Chin. Ann. of Math.
12B: 2(1991), 230—234.

ON A PROBLEM OF WHITNEY**

GAN DANYAN (干丹岩)*

Abstract

In 1944 H. Whitney raised a problem: Let M be an open smooth n-manifold. Does there exist an imbedding of M into $\mathbf{R}^{2n}$ with no limit point set? Introducing a sort of Morse number for open manifolds and using Whitney trick, the author gives a direct proof of the affirmative answer to it.

In [8] H. Whitney proved the famous harder imbedding theorem: every smooth n-dimensional manifold M can be imbedded in $2n$-dimen-sional Euclidean space $\mathbf{R}^{2n}$. And he raised the following

PROBLEM: Does there exist an imbedding, for M open, with no limit set?

It is regarded to be solved[6] as a corollary of a theorem of M. W. Hirsch[2]. But J. Milnor[5, p.120] still reminds us to prove it.

Using Morse theory and Whitney trick we will give a direct proof of it. And as a byproduct and a preparation we will obtain a necessary and sufficient condition for an open manifold to be bounded.

§ 1. Morse Function and Being Bounded

Every smooth manifold in question is assumed to be probably with boundary, and we suppose the boundary is compact except for § 4.

Let M be a connected open n-dimensional smooth manifold. Take the one-point-compactification of M, it will be denoted by $\overline{M}=M\cup\{*\}$, where $*$ is the infinite point. Since M has countable basis, its one-point-compactification $\overline{M}$ is metrizable, and $\{*\}$ is a closed subset and hence a G_δ set in $\overline{M}$. By Urysohn lemma[1], there exists a continuous function $f:\overline{M}\to[0,1]$ with $f^{-1}(0)=\partial M$ and $f^{-1}(1)=*$.

Define ε: $M\to[0,1)$ by

$$\varepsilon(x):=\frac{1}{2}\min\{f(x),1-f(x)\}.$$

Manuscript received May 27, 1988. Revised October 20, 1988.

* Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, China.

** The project supported by National Natural Science Foundation of China.

Taking a smooth $\varepsilon/2$-approximation of f on M, and then extending it to $\{*\}\to\{1\}$, we obtain g: $\overline{M}\to[0, 1]$. g is continuous on $\overline{M}$ and smooth on M with $g^{-1}(0)=\partial M$ and $g^{-1}(1)=*$. By Theorem 3.1.2 of [3] we can take a smooth $\varepsilon/2$-approximation of g over M with no degenerate critical points, and extending it to $\overline{M}$ continuously we then obtain $h:\overline{M}\to[0, 1]$ with $h^{-1}(0)=\partial M$ and $h^{-1}(1)=*$. Since nondegenerate critical points are isolated critical points, function h has at most countably many critical points. Therefore, we have

Lemma 1. *For every connected open smooth manifold M, there exists a function $f:\overline{M}\to[0, 1]$, such that*

(a) *f is continuous;*

(b) *$f|M$ is smooth over M;*

(c) *f has no degenerate critical points and at most countably many nondegenerate critical points;*

(d) *$f^{-1}(0)=\partial M$ and $f^{-1}(1)=*$.*

A function satisfying the conditions in Lemma 1 is called a Morse function on $\overline{M}$. The Morse number μ of M is the minimum over all Morse functions f on $\overline{M}$ of the number of critical points of f. If $\mu(M)$ is finite, M is said to be of finite type; otherwise M is said to be of infinite type.

An open smooth n-dimensional manifold M is said to be boundable, if there exists a compact smooth n-dimensional manifold N and a smooth imbedding i: $M\hookrightarrow N$ such that $N\setminus i(M)\subset\partial N$. Such a manifold N is unique up to h-cobordism and is called a bounded manifold of M. For simplicity we regard M as a submanifold of its bounded manifold by identifying M and $i(M)$. Thus we have $N\setminus M\subset\partial N$.

Theorem 2. *A connected open smooth manifold is boundable if and only if it is of finite type.*

Proof Clearly every boundable manifold is of finite type.

Let M be a connected open smooth n-dimensional manifold and let M be of finite type. Take the one-point-compactification $\overline{M}$ of M and a Morse function f on $\overline{M}$ with only finitely many non-degenerate critidal points. Let a be a regular value of f with $0<a<1$. Denote $M^a=f^{-1}[0, a]$ and $L^a=f^{-1}(a)$. M^a is a compact submanifold of M; L^a is an $(n-1)$-dimensional submanifold of M; and $\partial M^a=\partial M \amalg L^a$. We need a lemma.

Lemma 3. *For $a<1$ and bigger than all critical values of f, $f^{-1}[a, 1)$ is diffeomorphic to $L^a\times[a, 1)$.*

Proof For every b with $a<b<1$, Theorem 3.4 of [4] implies $f^{-1}[a, b]$ is diffeomorphic to $L^a\times[a, b]$. Extending this diffeomorphism as $b\to 1$ we obtain the required diffeomorphism, written $f^{-1}[a, 1) \xlongequal{\text{DIFF.}} L^a\times[a, 1)$.

Corollary 4. *For a as assumed above $f^{-1}[a, 1]$ is homeomorphic to the cone $C(*, L^a)$ with vertex $*$ and base L^a.*

Continue the proof of Theorem 2. Using Lemma 3, we have

$$f^{-1}[a, b) \overset{\text{DIFF.}}{=\!=\!=} L^a \times [a, b) \overset{\text{DIFF.}}{=\!=\!=} L^a \times [a, 1) \overset{\text{DIFF.}}{=\!=\!=} f^{-1}[a, 1).$$

Thus $f^{-1}[0, b) \overset{\text{DIFF.}}{=\!=\!=} f^{-1}[0, 1)$. Therefore $M^b = f^{-1}[0, b]$ can be considered as the bounded manifold of $M = f^{-1}[0, 1)$, i. e. M is boundable.

§ 2. Closed Imbedding of Finite Type Manifolds

An n-dimensional cobordism is a triple (W, V_0, V_1), where W is a compact n-dimensional smooth manifold, V_0 and V_1 are two compact boundaryless $(n-1)$-dimensional smooth manifolds, such that $\partial W = V_0 \amalg V_1$.

Lemma 5. *Let (W, V_0, V_1) be an n-dimensional cobordism. Let i_α $(\alpha = 0, 1)$ be a smooth imbedding $V_\alpha \hookrightarrow \mathbf{R}^{2n-1} \times \{\alpha\}$ provided $n \neq 2$ and a smooth imbedding $V_\alpha \hookrightarrow \mathbf{R}^2 \times \{0\} \times \{\alpha\}$ such that each component (topological circle) of $i_\alpha(V_\alpha)$ contains no other component in its interior region, provided $n = 2$. Then the imbedding $i_0 \amalg i_1$ can be extended to a neat smooth imbedding $(W, V_0, V_1) \hookrightarrow (\mathbf{R}^{2n-1} \times [0, 1], \mathbf{R}^{2n-1} \times \{0\}, \mathbf{R}^{2n-1} \times \{1\})$, which is perpendicular to $\mathbf{R}^{2n-1} \times \{0\} \amalg \mathbf{R}^{2n-1} \times \{1\}$ over $V_0 \amalg V_1$.*

Proof For $n = 1$, the lemma is trivial.

For $n = 2$, we assume that W is connected without loss of generality. W can be considered as the result of puncturing finite holes on a connected closed surface $\hat{W}$. The boundaries of the holes are circles and constitute $V_0 \amalg V_1$. We assume that V_0 and V_1 are contained in discs D_0 and D_1 of $\hat{W}$ respectively. Using the results in section 5 of [7], we imbed the sphere, projective plane, Klein bottle and add the necessary number of handles to obtain $\hat{W}$. We may get an imbedding of $\hat{W}$ into $\mathbf{R}^3 \times (0, 1)$ perpendicular to $\mathbf{R}^3 \times \left\{\frac{1}{3}\right\}$ and $\mathbf{R}^3 \times \left\{\frac{2}{3}\right\}$ such that the portions in $\mathbf{R}^3 \times \left[0, \frac{1}{3}\right]$ and $\mathbf{R}^3 \times \left[\frac{2}{3}, 1\right]$ are D_0 and D_1 respectively, and with $\partial D_0 \subset \mathbf{R}^2 \times \{0\} \times \left\{\frac{1}{3}\right\}$ and $\partial D_1 \subset \mathbf{R}^2 \times \{0\} \times \left\{\frac{2}{3}\right\}$. The restriction of this imbedding to $\hat{W} \backslash \mathrm{Int.}\,(D_0 \amalg D_1)$ is a neat imbedding into $\mathbf{R}^3 \times \left[\frac{1}{3}, \frac{2}{3}\right]$ and perpendicular over $\partial D_0 \amalg \partial D_1$. Furthermore, we can extend it over W and obtain a required imbedding. Since the domain $D_\alpha \cap W$ $(\alpha = 0, 1)$ of the extension is a punctured sphere, one of the boundary circles of which is ∂D_α and the others constitute V_α.

Now we suppose $n > 2$. Take collars (for simplicity, we omit the symbol of smooth imbeddings) $V_0 \times [0, \delta]$ and $V_1 \times [1-\delta, 1]$ of V_0 and V_1 in W respectively, where $\delta > 0$ is small enough. Map $V_0 \times [0, \delta]$ and $V_1 \times [1-\delta, 1]$ into $\mathbf{R}^{2n-1} \times [0,$

$\delta/2]$ and $\mathbf{R}^{2n-1}\times[1-\delta/2, 1]$ parameter-preservingly with respect to the parameter t for $t\in[0, \delta/2]$ and $[1-\delta/2, 1]$ respectively, into two cones in $\mathbf{R}^{2n-1}\times[\delta/2, \delta]$ and $\mathbf{R}^{2n-1}\times[1-\delta, 1-\delta/2]$ with bases the imbedding images $i_0(V_0)\times\{\delta/2\}$ in $\mathbf{R}^{2n-1}\times\{\delta/2\}$ and $i_1(V_1)\times\{1-\delta/2\}$ in $\mathbf{R}^{2n-1}\times\{1-\delta/2\}$ and vertices in $\mathbf{R}^{2n-1}\times\{\delta\}$ and $\mathbf{R}^{2n-1}\times\{1-\delta\}$ parameter-preservingly for $t\in[\delta/2, \delta]$ and $[1-\delta, 1-\delta/2]$ respectively. Then extend the map defined on $V_0\times[0, \delta]\amalg V_1\times[1-\delta, 1]$ to a continuous map $W\to\mathbf{R}^{2n-1}\times[0, 1]$ such tha t the image of $W\backslash(V_0\times[0, \delta]\amalg V_1\times[1-\delta, 1])$ is contained in $\mathbf{R}^{2n-1}\times(\delta, 1-\delta)$. By Theorem 6 of [9], there is a sufficiently close smooth approximation leaving fixed on $V_0\times[0, \delta_1)\amalg V_1\times(1-\delta_1, 1]$ for some δ_1 with $0<\delta_1<\delta$, which is an immersion. Then delete the selfintersections as in Theorem 5 of [8], we obtain a smooth imbedding, which leaves fixed and maps parameter-preservingly on $V_0\times[0, \delta_1)\amalg V_1\times(1-\delta_1, 1]$. Therefore this imbedding is neat and perpendicular over $V_0\amalg V_1$.

Using this lemma we can prove

Threoem 6. *Let M be a connected open smooth n-manifold of finite type. Then M can be imbedded in $\mathbf{R}^{2n}$ with empty limit set.*

Proof By Theorem 2, M is boundbale, and let W be the bounded manifold of M, which is compact and connected. Denote $V_0=\partial M$ and $V_1=\partial W\backslash\partial M$, we obtain an n-dimensional cobordism (W, V_0, V_1). We imbed V_0 and V_1 into $\mathbf{R}^{2n-1}\times\{0\}$ and $\mathbf{R}^{2n-1}\times\{1\}$ respectively satisfying the hypotheses in Lemma 5. Lemma 5 implies that W can be imbedded smoothly into $\mathbf{R}^{2n-1}\times[0, 1]$, so that the imbedding is perpendicular to $\mathbf{R}^{2n-1}\times\{1\}$ over V_1. Then pushing the hyperplane $\mathbf{R}^{2n-1}\times\{1\}$ in $\mathbf{R}^{2n}$ along the $2n$-th axis to the infinity, we obtain a smooth neat imbedding from M into $\mathbf{R}^{2n-1}\times[0, \infty)$. This imbedding has no limit point in $\mathbf{R}^{2n}$.

§ 3. Closed Imbedding of Infinfte Type Manifolds

Lemma 7. *Let M be a connected open n-manifold of infinite type. Let $\overline{M}$ be the one-point-compactification of M. Then there exists a Morse function f on $\overline{M}$ such that f has distinct values at distinct critical points, and all critical points can be numbered: $x_1, x_2, \cdots, x_k, \cdots$, so that*

$$f(x_1)<f(x_2)<\cdots<f(x_k)<\cdots\to1.$$

Proof Since the critical points of a Morse function on $\overline{M}$ are isolated critical and countably many, and have no limit point in M, we see that the critical values of a Morse functon constitute a sequence converging to 1, and each critical value corresponds to only finite many critical points in M.

Take a Morse function on $\overline{M}$. By means of the method in Lemma 2.8 of [4], we can modify it slightly and obtain a new Morse function with the same critical points such that each critical value corresponds to only one critical point. Denote it by f. Then we can label all critical points $\{x_k\}$ according to the order of their critical values, so $f(x_1)<f(x_2)<\cdots<f(x_k)<\cdots\to 1$.

As a corollary, we have

Lemma 8. *Let M be a connected open smooth n-manifold of infinite type. Then M can be expressed as a composition of countably many cobordisms $c_1, c_2, \cdots, c_k, \cdots$:*

$$M=c_1c_2\cdots c_k\cdots$$

where each c_k has only one critical point.

Proof Take the Morse function f in Lemma 7. Then take $\{a_k\}$ such that $a_0=0$; $f(x_k)<a_k<f(x_{k+1})$, for $k=1, 2, \cdots$. Set $W_k=f^{-1}[a_{k-1}, a_k]$, $V_k=f^{-1}(a_k)$, $c_k=(W_k, V_{k-1}, V_k)$, $k=1, 2, \cdots$. Similarly to § 4 of [4] we can then prove Lemma 8.

Theorem 9. *Let M be a connected open smooth n-manifold of infinite type. Then M is imbeddable in $\mathbf{R}^{2n}$ with empty limit set.*

Proof Take c_k, $k=1, 2, \cdots$ in Lemma 8. For each $k\geq 0$, choose a smooth imbedding $V_k\hookrightarrow\mathbf{R}^{2n-1}\times\{k\}$ satisfying the hypotheses in Lemma 5. By Lemma 5 then take a smooth imbedding j_k: $c_k=(W_k, V_{k-1}, V_k)\hookrightarrow(\mathbf{R}^{2n-1}\times[k-1, k], \mathbf{R}^{2n-1}\times\{k-1\}, \mathbf{R}^{2n-1}\times\{k\})$ which is perpendicular to $\mathbf{R}^{2n-1}\times\{k-1\}$ and $\mathbf{R}^{2n-1}\times\{k\}$ on V_{k-1} and V_k respectively. Define

$$j: M=c_1c_2\cdots c_k\cdots\to\mathbf{R}^{2n}$$

by

$$j|c_k=j_k, \ k=1, 2, \cdots, \cdots.$$

Then j is a smooth imbedding with no limit ponits.

§ 4. The Case of Noncompact Boundary

Let M be a connected open smooth n-manifold with noncompact boundary ∂M. Doubling M we obtain an open manifold without boundary. The preceding results imply that the doubled manifold can be imbedded into $\mathbf{R}^{2n}$ with no limit points, the restriction of this imbedding to M is desired.

References

[1] Dugundji, J., Topology, Allyn & Bacon, Inc., Boston, 1966.

[2] Hirsch, M., On imbedding differentiable manifolds in Euclidean space, *Annals Math.*, **73** (1961), 566–571.

[3] Hirsch, M., Differential Topology, Springer-Verlag, 1976.

[4] Milnor, J., Lectures on the h-cobordism theorem, Princeton Univ. Press, 1965.

[5] Milnor, J., & Stasheff, J., Characteristic classes, Princeton Univ. Press, 1974.

[6] Narasimhan, R., Analysis on real and complex manifolds, North-Holland Publishing Company, 1968.

[7] Whitney, H., On the topology of differentiable manifolds, in Lectures in Topology, Univ. of Michigan Press, 1941, 101–141.

[8] Whitney, H., The selfintersections of smooth n-manifold in $2n$-space, *Ann. Math.*, **45** (1944), 220–246.

[9] Whitney, H., The singularities of a smooth n-manifold in $(2n-1)$-space, *Ann. Math.*, **45** (1944), 247–293.