(h(N^c + n_{a,m}^c) - h(n_{a,m}^c)) \\ \qquad = -c^{-1}((U^c + u_{a,m}^c)\times G^c + U^c \times B_{a,m}^c) - R_u^c, \\ \partial_t F^c - c\nabla\times G^c = (N^c + n_{a,m}^c)U^c + N^c u_{a,m}^c - R_E^c, \quad \mathrm{div} F^c = -N^c, \\ \partial_t G^c + c\nabla \times F^c = -R_B^c, \quad \mathrm{div}\, G^c = 0, \\ t = 0: \quad (N^c, U^c, F^c, G^c) = \Big(n_0^c - \sum_{j=0}^m c^{-j} n_j, u_0^c - \sum_{j=0}^m c^{-j} u_j, E_0^c \\ \qquad\qquad - \sum_{j=0}^m c^{-j} E^j(\cdot, 0), B_0^c - \sum_{j=0}^m c^{-j} B^j(\,\cdot\,,0)\Big). \end{cases} \tag{4.5}$$

Set

$$W^c = \begin{pmatrix} N^c \\ U^c \end{pmatrix}, \quad W_0^c = \begin{pmatrix} n_0^c - \sum_{j=0}^m c^{-j} n_j \\ u_0^c - \sum_{j=0}^m c^{-j} u_j \end{pmatrix},$$

$$(F_0^c, G_0^c) = \Big(E_0^c - \sum_{j=0}^m c^{-j} E^j(\,\cdot\,,0), B_0^c - \sum_{j=0}^m c^{-j} B^j(\,\cdot\,,0)\Big),$$

$$A_i(W^c) = (U^c + u_{a,m}^c)_i I + \begin{pmatrix} 0 & (N^c + n_{a,m}^c)e_i^T \\ h'(N^c + n_{a,m}^c)e_i & 0 \end{pmatrix},$$

$$H_1(W^c) = \begin{pmatrix} (U^c\cdot\nabla)n_{a,m}^c + N^c \mathrm{div}\, u_{a,m}^c \\ (U^c\cdot\nabla)u_{a,m}^c + (h'(N^c + n_{a,m}^c) - h'(n_{a,m}^c))\nabla n_{a,m}^c \end{pmatrix},$$

$$R^c = \begin{pmatrix} -R_n^c \\ -R_u^c \end{pmatrix}, \quad H_2(F^c) = \begin{pmatrix} 0 \\ F^c \end{pmatrix}, \quad H_3(W^c, G^c) = \begin{pmatrix} 0 \\ (U^c + u_{a,m}^c)\times G^c + U^c \times B_{a,m}^c \end{pmatrix},$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$ and y_i denotes the i-th component of $y \in \mathbb{R}^3$. Also, note that from (4.2) and $(4.5)_1$, the redundant equations $\mathrm{div} F^c = -N^c$ and $\mathrm{div}\, G^c = 0$ in the system (4.5) hold only if they are satisfied by the initial conditions. Thus the problem

(4.5) for the unknowns (W^c, F^c, G^c) can be rewritten as

$$\begin{cases} \partial_t W^c + \sum_{i=1}^{3} A_i(W^c)\partial_{x_i} W^c + H_1(W^c) + H_2(F^c) = -c^{-1}H_3(W^c, G^c) + R^c, \\ \partial_t F^c - c\nabla \times G^c = (N^c + n^c_{a,m})U^c + N^c u^c_{a,m} - R^c_E, \\ \partial_t G^c + c\nabla \times F^c = -R^c_B, \\ t = 0: \quad (W^c, F^c, G^c) = (W^c_0, F^c_0, G^c_0), \end{cases} \tag{4.6}$$

with $\operatorname{div} F^c(x,0) = -N^c(x,0)$ and $\operatorname{div} G^c(x,0) = 0$.

It is not difficult to see that the equations of W^c in (4.6) are symmetrizable hyperbolic, i.e. if we introduce

$$A_0(W^c) = \begin{pmatrix} h'(N^c + n^c_{a,m}) & 0 \\ 0 & (N^c + n^c_{a,m})I_{3\times 3} \end{pmatrix},$$

which is positively definite when $N^c + n^c_{a,m} \geq C > 0$ for $c \gg 1$, then $\widetilde{A}_i(W^c) = A_0(W^c)A_i(W^c)$ are symmetric for all $1 \leq i \leq 3$. Here the condition $N^c + n^c_{a,m} \geq C$ for $c \gg 1$ follows from $n^0 \geq \frac{\delta}{2}$ (see Proposition 3.1) and the estimate (4.15) (see Theorem 4.1 below).

4.2 Convergence of the expansions

It is clear that the existence and uniqueness of smooth solutions of (2.1)–(2.5) is equivalent to that of (4.5) or (4.6). Then in order to rigorously justify the asymptotic expansion (2.6), it suffices to prove the existence of the smooth solutions to (4.5) or (4.6) and to obtain their uniform estimates with respect to the light speed c. This will be done by the iteration techniques for symmetrizable hyperbolic problems.

More precisely, we solve the nonlinear problem (4.6) by the following iteration for linear problems (see [9]):

$$\begin{cases} \partial_t W^{c,k+1} + \sum_{i=1}^{3} A_i(W^{c,k})\partial_{x_i} W^{c,k+1} + H_1(W^{c,k}) + H_2(F^{c,k}) = R^c - c^{-1}H_3(W^{c,k}, G^{c,k}), \\ \partial_t F^{c,k+1} - c\nabla \times G^{c,k+1} = (N^{c,k} + n^c_{a,m})U^{c,k} + N^{c,k}u^c_{a,m} - R^c_E, \\ \partial_t G^{c,k+1} + c\nabla \times F^{c,k+1} = -R^c_B, \\ t = 0: \quad (W^{c,k+1}, F^{c,k+1}, G^{c,k+1}) = (W^c_0, F^c_0, G^c_0), \quad k = 0, 1, \cdots \end{cases} \tag{4.7}$$

with

$$W^{c,0}(t,x) = W^c_0(x), \quad (F^{c,0}, G^{c,0})(x,t) = (F^c_0, G^c_0)(x).$$

For studying the problems (4.6) and (4.7), we introduce the Sobolev's norms

$$\|W(t)\|_l = \Big(\sum_{|\alpha| \leq l} \|\partial^\alpha_x W(t)\|^2_{L^2(\mathbb{T}^3)} \Big)^{\frac{1}{2}}, \quad \|W\|_{l,T} = \sup_{0 \leq t \leq T} \|W(t)\|_l, \quad l \in \mathbb{N}^*.$$

The key point for proving the existence of smooth solutions and the convergence of the expansions as $c \to \infty$ is the following a priori estimate.

Lemma 4.1 *Let s_0 and l be two integers such that $\frac{5}{2} < l \le s_0$. Assume* (H) *holds with $N = m$, $s = m + s_0 + 3$ and*

$$\|(W_0^c, F_0^c, G_0^c)\|_l \le D_1 c^{-m} \tag{4.8}$$

for some integer $m \ge 1$ and constant $D_1 > 0$ independent of c. Then there are constants $D_2 > 0$, $D_3 > 0$, $c_0 > 0$ and $T_2 \in (0, T_1]$, such that for all $c \ge c_0$ the solutions $(W^{c,k}, F^{c,k}, G^{c,k})$ of (4.7) *satisfy*

$$\|(W^{c,k}, F^{c,k}, G^{c,k})\|_{l,T_2} \le D_2 c^{-m}, \quad \forall\, k \in \mathbb{N}, \tag{4.9}$$

$$\|\partial_t W^{c,k}\|_{l-1,T_2} \le D_3 c^{-m}, \quad \forall\, k \in \mathbb{N}. \tag{4.10}$$

As the proof of Lemma 4.1 involves only elementary calculations it is postponed to Appendix.

Returning to the problem (2.1)–(2.5) or (4.5), we conclude

Theorem 4.1 *For any fixed integers $s_0 > \frac{5}{2}$ and $m \ge 1$, let the assumption* (H) *with $N = m$ and $s = m + s_0 + 3$ hold. Furthermore, suppose that*

$$(E_j, B_j)(x) = (E^j, B^j)(x, 0), \quad 0 \le j \le m \tag{4.11}$$

with $((E^j, B^j)(x, 0))_{0 \le j \le m}$ being given as in Sections 2–3*,*

$$\operatorname{div} E_0^c = b - n_0^c, \quad \operatorname{div} B_0^c = 0, \quad x \in \mathbb{T}^3, \tag{4.12}$$

$$\Big\|(n_0^c, u_0^c, E_0^c, B_0^c) - \sum_{j=0}^{m} c^{-j}(n_j, u_j, E_j, B_j)\Big\|_{s_0} \le C c^{-m}. \tag{4.13}$$

Then the problem (2.1)–(2.5) *has a unique solution*

$$(n^c, u^c, E^c, B^c) \in C([0, T_2], H^{s_0}(\mathbb{T}^3)) \cap C^1([0, T_2], H^{s_0 - 1}(\mathbb{T}^3)) \tag{4.14}$$

satisfying

$$\Big\|(n^c, u^c, E^c, B^c) - \sum_{j=0}^{m} c^{-j}(n^j, u^j, E^j, B^j)\Big\|_{s_0, T_2} \le C c^{-m}, \tag{4.15}$$

where $(n^j, u^j, E^j, B^j)_{0 \le j \le m}$ are solutions to the problems (2.7)–(2.8) *and $C > 0$ is a constant independent of c.*

Proof First, the uniform estimates (4.9)–(4.10) together with (4.7) yield the boundedness of the sequence $(W^{c,k}, F^{c,k}, G^{c,k})_{k\in\mathbb{N}}$ in $L^\infty([0, T_2], H^{s_0}(\mathbb{T}^3)) \cap W^{1,\infty}([0, T_2], H^{s_0-1}(\mathbb{T}^3))$, since $m \ge 1$. Or the injection $H^{s_0}(\mathbb{T}^3) \subset C^1(\mathbb{T}^3)$ is compact due to $s_0 > \frac{5}{2}$. Then Aubin's lemma (see [14]) implies that $(W^{c,k}, F^{c,k}, G^{c,k})_{k\in\mathbb{N}}$ is compact in $C([0, T_2], C^1(\mathbb{T}^3))$. Consequently, up to a subsequence, $(W^{c,k}, F^{c,k}, G^{c,k})_{k\in\mathbb{N}}$ converges to some (W^c, F^c, G^c) in the space $C([0, T_2], C^1(\mathbb{T}^3))$ as $k \to +\infty$. Combining this with the boundedness results (4.9)–(4.10), we obtain $(W^c, F^c, G^c) \in C([0, T_2], H^{s_0}(\mathbb{T}^3)) \cap \mathrm{Lip}([0, T_2], H^{s_0-1}(\mathbb{T}^3))$. In addition, a similar argument as in [9, Theorem 2.1(b)] gives $(W^c, F^c, G^c) \in C^1([0, T_2], H^{s_0-1}(\mathbb{T}^3))$. Passing to the limit $k \to \infty$ in the system (4.7) shows that (W^c, F^c, G^c) is a classical solution to the problem (4.6). By the transform (4.4), this shows the existence and uniqueness of smooth solutions $(n^c, u^c, E^c, B^c) \in C([0, T_2], H^{s_0}(\mathbb{T}^3)) \cap C^1([0, T_2], H^{s_0-1}(\mathbb{T}^3))$ to the problem (2.1)–(2.5). The uniqueness of smooth solutions follows from a standard argument, and implies the convergence of the whole sequence $(W^{c,k}, F^{c,k}, G^{c,k})_{k\in\mathbb{N}}$ to (W^c, F^c, G^c).

Next, from (4.3), we know that to have the estimate (4.15), we need

$$(n^c_{a,m}, u^c_{a,m}, E^c_{a,m}, B^c_{a,m}) = \sum_{j=0}^{m} c^{-j}(n^j, u^j, E^j, B^j) \in C^1([0,T_1], H^{s_0+1}(\mathbb{T}^3)),$$

which can be guaranteed by the assumption $s = m+s_0+3$ and Propositions 3.1–3.3. Moreover, the assumptions (4.11) and (4.13) imply (4.8) while the assumptions (4.11) and (4.12) give $\mathrm{div}F^c(x,0) = -N^c(x,0)$ and $\mathrm{div}\, G^c(x,0) = 0$ according to the transform (4.4). Finally, the estimate (4.15) can be easily derived from the estimate (4.9). This ends the proof of Theorem 4.1.

5 A Study of Initial Layers

5.1 Asymptotic expansions

In Theorem 4.1, the compatibility condition (4.11) is made on the initial data (E^c_0, B^c_0). This condition means that the initial profiles $(E^j, B^j)(\cdot, 0)$ are determined through the resolution of the problems (2.7)–(2.8) for (n^j, u^j, E^j, B^j). Then (E^c_0, B^c_0) are not given explicitly. If the condition (4.11) does not hold, the phenomenon of initial layers occurs. In this section, we consider the limit $c \to 0$ in the problem (2.1)–(2.5) with general initial data. We show that the condition (4.11) can be removed by constructing a first correction of the initial conditions.

Let the initial data $(n^c_0, u^c_0, E^c_0, B^c_0)$ have the asymptotic expansion of the form:

$$(n^c_0, u^c_0, E^c_0, B^c_0) = (n_0, u_0, E_0, B_0) + c^{-1}(n_1, u_1, E_1, B_1) + O(c^{-1}). \tag{5.1}$$

In view of the Euler equations, there do not exist first order initial layers on the variables (n, u). Then we may take the following ansatz:

$$(n^c, u^c)(x,t) = (n^0, u^0)(x,t) + c^{-1}[(n^1, u^1)(x,t) + (n^1_I, u^1_I)(x,\tau)] + O(c^{-1}), \tag{5.2}$$

$$\begin{aligned}(E^c, B^c)(x,t) = {} & (E^0, B^0)(x,t) + (E^0_I, B^0_I)(x,\tau)\\ & + c^{-1}[(E^1, B^1)(x,t) + (E^1_I, B^1_I)(x,\tau)] + O(c^{-1}),\end{aligned} \tag{5.3}$$

where $\tau = ct \in \mathbb{R}$ is the fast variable and the subscript "I" stands for the initial layer variables. Notice that the Maxwell equations (2.3)–(2.4) are linear for E and B. Then the profiles (E^j, B^j) and (E^j_I, B^j_I) can be treated separately for $j = 0, 1$.

Substituting the expressions (5.1)–(5.3) into the problem (2.1)–(2.5), we have

(1) The leading profiles (n^0, u^0, E^0, B^0) satisfy the problem (2.7) in which $B^0 = 0$, i.e.,

$$\begin{cases} \partial_t n^0 + \mathrm{div}\,(n^0 u^0) = 0, \\ \partial_t u^0 + (u^0 \cdot \nabla) u^0 + \nabla h(n^0) = -E^0, \\ \mathrm{div}\, E^0 = b(x) - n^0, \quad \nabla \times E^0 = 0, \\ B^0 = 0, \\ t = 0: \quad (n^0, u^0) = (n_0, u_0). \end{cases} \tag{5.4}$$

(2) The leading initial layers (E_I^0, B_I^0) satisfy

$$\begin{cases} \partial_\tau E_I^0 - \nabla \times B_I^0 = 0, \\ \partial_\tau B_I^0 + \nabla \times E_I^0 = 0, \\ \operatorname{div} E_I^0 = 0, \quad \operatorname{div} B_I^0 = 0, \\ t = 0: \quad (E_I^0, B_I^0) = (E_0 - E^0(\,\cdot\,, 0), B_0) \end{cases} \tag{5.5}$$

for which we need

$$\operatorname{div} E_I^0(x, 0) = \operatorname{div} B_I^0(x, 0) = 0.$$

Therefore

$$\operatorname{div} E_0 = b - n_0, \quad \operatorname{div} B_0 = 0. \tag{5.6}$$

(3) The second order profiles (n^1, u^1, E^1, B^1) satisfy the equations in (3.1). Since $B^0 = 0$, we may take

$$n^1 = 0, \quad u^1 = 0, \quad E^1 = 0. \tag{5.7}$$

Then B^1 is determined by

$$\operatorname{div} B^1 = 0, \quad \nabla \times B^1 = \partial_t E^0 - n^0 u^0. \tag{5.8}$$

(4) The second order initial layers $(n_I^1, u_I^1, E_I^1, B_I^1)$ satisfy

$$n_I^1 = 0, \quad \partial_\tau u_I^1 + E_I^0 = 0, \quad u_I^1(\,\cdot\,, 0) = u_1, \tag{5.9}$$

$$\begin{cases} \partial_\tau E_I^1 - \nabla \times B_I^1 = 0, \\ \partial_\tau B_I^1 + \nabla \times E_I^1 = 0, \\ \operatorname{div} E_I^1 = 0, \quad \operatorname{div} B_I^1 = 0, \\ t = 0: \quad (E_I^1, B_I^1) = (E_1, B_1 - B^1(\,\cdot\,, 0)) \end{cases} \tag{5.10}$$

for which we need

$$\operatorname{div} E_1 = 0, \quad \operatorname{div} B_1 = 0. \tag{5.11}$$

5.2 Convergence

According to the asymptotic expansions discussed above, set

$$\begin{cases} n_a^c(x,t) = n^0(x,t), \quad u_a^c(x,t) = u^0(x,t) + c^{-1} u_I^1(x, ct), \\ E_a^c(x,t) = E^0(x,t) + E_I^0(x, ct) + c^{-1} E_I^1(x, ct), \\ B_c^a(x,t) = B^0(x,t) + B_I^0(x, ct) + c^{-1}(B^1(x,t) + B_I^1(x, ct)). \end{cases} \tag{5.12}$$

Then a straightforward computation gives

$$\begin{cases} \partial_t n_a^c + \operatorname{div}(n_a^c u_a^c) = R_n^c, \\ \partial_t u_a^c + (u_a^c \cdot \nabla) u_a^c + \nabla h(n_a^c) = E_a^c + c^{-1} u_a^c \times B_a^c + R_u^c, \\ \partial_t E_a^c - c\nabla \times B_a^c = n_a^c u_a^c + R_E^c, \quad \operatorname{div} E_a^c = b - n_a^c, \\ \partial_t B_a^c + c\nabla \times E_a^c = R_B^c, \quad \operatorname{div} B_a^c = 0, \\ \mathbf{m}(E_a^c) = \mathbf{m}(B_a^c) = 0, \\ t = 0: \quad (n_a^c, u_a^c, E_a^c, B_a^c) = (n_0, u_0 + c^{-1} u_1, E_0 + c^{-1} E_1, B_0 + c^{-1} B_1), \end{cases} \tag{5.13}$$

where the expressions of the remainders R_n^c, R_u^c, R_E^c and R_B^c are given by

$$R_n^c = c^{-1}\mathrm{div}(n^0 u_I^1) = c^{-1}\nabla n^0 \cdot u_I^1, \tag{5.14}$$

$$\begin{aligned} R_u^c &= c^{-1}[(u^0\cdot\nabla)u_I^1 + (u_I^1\cdot\nabla)u^0 + E_I^1 + u^0\times B_I^0] \\ &\quad + c^{-2}[(u_I^1\cdot\nabla)u_I^1 + (u^0 + c^{-1}u_I^1)(B^1 + B_I^1) + u_I^1 \times B_I^0], \end{aligned} \tag{5.15}$$

$$R_E^c = -c^{-1} n^0 u_I^1, \quad R_B^c = c^{-1}\partial_t B^1. \tag{5.16}$$

Now we seek for an error estimate similar to (4.3) for $(R_n^c, R_u^c, R_E^c, R_B^c)$. To this end, let us interpret the regularity assumptions made in Theorem 4.1 as

(H$'$) $s > \frac{7}{2}$, $n_1 = 0$ and

$$b,\, n_0,\, u_0,\, E_0,\, B_0,\, u_1,\, E_1,\, B_1 \in H^s(\mathbb{T}^3), \quad n_0 \ge \delta \quad \text{in } \mathbb{T}^3, \quad \mathbf{m}(b - n_0) = 0.$$

Then Proposition 3.1 and Lemma 3.1 show that there are unique functions (n^0, u^0, E^0, B^0) with $B^0 = 0$ and B^1, such that

$$\begin{cases} n^0,\, u^0 \in C([0,T_0], H^s(\mathbb{T}^3)) \cap C^1([0,T_0], H^{s-1}(\mathbb{T}^3)), \\ E^0 \in C([0,T_0], H^{s+1}(\mathbb{T}^3)) \cap C^1([0,T_0], H^s(\mathbb{T}^3)), \\ B^1 \in C([0,T_1], H^{s+1}(\mathbb{T}^3)) \cap C^1([0,T_1], H^s(\mathbb{T}^3)), \end{cases} \tag{5.17}$$

with $\mathbf{m}(E^0) = \mathbf{m}(B^1) = 0$ and $n^0 \ge \frac{\delta}{2}$ in $\mathbb{T}^3 \times [0, T_0]$. Note that the assumption $s > \frac{7}{2}$ is needed to guarantee that the injection $H^{s-1}(\mathbb{T}^3) \subset C^1(\mathbb{T}^3)$ is compact (see the proof of Theorem 4.1).

On the other hand, the initial layers (E_I^0, B_I^0) and (E_I^1, B_I^1) satisfy the same linear Maxwell equations. Then there exist global solutions (E_I^0, B_I^0) and (E_I^1, B_I^1) of the problems (5.5) and (5.10). They satisfy the energy estimates

$$\int_{\mathbb{T}^3} (|\partial^\alpha E_I^j(x,\tau)|^2 + |\partial^\alpha B_I^j(x,\tau)|^2)dx = \int_{\mathbb{T}^3} (|\partial^\alpha E_I^j(x,0)|^2 + |\partial^\alpha B_I^j(x,0)|^2)dx, \quad \forall\, \tau > 0$$

for $j = 0, 1$ and all $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s$. Then we deduce that

$$(E_I^j, B_I^j) \in C([0,+\infty), H^s(\mathbb{T}^3)), \quad j = 0, 1, \tag{5.18}$$

with the uniform estimate

$$\|E_I^j(\,\cdot\,,\tau)\|_{H^s(\mathbb{T}^3)} + \|B_I^j(\,\cdot\,,\tau)\|_{H^s(\mathbb{T}^3)} \le K_1, \quad \forall\, \tau > 0, \;\; j = 0, 1, \tag{5.19}$$

where $K_1 > 0$ is a constant independent of τ.

Finally, from (5.9) we obtain

$$u_I^1(x,\tau) = u_1(x) - \int_0^\tau E_I^0(x,s)ds, \tag{5.20}$$

which is non-local with respect to the fast variable τ. In order to establish a uniform estimate of u_I^1 with respect to $\tau > 0$, we set

$$\xi(x,\tau) = \int_0^\tau E_I^0(x,s)ds, \quad \eta(x,\tau) = \int_0^\tau B_I^0(x,s)ds. \tag{5.21}$$

From (5.5), it is easy to verify that ξ and η solve the problem on $\mathbb{T}^3 \times [0, +\infty)$

$$\begin{cases} \partial_\tau \xi - \nabla \times \eta = E_I^0(\,\cdot\,, 0), \\ \partial_\tau \eta + \nabla \times \xi = B_I^0(\,\cdot\,, 0), \\ \operatorname{div} \xi = \operatorname{div} \eta = 0, \\ t = 0: \quad \xi = \eta = 0. \end{cases}$$

Since

$$\operatorname{div} E_I^0(x, 0) = \operatorname{div} E_I^0(x, 0) = 0,$$

there exist functions ϕ, $\psi \in C([0, +\infty), H^{s+1}(\mathbb{T}^3))$ such that

$$E_I^0(x, 0) = \nabla \times \phi(x), \quad B_I^0(x, 0) = -\nabla \times \psi(x).$$

Therefore, ξ and η satisfy, in $\mathbb{T}^3 \times [0, +\infty)$

$$\begin{cases} \partial_\tau (\xi + \psi(x)) - \nabla \times (\eta + \phi(x)) = 0, \\ \partial_\tau (\eta + \phi(x)) + \nabla \times (\xi + \psi(x)) = 0, \\ \operatorname{div} \xi = \operatorname{div} \eta = 0, \\ t = 0: \quad \xi = \eta = 0. \end{cases}$$

This yields the energy estimate

$$\int_{\mathbb{T}^3} (|\partial^\alpha (\xi(x, \tau) + \psi(x))|^2 + |\partial^\alpha (\eta(x, \tau) + \phi(x))|^2) dx = \int_{\mathbb{T}^3} (|\partial^\alpha \phi(x))|^2 + |\partial^\alpha \psi(x))|^2) dx$$

for all $\tau > 0$ and $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s + 1$, from which we deduce a uniform estimate of ξ with respect to τ. Thus, from (5.20) we have

$$\|u_I^1(\,\cdot\,, \tau)\|_{H^s(\mathbb{T}^3)} \le K_2, \quad \forall\, \tau > 0, \tag{5.22}$$

where $K_2 > 0$ is a constant independent of τ.

From (5.14)–(5.19), (5.22) and Moser-type inequalities (see [9, p. 43]), we obtain the following error estimate.

Lemma 5.1 *Let the assumptions* (H′)*,* (5.6) *and* (5.13) *hold. Then we have*

$$\sup_{0 \le t \le T_1} \|(R_n^c, R_u^c, R_E^c, R_B^c)(\,\cdot\,, t)\|_{s-1} \le K_3 c^{-1}, \tag{5.23}$$

where $K_3 > 0$ *is a constant independent of* c*.*

Repeating the same arguments as in Section 4, we obtain finally

Theorem 5.1 *Under the assumptions of Lemma* 5.1 *and*

$$\|(n_0^c, u_0^c, E_0^c, B_0^c) - ((n_0, u_0, E_0, B_0) + c^{-1}(0, u_1, E_1, B_1))\|_{s-1} \le K_4 c^{-1}, \tag{5.24}$$

there is $T_2 \in (0, T_1]$ *such that the problem* (2.1)–(2.5) *has a unique solution* $(n^c, u^c, E^c, B^c) \in C([0, T_2]; H^{s-1}(\mathbb{T}^3)) \cap C^1([0, T_2]; H^{s-2}(\mathbb{T}^3))$ *satisfying*

$$\|(n^c, u^c, E^c, B^c) - (n_a^c, u_a^c, E_a^c, B_a^c)\|_{s-1, T_2} \le K_5 c^{-1}, \tag{5.25}$$

where $K_4 > 0$ *and* $K_5 > 0$ *are constants independent of* c*.*

Appendix Proof of Lemma 4.1

Let $\alpha \in \mathbb{N}^3$ with $|\alpha| \le l$, from (4.7) we know that

$$(W_\alpha^{c,k+1}, F_\alpha^{c,k+1}, G_\alpha^{c,k+1}) = \partial_x^\alpha (W^{c,k+1}, F^{c,k+1}, G^{c,k+1})$$

satisfies the following problem

$$\begin{cases} A_0(W^{c,k})\partial_t W_\alpha^{c,k+1} + \sum_{i=1}^{3} \widetilde{A}_i(W^{c,k})\partial_{x_i} W_\alpha^{c,k+1} = R_{1,\alpha}^{c,k}, \\ \partial_t F_\alpha^{c,k+1} - c\nabla \times G_\alpha^{c,k+1} = R_{2,\alpha}^{c,k}, \\ \partial_t G_\alpha^{c,k+1} + c\nabla \times F_\alpha^{c,k+1} = -\partial_x^\alpha R_B^c, \\ t = 0: \quad (W_\alpha^{c,k+1}, F_\alpha^{c,k+1}, G_\alpha^{c,k+1}) = \partial_x^\alpha (W_0^c, F_0^c, G_0^c), \end{cases} \tag{A1}$$

where

$$\begin{aligned} R_{1,\alpha}^{c,k} &= A_0(W^{c,k})\partial_x^\alpha \Big(R^c - H_1(W^{c,k}) - H_2(F^{c,k}) - \frac{1}{c} H_3(W^{c,k}, G^{c,k}) \Big) \\ &\quad + \sum_{i=1}^{3} A_0(W^{c,k})(A_i(W^{c,k})\partial_{x_i} W_\alpha^{c,k+1} - \partial_x^\alpha (A_i(W^{c,k})\partial_{x_i} W^{c,k+1})), \\ R_{2,\alpha}^{c,k} &= \partial_x^\alpha ((N^{c,k} + n_{a,m}^c)U^{c,k} + N^{c,k} u_{a,m}^c - R_E^c). \end{aligned}$$

Estimates (4.9)–(4.10) are obviously true for $k = 0$ with any $T_2 > 0$. By induction on k, suppose (4.9)–(4.10) hold for some $k \ge 1$ where $D_2 > 0$ and $T_2 > 0$ are to be fixed, and we want to prove (4.9) for $k + 1$, i.e.,

$$\|(W^{c,k+1}, F^{c,k+1}, G^{c,k+1})\|_{l,T_2} \le D_2 c^{-m},$$

which implies, together with $(4.7)_1$, that

$$\|\partial_t W^{c,k+1}\|_{l-1,T_2} \le D_3 c^{-m}.$$

In what follows we let D_i ($i \ge 4$) be various positive constants independent of c, $k \in \mathbb{N}$, D_2 and D_3.

First, (4.9) implies that the matrix $A_0(W^{c,k})$ is positively definite uniformly with respect to c, k and D_2 and

$$\|(W^{c,k}, F^{c,k}, G^{c,k})\|_{l,T_2} \le 1, \quad \|\partial_t W^{c,k}\|_{l-1,T_2} \le 1$$

for all $c \ge c_0$ with some $c_0 > 0$. It follows that,

$$\mathrm{div} A(W^{c,k}) = \partial_t A_0(W^{c,k}) + \sum_{i=1}^{3} \partial_{x_i} \widetilde{A}_i(W^{c,k})$$

satisfies

$$\|\mathrm{div} A(W^{c,k})\|_{L^\infty([0,T_2]\times\mathbb{T}^3)} \le D_4,$$

since $l > \frac{5}{2}$. Employing the classical energy estimate of symmetric hyperbolic equations to the problem $(\mathrm{A1})_1$, we obtain

$$\sup_{0\le t\le T_2} \|W_\alpha^{c,k+1}(t)\|_{L^2(\mathbb{T}^3)} \le D_5 e^{D_5 T_2} \Big(\|\partial_x^\alpha W_0^c\|_{L^2(\mathbb{T}^3)} + \int_0^{T_2} \|R_{1,\alpha}^{c,k}(\tau)\|_{L^2(\mathbb{T}^3)} d\tau \Big). \tag{A2}$$

Next, by the vector analysis formula

$$\operatorname{div}(f \times g) = (\nabla \times f) \cdot g - (\nabla \times g) \cdot f,$$

the singular term $O(c)$ appearing in Sobolev's energy estimates vanishes, i.e.,

$$\int_{\mathbb{T}^3} (-c\nabla \times G_\alpha^{c,k+1} \cdot F_\alpha^{c,k+1} + c\nabla \times F_\alpha^{c,k+1} \cdot G_\alpha^{c,k+1})dx = c\int_{\mathbb{T}^3} \operatorname{div}(F_\alpha^{c,k+1} \times G_\alpha^{c,k+1})dx = 0.$$

Hence, we get from $(\mathrm{A}1)_{2,3}$ that

$$\begin{aligned}&\sup_{0\le t\le T_2} \|(F_\alpha^{c,k+1}, G_\alpha^{c,k+1})(t)\|_{L^2(\mathbb{T}^3)}\\ &\le D_6\Big(\|\partial_x^\alpha(F_0^c, G_0^c)\|_{L^2(\mathbb{T}^3)} + \int_0^{T_2} \|(R_{2,\alpha}^{c,k}, \partial_x^\alpha R_B^c)(\tau)\|_{L^2(\mathbb{T}^3)}d\tau\Big).\end{aligned} \tag{A3}$$

Combining (A2) and (A3) together, we get

$$\begin{aligned}&\sup_{0\le t\le T_2} \|(W_\alpha^{c,k+1}, F_\alpha^{c,k+1}, G_\alpha^{c,k+1})(t)\|_{L^2(\mathbb{T}^3)}\\ &\le D_6\Big(\|\partial_x^\alpha(F_0^c, G_0^c)\|_{L^2(\mathbb{T}^3)} + \int_0^{T_2} \|(R_{2,\alpha}^{c,k}, \partial_x^\alpha R_B^c)(\tau)\|_{L^2(\mathbb{T}^3)}d\tau\Big)\\ &\quad + D_5e^{D_5T_2}\Big(\|\partial_x^\alpha W_0^c\|_{L^2(\mathbb{T}^3)} + \int_0^{T_2} \|R_{1,\alpha}^{c,k}(\tau)\|_{L^2(\mathbb{T}^3)}d\tau\Big).\end{aligned} \tag{A4}$$

By the definition of $R_{i,\alpha}^{c,k}$ $(i = 1, 2)$, (4.3), the classical Moser-type inequality (see [7, 8]) and Sobolev's embedding lemma with $l > \frac{5}{2}$, we deduce

$$\int_0^{T_2} \|(R_{2,\alpha}^{c,k}, \partial_x^\alpha R_B^c)(\tau)\|_{L^2(\mathbb{T}^3)}d\tau \le C(D_2)T_2c^{-m}, \tag{A5}$$

$$\int_0^{T_2} \|R_{1,\alpha}^{c,k}(\tau)\|_{L^2(\mathbb{T}^3)}d\tau \le D_7T_2c^{-m} + C(D_2)T_2(c^{-m} + \|W^{c,k+1}\|_{l,T_2}). \tag{A6}$$

Here the constant $C(D_2) > 0$ may depend on D_2. Substituting (A5)–(A6) into (A4) and using (4.8), we get

$$\begin{aligned}\|(W^{c,k+1}, F^{c,k+1}, G^{c,k+1})\|_{l,T_2} &\le (D_8(1 + e^{D_5T_2}) + D_5D_7e^{D_5T_2}T_2)c^{-m} + D_6C(D_2)T_2c^{-m}\\ &\quad + D_5e^{D_5T_2}C(D_2)T_2(c^{-m} + \|W^{c,k+1}\|_{l,T_2}).\end{aligned}$$

Now we choose $T_2 > 0$ such that

$$e^{D_5T_2} \le 2, \quad e^{D_5T_2}T_2 \le 1, \quad C(D_2)T_2 \le 1, \quad D_5e^{D_5T_2}C(D_2)T_2 \le \frac{1}{2}.$$

Then

$$\|(W^{c,k+1}, F^{c,k+1}, G^{c,k+1})\|_{l,T_2} \le D_2c^{-m},$$

with $D_2 = 2(3D_8 + D_5D_7 + D_6 + 1)$. This completes the proof of Lemma 4.1.

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