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COHOMOLOGY VANISHING IN HILBERT SPACES****

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Abstract

The authors give two cohomology vanishing theorems for domains, which are not pseudoconvex, and characterize the holomorphy of domains with smooth boundaries in separable Hilbert spaces through cohomology vanishing.

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§ 1 . Introduction

Oka [40] proved that any additive Cousin problem is solvable in a domain of holomorphy in $\mathbf{C}^n$. Oka [41]-Cartan [2, 3]-Serre [46] generalized this as the theorem B for analytic coherent sheaves over Stein spaces. J. P. Serre [47] proved that a domain D in $\mathbf{C}^n$ is a domain of holomorphy if $\mathrm{H}^p(D,\mathcal{O}) = 0$ holds for all positive integer p smaller than n, $\mathcal{O}$ being the structure sheaf of D. G. Scheja [45] proved the bijectivity of the canonical homomorphism $\mathrm{H}^p(X,\mathcal{F}) \to \mathrm{H}^p(X - A,\mathcal{F})$ of the cohomology of a complex space X with coefficients in an analytic coherent sheaf $\mathcal{F}$ in that of the complement of an analytic set A with respect to X, when $\mathrm{codh}_x\mathcal{F} \geq \dim_x A + p + 2$ at each point $x \in X$. Andreotti-Grauert [1] proved the bijectivity of the canonical homomorphism $\mathrm{H}^p(X,\mathcal{F}) \to \mathrm{H}^p(X - B_c,\mathcal{F})$ of the cohomology of a complex space X defined by a strongly q-pseudoconvex function φ with coefficients in an analytic coherent sheaf $\mathcal{F}$ in that of the complement of a q-convex open set $B_c := \{x \in X;\ \varphi(x) < c\}$ for $c < 0$ with respect to the complex space X, when $p \leq \mathrm{dih}(\mathcal{F}) - q - 1$. J. Kajiwara [18] proved that a domain D with real 1 codimensional continuous boundary in a Stein mainfold S is Stein, if and only if , for any analytic polycylinder P in S, there holds $\mathrm{H}^1(D \cap P,\mathcal{O}) = 0$.

In the infinite dimensional case, S. Dineen [8] proved $\mathrm{H}^1(\Omega,\mathcal{O}) = 0$ for the structure sheaf $\mathcal{O}$ over a pseudoconvex domain Ω in a $\mathbf{C}$-linear space E equipped with the finite open topology. So, the pseudoconvexity implies the cohomology vanishing similarly to the finite

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dimensional case. By L. Gruman [13], pseudoconvex domains are domains of holomorphy in the locally convex space equipped with the finite open topology or in Hilbert spaces.

Kajiwara-Shon [26] proved, however, for a pseudoconvex domain Ω in the **C**-linear locally convex space E equipped with the finite open topology, for an analytic subset A of Ω and for any positive integer $p \leq \operatorname{codim} A - 2$, the cohomology vanishing $\mathrm{H}^p(\Omega - A, \mathcal{O}) = 0$. The complement A of the open set $\Omega - A$ with respect to the open set Ω has no interior point and in case that codim $= \infty$ the cohomology vanishing of all positive degree p does not imply the holomorphy of the domain. Moreover, S. Ohgai [39] proved, for any positive integers p and q, for a pseudoconvex domain Ω in the **C**-linear locally space E equipped with the finite open topology, for a strongly q-convex C^∞ function φ on Ω and for a negative number c, the cohomology vanishing $\mathrm{H}^p(\{x \in \Omega;\ \varphi(x) > c\}, \mathcal{O}) = 0$. In this case, the complement $\{x \in \Omega; \varphi(x) \leq c\}$ of the open set $\{x \in \Omega; \varphi(x) > c\}$ with respect to the open set Ω may have interior points $\in \{x \in \Omega; \varphi(x) > c\}$.

So, in the infinite dimensional case, i.e. for $n = \infty$, the theorem of Serre [47] does not hold and the vanishing of cohomology of all positive degree with coefficients in the structure sheaf $\mathcal{O}$ of the domain Ω assures neither the pseudoconvexity nor the holomorphy of the domain Ω.

X. D. Li [34] proved the holomorphy of a domain Ω with a continuous boundary from the validity of Oka's principle for intersections $\Omega \cap P$ of the domain Ω and polydiscs P in the **C**-linear locally convex space E equipped with the finite open topology.

Recently, L. Lempert [33] solved $\bar{\partial}$-equations in pseudoconvex domains of spaces belonging to the category of Banach spaces, which are equipped with Schauder basis and satisfy his hypothesis (X), and gave cohomology vanishing theorems for pseudoconvex domains in those Banach spaces. The Banach space ℓ^p belongs to the above category of Banach spaces for $p \geq 1$.

The aim of the present paper is to establish two cohomology vanishing theorems of types Scheja and Andreotti-Grauert and to derive the holomorphy, i.e., the pseudoconvexity of a domain Ω from the cohomology vanishing $\mathrm{H}^1(\Omega \cap P) = 0$ for any pseudoconvex domain P in the separable Hilbert space, which is isomorphic with ℓ^2 and is contained in the above category of Lempert.

§ 2 . Theorem of Scheja's Type

Let E be a **C**-linear Hausdorff space and Λ be the set of finite dimensional affine subspaces of E. A complex valued function h on an open subset D of E is said to be Gâtaux holomorphic if, for any $S_f \in \Lambda$, the restriction of h to $D \cap S_f$ is holomorphic on the open subset $D \cap S_f$ of the finite dimensional space S_f. A complex valued Gâtaux holomorphic function h on an open subset D of E is said to be holomorphic if h is continuous on D. The sheaf $\mathcal{O}$ of germs of holomorphic functions over E is called the structure sheaf of E. An open subset D of E is said to be pseudoconvex if, for any $S_f \in \Lambda$, the intersection $D \bigcap S_f$ is a pseudoconvex open set of the finite dimensional complex space S_f. By Ph. Noverraz [37], the above definition of pseudoconvexity is equivalent to several definitions of the pseudoconvexity.

We extend the theorem of G. Scheja [45] to a separable Hilbert space in the following theorem for the structure sheaf.

Theorem 2.1. *Let S a separable Hilbert space, Ω be a pseudoconvex domain in S, $\mathcal{O}$ be the sheaf of germs of holomorphic functions over Ω and A be an analytic subset of Ω. Assume that there exists a positive integer n_0 such that in any finite dimensional affine* *subspace S_f, with dimension not smaller than n_0, the codimension of $A \cap S_f$ is larger than 2 at each point of $A \cap S_f$. Then, there holds* $\mathrm{H}^1(\Omega - A, \mathcal{O}) = 0$.

Proof. Convex open coverings are cofinal between open coverings of the locally convex open set $\Omega - A$. So, let $\mathcal{U} := \{U_i; i \in I\}$ be an open covering of $\Omega - A$ such that each U_i is an open convex set in $\Omega - A$ and $\{f_{ij}; i, j \in I\}$ be a 1-cocycle of the covering $\mathcal{U}$ with coefficients in the structure sheaf $\mathcal{O}$. Since each U_i is pseudoconconvex, for any positive integer μ and non negative integer s with $s \leq \mu - 1$ and for each $i_0, i_1, \cdots, i_s \in I$, we have

$$\mathrm{H}^\mu(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_s}, \mathcal{O}) = 0$$

by [33]. So, $\mathcal{U}$ is a Leray covering of $\Omega - A$ and, by Lemma L_p of [45], we have

$$\mathrm{H}^1(\mathcal{U}, \mathcal{O}) = \mathrm{H}^1(\Omega - A, \mathcal{O}).$$

To begin with finite n pieces of orthonormal vectors of S, which span the affine subspace S_f, we put them in the opening of a complete orthonormal basis of S, we regard S as the space ℓ^2 of sequences $z = (z_1, z_2, \cdots, z_k, \cdots)$ of complex numbers z_k which satisfy

$$\|z\| := \sqrt{\sum_{k=1}^{\infty} |z_k|^2} < +\infty,$$

regard $\mathbf{C}^n$ as its subspace

$$\{z = (z_1, z_2, \cdots, z_n, \cdots) \in \ell^2;\ z_k = 0\ (k > n)\}$$

and regard S_f as an affine subspace of $\mathbf{C}^n$. Without loss of generality, we may assume that $S_f = \mathbf{C}^n$.

Now, we put $\Omega_n := \Omega \cap \mathbf{C}^n$ and $\mathcal{U}_n := \{U_i \cap \mathbf{C}^n; i \in I\}$. Let $\mathcal{O}_n$ be the sheaf of germs of holomorphic functions over Ω_n and $d_n(z)$ be the distance between a point $z \in \Omega_n$ and the boundary $\partial\Omega_n$. Since we have

$$\mathrm{H}^1(\mathcal{U}_n, \mathcal{O}_n) = \mathrm{H}^1(\Omega_n - A, \mathcal{O}_n) = 0 \qquad \text{for} \quad n \geq n_0$$

by [45], for any $i \in I$, there exist holomorphic functions f_{i,n_0} on U_{i,n_0} and g_{i,n_0+1} on U_{i,n_0+1} such that we have

$$\begin{aligned} f_{ij} &= f_{j,n_0} - f_{i,n_0} && \text{on each} \ \ U_{i,n_0} \cap U_{j,n_0}, \\ f_{ij} &= g_{j,n_0+1} - g_{i,n_0+1} && \text{on each} \ \ U_{i,n_0+1} \cap U_{j,n_0+1}. \end{aligned}$$

Put $f_{n_0} := f_{i,n_0} - g_{i,n_0+1}$ on U_{i,n_0}. Then f_{n_0} is a well-defined holomorphic function on Ω_{n_0}.

For any positive integer n with $n \geq n_0$, we put

$$\begin{aligned} &\Omega_n := \{z = (z_\nu) \in \Omega; z_\nu = 0 (\nu > n)\}, \\ &K_n := \Big\{z \in \Omega_n; -\log d_n(z) + \log|z_n| \leq -1, \sum_{\mu=1}^{n} |z_\mu|^2 \leq n^2, -\log d_n(z) \leq \log n\Big\}. \end{aligned} \tag{2.1}$$

Following the method of L. Gruman [13], which proved that a pseudoconvex domain in S is a domain of holomorphy in S, we use a solution v_{n_0+1} with L^2 estimate of a $\bar\partial$-equation $\bar\partial v_{n_0+1} = z_{n_0+1}^{-2} \bar\partial \phi_{n_0+1} \wedge f_{n_0} \circ \pi_{n_0+1}$ for the canonical projection $\pi_{n_0+1}(z_1, z_2, \cdots, z_{n_0+1}) = (z_1, z_2, \cdots, z_{n_0})$ and for a real valued function ϕ_{n_0+1} of class C^∞ on Ω_{n_0+1}, which takes the value 1 in a neighborhood in Ω_{n_0+1} of the union of the compact set K_{n_0+1} and the subset Ω_{n_0} of Ω_{n_0+1} and the value 0 in a neighborhood of the subset $\Omega_{n_0+1}-\pi_{n_0+1}^{-1}(\Omega_{n_0})$ of Ω_{n_0+1}, and we construct a holomorphic function

$$f_{n_0+1} := \phi_{n_0+1} f_{n_0} \circ \pi_{n_0+1} - z_{n_0+1}^2 v_{n_0+1} \qquad \text{on} \ \ \Omega_{n_0+1},$$

which is a revision of C^∞ extension $\phi_{n_0+1} f_{n_0} \circ \pi_{n_0+1}$ to Ω_{n_0+1} of the preceding holomorphic function f_{n_0} on Ω_{n_0} by $\bar{\partial}$-analysis of Hörmander [16] and satisfies

$$|f_{n_0} \circ \pi_{n_0+1}(z_1, z_2, \cdots, z_{n_0+1}) - f_{n_0+1}(z_1, z_2, \cdots, z_{n_0+1})| \le |z_{n_0+1}|^2 \tag{2.2}$$

for $(z_1, z_2, \cdots, z_{n_0+1}) \in K_{n_0+1} \cap \pi_{n_0+1}^{-1}(\Omega_{n_0})$. We revise the cochain $\{g_{i,n_0+1}; i \in I\} \in Z^0(\mathcal{U}_{n_0+1}, \mathcal{O}_{n_0+1})$, as $\{f_{i,n_0+1}; i \in I\} \in Z^0(\mathcal{U}_{n_0+1}, \mathcal{O}_{n_0+1})$ by putting $f_{i,n_0+1} := g_{i,n_0+1} + f_{n_0+1}$ on U_{i,n_0+1} for each $i \in I$. Then, the revised cochain $\{f_{i,n_0+1}; i \in I\}$ satisfies

$$|f_{i,n_0} \circ \pi_{n_0+1}(z_1, z_2, \cdots, z_{n_0+1}) - f_{i,n_0+1}(z_1, z_2, \cdots, z_{n_0+1})| \le |z_{n_0+1}|^2 \tag{2.3}$$

at each point $(z_1, z_2, \cdots, z_{n_0+1}) \in U_{i,n_0+1} \cap K_{n_0+1} \cap \pi_{n_0+1}^{-1}(\Omega_{n_0})$ for any $i \in I$, the majoration $|z_{n_0+1}|^2$ being independent of $i \in I$.

Now, as an assumption of induction with respect to $n \ge n_0$, we assume that there exists a sequence $\{\{f_{i,m}; i \in I\}; m = n_0+1, n_0+2, \cdots, n\}$ of 0-cochains $\{f_{i,m}; i \in I\} \in C^0(\mathcal{U}_m, \mathcal{O}_m)$ such that we have $f_{ij} = f_{j,m} - f_{i,m}$ on each $U_{i,m} \cap U_{j,m}$ and the preceding 0-cochain $\{f_{i,m}; i \in I\}$ and the rear 0-cochain $\{f_{i,m+1}; i \in I\}$ satisfy the inequality

$$|f_{i,m} \circ \pi_{m+1}(z_1, z_2, \cdots, z_{m+1}) - f_{i,m+1}(z_1, z_2, \cdots, z_{m+1})| \le |z_{m+1}|^2 \tag{2.4}$$

at each point $(z_1, z_2, \cdots, z_{m+1}) \in U_{i,m+1} \cap K_{m+1} \cap \pi_{m+1}^{-1}(\Omega_m)$ for any $i \in I$.

Since we have

$$H^1(\mathcal{U}_{n+1}, \mathcal{O}_{n+1}) = H^1(\Omega_{n+1} - A, \mathcal{O}_{n+1}) = 0$$

by [45], for any $i \in I$, there exist holomorphic functions $g_{i,n+1}$ on $U_{i,n+1}$ such that we have

$$f_{ij} = g_{j,n+1} - g_{i,n+1} \qquad \text{on} \ \ U_{i,n_0+1} \cap U_{j,n_0+1}.$$

Put $f_n := f_{i,n} - g_{i,n+1}$ on $U_{i,n}$. Then f_n is a well-defined holomorphic function on Ω_n.

Again, following the method of L. Gruman [13], we use a solution v_{n+1} with L^2 estimate of a $\bar{\partial}$-equation $\bar{\partial} v_{n+1} = z_{n+1}^{-2} \bar{\partial} \phi_{n+1} \wedge f_n \circ \pi_{n+1}$ for the canonical projection $\pi_{n+1}(z_1, z_2, \cdots, z_{n+1}) = (z_1, z_2, \cdots, z_n)$ and for a real valued function ϕ_n of class C^∞ on Ω_{n+1}, which takes the value 1 in a neighborhood in Ω_{n+1} of the union of the compact set K_{n+1} and the subset Ω_n of Ω_{n+1} and the value 0 in a neighborhood of the subset $\Omega_{n+1} - \pi_{n+1}^{-1}(\Omega_n)$ of Ω_{n+1}, and we construct a holomorphic function

$$f_{n+1} := \phi_{n+1} f_n \circ \pi_{n+1} - z_{n+1}^2 v_{n+1} \qquad \text{on} \ \ \Omega_{n+1},$$

which is a revision of C^∞ extension $\phi_{n+1} f_n \circ \pi_{n+1}$ to Ω_{n+1} of the preceding holomorphic function f_n on Ω_n by $\bar{\partial}$-analysis of Hörmander [16], and satisfies

$$|f_n \circ \pi_{n+1}(z_1, z_2, \cdots, z_{n+1}) - f_{n+1}(z_1, z_2, \cdots, z_{n+1})| \le |z_{n+1}|^2 \tag{2.5}$$

at each point $(z_1, z_2, \cdots, z_{n+1}) \in K_{n+1} \cap \pi_n^{-1}(\Omega_n)$. We revise the cochain $\{g_{i,n+1}; i \in I\} \in Z^0(\mathcal{U}_{n+1}, \mathcal{O}_{n+1})$, as $\{f_{i,n+1}; i \in I\} \in Z^0(\mathcal{U}_{n+1}, \mathcal{O}_{n+1})$ by putting $f_{i,n+1} := g_{i,n+1} + f_{n+1}$ on $U_{i,n+1}$ for each $i \in I$. Then, the revised cochain $\{f_{i,n+1}; i \in I\} \in C^0(\mathcal{U}_{n+1}, \mathcal{O}_{n+1})$ satisfies

$$|f_{i,n} \circ \pi(z_1, z_2, \cdots, z_{n+1}) - f_{i,n+1}(z_1, z_2, \cdots, z_{n+1})| \le |z_{n+1}|^2 \tag{2.6}$$

at each point $(z_1, z_2, \cdots, z_{n+1}) \in U_{i,n+1} \cap K_{n+1} \cap \pi_{n+1}^{-1}(\Omega_n)$ for any $i \in I$, the majoration $|z_{n+1}|^2$ being independent of $i \in I$.

Thus, by induction, we have constructed a sequence $\{\{f_{i,m}; i \in I\}; m = n_0+1, n_0+2, \cdots, n, \cdots\}$ of 0-cochains $\{f_{i,m}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_m, \mathcal{O}_m)$ such that $f_{ij} = f_{j,m} - f_{i,m}$ on each $U_{i,m} \cap U_{j,m}$ and the preceding 0-cochain $\{f_{i,m}; i \in I\}$ and the rear 0-cochain $\{f_{i,m+1}; i \in I\}$ satisfies the inequality (2.4). Then, for each $i \in I$, the sequence $\{\{f_{i,n}; i \in I\}; n \geq n_0\}$ of holomorphic functions $f_{i,n}$ on $U_{i,n}$ converges to a holomorphic function $f_i(z)$ on U_i and the 0-cochain $\{f_{i,n}; i \in I\}$ satisfies $f_{ij} = f_j - f_i$ on each U_i.

§ 3. Theorem of Type of Andreotti-Grauert

Let E be a $\mathbf{C}$-linear Hausdorff space and $\mathcal{O}$ be its structure sheaf. Let D be a domain of E. A real valued C^∞ function φ on D is said to be strongly q-convex if there exists a positive integer n_0 such that, for any integer n with $n \geq n_0$ and for any n-dimensional $\mathbf{C}$-linear subspace S_f of E, the Levi form of the restriction $\varphi|D \cap S_f$ of the function φ to the n-dimensional open set $D \cap S_f$ has at least $n - q + 1$ pieces of positive eigenvalues at every point of $D \cap S_f$.

Theorem 3.1. *Let Ω be a pseudoconvex domain in a separable Hilbert space S, $\mathcal{O}$ be its structure sheaf, q be a positive integer and φ be a strongly q-convex function on Ω such that, for positive numbers c_1 and c_2 with $c_1 < c_2$, and for any finite dimensional affine subspace S_f of S, the set $\{z \in \Omega \cap S_f; c_1 < \varphi(z) < c_2\}$ is relatively compact in $\Omega \cap S_f$. Let c be a negative number. We put $Y := \{z \in \Omega; \varphi(z) > c\}$. Then we have*

$$\mathrm{H}^1(Y, \mathcal{O}) = 0. \tag{3.1}$$

Proof. Taking a complete orthonormal basis of S, we may assume that the separable Hilbert space S is the space of square summable sequences $z := (z_1, z_2, \cdots, z_n, \cdots)$ of complex numbers z_n with the norm

$$\|z\| := \sqrt{\sum_{n=1}^{\infty} |z_n|^2}$$

as in the preceding section. For any positive integer n, we regard the complex space $\mathbf{C}^n$ as the n-dimensional subspace

$$\{z = (z_1, z_2, \cdots, z_n, \cdots) \in S; z_k = 0 \ (k > n)\}$$

and we put $\Omega_n := \Omega \cap \mathbf{C}^n$. Under these notations, by the assumption of the existence of the strongly q-convex function φ on Ω, there exists a positive integer n_0 such that, for any integer $n \geq n_0$, the Levi form of φ has at least $n - q + 1$ pieces of positive eigenvalues at each point of Ω_n.

Let $\mathcal{U} = \{U_i; i \in I\}$ be an open covering of Ω such that each open set U_i is convex. Then, for any positive integer p and any integer r with $0 \leq r \leq p-1$ by L. Lempert [33], we have

$$\mathrm{H}^{p-r}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_r}, \mathcal{O}) = 0$$

for the pseudoconvex domain $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_r}$ of the separable Hilbert space $S \cong \ell^2$. Then, the cannonical homomorphism $\mathrm{H}^p(\mathcal{U}, \mathcal{O}) \to \mathrm{H}^p(\Omega, \mathcal{O})$ is bijective by Lemma L_p of G. Scheja [45], in other words, $\mathcal{U}$ is a Leray covering of Ω. So, it suffices to prove the vanishing $\mathrm{H}^p(\mathcal{U}, \mathcal{O}) = 0$ of cohomology of the covering $\mathcal{U}$ with coefficients in the structure sheaf $\mathcal{O}$.

Let $\{f_{ij}; i, j \in I\}$ be an element of $\mathrm{Z}^1(\mathcal{U}, \mathcal{O})$, i.e., a 1-cocycle of the covering $\mathcal{U}$ with coefficients in the structure sheaf $\mathcal{O}$. For any positive integer n, we put $U_{i,n} := U_i \cap \mathbf{C}^n$ for each $i \in I$ and $\mathcal{U}_n = \{U_{i,n}; i \in I\}$. We prove by induction with respect to an integer $n \geq n_0$ a proposition P_n which asserts that there exists a sequence $\{\{f_{i,m}; i \in I\}; m = n_0 + 1, n_0 + 2, \cdots, n\}$ of $\{f_{i,m}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_m, \mathcal{O}_m)$ such that, for any integer m with $n_0 \leq m \leq n$, the restriction $\{f_{ij,m}; i, j \in I\}$ of the 1-cocycle $\{f_{ij}; i, j \in I\}$ to the covering $\mathcal{U}_m := \{U_i \cap \mathbf{C}^m; i \in I\}$ is the coboundary of a 0-cochain $\{f_{i,m}; i \in I\}$ of the covering $\mathcal{U}_m$ and that, for any $m < n$, each rear $\{f_{i,m+1}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_{m+1}, \mathcal{O}_{m+1})$ is an extension of the preceding $\{f_{i,m}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_m, \mathcal{O}_m)$ satisfying (2.4) at each point $(z_1, z_2, \cdots, z_{m+1}) \in U_{i,m+1} \cap K_{m+1} \cap \pi_{m+1}^{-1}(\Omega_m)$ for any $i \in I$.

We assume the validity of the proposition P_n and prove the validity of the rear proposition P_{n+1}. Since the Levi form of the restriction of the function φ to Ω_{n+1} has at least $n+1-q+1$ pieces of positive eigenvalues at every point of Ω_{n+1}, we have $n+1-q+1 \geq 1$, i.e., $1 \leq \mathrm{dih}(\mathcal{O}_{n+1}) - q + 1$. By [1], we have the isomorphism

$$\mathrm{H}^p(\Omega_{n+1}, \mathcal{O}_{n+1}) \cong \mathrm{H}^p(\Omega_{n+1} \cap Y, \mathcal{O}_{n+1}). \tag{3.2}$$

Since the pseudoconvex covering $\mathcal{U}_{n+1}$ is a Leray covering from Lemma L_p of G. Scheja [45] by the theorem B of Oka [41]-Cartan [2, 3]-Serre [46], we have the cohomology vanishing

$$\mathrm{H}^1(\Omega_{n+1}, \mathcal{O}_{n+1}) = 0.$$

Hence the 1-cocycle $\{f_{ij,n+1}\} \in \mathrm{Z}^1(\mathcal{U}_{n+1}, \mathcal{O}_{n+1}) \cong \mathrm{B}^p(\mathcal{U}_{n+1}, \mathcal{O}_{n+1})$ is the coboundary of a 0-cochain $\{g_{i,n+1}; i \in I\}$ of the covering $\mathcal{U}_{n+1}$, i.e., there holds

$$f_{ij,n+1} = g_{j,n+1} - g_{i,n+1} \qquad \text{on each } \; U_{ij,n+1} := U_i \cap U_j \cap \mathbf{C}^{n+1}.$$

The preceding function $f_{i,n}$ is holomorphic in $U_{i,n} := U_i \cap \mathbf{C}^n$. We put $f_n := f_{i,n} - g_{i,n+1}$ on $U_{i,n}$. Then f_n is a well-defined holomorphic function on $Y \cap \Omega_n$. Since we have the isomorphism

$$\mathrm{H}^0(\Omega_n, \mathcal{O}_n) \cong \mathrm{H}^0(Y \cap \Omega_n, \mathcal{O}_n)$$

by [1], f_n is holomorphically continued to a holomorphic function on $Y \cap \mathbf{C}^n$, which is denoted by the same symbol f_n. Now, under the same notations in the proof of the preceding theorem, we extend the holomorphic function f_n to a holomorphic function f_{n+1} following the method of L. Gruman [13] and using $\bar{\partial}$-analysis of Hörmander [16], so we have

$$|f_n \circ \pi(z_1, z_2, \cdots, z_{n+1}) - f_{n+1}(z_1, z_2, \cdots, z_{n+1})| \leq |z_{n+1}|^2 \tag{3.3}$$

at each point $(z_1, z_2, \cdots, z_{n+1}) \in K_{n+1} \cap \pi_{n+1}^{-1}(\Omega_n)$. We revise the cochain $\{g_{i,n+1}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_{n+1}, \mathcal{O}_{n+1})$ as $\{f_{i,n+1}; i \in I\} \in \mathrm{Z}^0(\mathcal{U}_{n+1}, \mathcal{O}_{n+1})$ by putting $f_{i,n+1} := g_{i,n+1} + f_{n+1}$ on $U_{i,n+1}$ for each $i \in I$. Then, the revised cochain $\{f_{i,n+1}; i \in I\}$ satisfies

$$|f_{i,n} \circ \pi(z_1, z_2, \cdots, z_{n+1}) - f_{i,n+1}(z_1, z_2, \cdots, z_{n+1})| \leq |z_{n+1}|^2 \tag{3.4}$$

at each point $(z_1, z_2, \cdots, z_{n+1}) \in U_{n+1} \cap K_{n+1} \cap \pi_{n+1}^{-1}(\Omega_n)$ for any $i \in I$, the majoration $|z_{n+1}|^2$ being independent of $i \in I$.

Thus, we have proved the proposition P_n for any integer $n \geq n_0$ and we have constructed a sequence $\{\{f_{i,m}; i \in I\}; m = n_0 + 1, n_0 + 2, \cdots, n, \cdots\}$ of 0-cochains $\{f_{i,m}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_m, \mathcal{O}_m)$ such that $f_{ij} = f_{j,m} - f_{i,m}$ on each $U_{ij,m}$ and the preceding 0-cochain $\{f_{i,m}; i \in I\}$ and the rear 0-cochain $\{f_{i,m+1}; i \in I\}$ satisfy the inequality (2.4). Since our space is equipped with the norm $\sqrt{\sum_{k=1}^{\infty} |z_k|^2}$, the sequence $\{f_{i,n}; n \geq n_0\}$ converges to a holomorphic function $f_i(z)$ on each U_i. And the coboundary of the 0-cochain $\{f_i; i \in I\} \in \mathrm{C}^0(\mathcal{U}, \mathcal{O})$ is the 1-cocycle $\{f_{ij}; i, j \in I\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{O})$.

§ 4. Characterization of Holomorphy of Domain

Let E be a $\mathbf{C}$-linear Hausdorff space and D be a domain of E. The boundary of D is said to be smooth, if, for any boudary point x of D, there exist an open neighborhood U of x in E and a real valued function β of class C^1 on U such that $d\beta \neq 0$ at x and that $D \cap U = \{x \in U; \beta(x) < 0\}$.

Theorem 4.1. *Let S be a separable Hilbert space and Ω be a domain with smooth boundary in the space S. Then Ω is a domain of holomorphy if and only if, for any pseudoconvex domain P in S, we have $\mathrm{H}^1(\Omega \cap P, \mathcal{O}) = 0$.*

Proof. Firstly, we prove that the domain Ω is a domain of holomorphy in the Hilbert space S. Let S_f be any finite dimensional affine subspace of S. Taking a suitable complete orthonormal basis of the separable Hilbert space S as in the proof of Theorem 2.1, we can identify S with the space ℓ^2 of sequences $z = (z_1, z_2, \cdots, z_k, \cdots)$ of complex numbers z_k such that

$$\|z\| := \sqrt{\sum_{k=1}^{\infty} |z_k|^2} < \infty$$

and regard $\mathbf{C}^n$ as its subspace

$$\{z = (z_1, z_2, \cdots, z_k, \cdots) \in \ell^2; z_k = 0 \ (k > n)\}$$

and regard S_f as an affine subspace of $\mathbf{C}^n$. Without loss of generality, we may assume that $S_f = \mathbf{C}^n$.

Let $\rho_n : S \cong \ell^2 \to \mathbf{C}^n$ be the projection defined by

$$\rho_n(z_1, z_2, \cdots, z_n, z_{n+1}, \cdots) = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n. \tag{4.1}$$

We put $\Omega_n := \Omega \cap \mathbf{C}^n$. Let Q_n be a pseudoconvex domain in $\mathbf{C}^n$. Let $z^{(0,n)} := (z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)}) \in \mathbf{C}^n$ be a boundary point of Ω_n in Q_n. We put $z^{(0)} := (z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)}, 0, 0, \cdots)$. Then, $z^{(0)}$ is also a boundary point of the domain Ω in $S \cong \ell^2$. There exist an open neighborhood U of $z^{(0)}$ in S and a real valued function β of class C^1 on U such that U is contained in $\rho_n^{-1}(Q_n)$, that $d\beta \neq 0$ at $z^{(0)}$ and that $\Omega \cap U = \{x \in U; \beta(x) < 0\}$. There exists a positive integer j such that either $\partial\beta/\partial x_j \neq 0$ at $z^{(0)}$ for the real part x_j of z_j or $\partial\beta/\partial y_j \neq 0$ at $z^{(0)}$ for the imaginary part y_j of z_j. In the latter case, we replace z_j by $\mathrm{i}z_j$. Moreover, we exchange z_j and z_1 and replace U by a closer neighborhood of $z^{(0)}$. We may assume $j < n$. Thus, without loss of generality, we may assume that $S_f = \mathbf{C}^n$ and that there exists a real valued function $g(y_1, z_2, z_3, \cdots)$ of class C^1 on a neighborhood V, which is a subset of U and is the ball of radius $3r_0$ centered by $z^{(0)}$, of the boundary point $z^{(0)}$ of S so as there holds

$$\Omega \cap V = \{z = (z_1, z_2, z_3, \cdots) \in V; x_1 < g(y_1, z_2, z_3, \cdots)\}. \tag{4.2}$$

Let B be the ball of radius $2r_0$ centered by $z^{(0)}$. We put $B_n := \rho_n(B)$ and will show that $\Omega_n \cap B_n$ is regular in the sense of [18], i.e., intersections of Ω_n and polycylinders in $\mathbf{C}^n$ are Cousin-I. In order to do so, for the given arbitrary pseudoconvex domain Q_n in $\mathbf{C}^n$, we will prove $\mathrm{H}^1(\Omega_n \cap B_n \cap Q_n, \mathcal{O}_n) = 0$.

For any non negative number t smaller than $1/2$, we denote by T_t and $T_{t,n}$ the translations

$$T_t(z_1, z_2, \cdots, z_n, z_{n+1}, \cdots) = (z_1 + r_0 t, z_2, \cdots, z_n, z_{n+1}, \cdots), \tag{4.3}$$

$$T_{t,n}(z_1, z_2, \cdots, z_n) = (z_1 + r_0 t, z_2, \cdots, z_n) \tag{4.4}$$

and put

$$P_t := \Big\{(z_1, z_2, \cdots) \in S; \sum_{k=1}^{\infty} |z_k - z_k^{(0)}|^2 < 4(1-t)^2 r_0^2\Big\}, \tag{4.5}$$

$$P_{t,n} := \Big\{(z_1, z_2, \cdots, z_n) \in \mathbf{C}^n; \sum_{k=1}^{n} |z_k - z_k^{(0)}|^2 < 4(1-t)^2 r_0^2\Big\}, \tag{4.6}$$

$$E_t := T_t^{-1}(\Omega \cap B \cap \rho_n^{-1}(Q_n) \cap P_t), \tag{4.7}$$

$$E_{t,n} := T_{t,n}^{-1}(\Omega_n \cap B_n \cap Q_n \cap P_{t,n}). \tag{4.8}$$

Then, for any positive number t smaller than $1/2$, $E_{t,n}$ is a relatively compact open subset of $E_{0,n} = \Omega_n \cap B_n \cap Q_n = \Omega_n \cap B_n$.

Now, let $\mathcal{U}_n := \{U_{i,n}; i \in I\}$ be any pseudoconvex covering of $E_{0,n}$ and $\{f_{ij,n}; i, j \in I\}$ be any 1-cocycle of the covering $\mathcal{U}_n$ with coefficients in the structure sheaf $\mathcal{O}_n$ of S_f, which is regarded as $\mathbf{C}^n$. Since the n-dimensional subset $E_{t,n}$ is relatively compact in the infinite dimensional open set $\Omega \cap B$ in S for any positive number t smaller than $1/2$, there exists a positive number τ such that, for the open convex set

$$W_t = \Big\{z = (z_1, z_2, z_3, \cdots) \in S; \sum_{k=n+1}^{\infty} |z_k|^2 < \tau^2\Big\} \tag{4.9}$$

in S, for any positive number t smaller than $1/2$, $\rho_n^{-1}(E_{t,n}) \cap W_t$ is contained in $\Omega \cap B$. We put $U_{t,i} := \rho_n^{-1}(U_{i,n}) \cap E_t \cap W_t$. Then, $\mathcal{U}_t := \{U_{t,i}; i \in I\}$ is a pseudoconvex covering of $E_t \cap W_t$. For any $i, j \in I$, each function $f_{ij,n} \circ \rho_n$ is holomorphic in $U_{t,i} \cap U_{t,j} \cap E_t \cap W_t$.

Since T_t maps $E_t \cap W_t$ biholomorphically onto $\Omega \cap B \cap \rho^{-1}(Q_n) \cap P_t \cap W_t$ and there holds

$$\mathrm{H}^1(\Omega \cap B \cap \rho^{-1}(Q_n) \cap P_t \cap W_t, \mathcal{O}) = 0$$

for the intersection of Ω and the pseudoconvex open set $B \cap \rho^{-1}(Q_n) \cap P_t \cap W_t$ by the assumption of our theorem, we have

$$\mathrm{H}^1(E_t \cap W_t, \mathcal{O}) \cong \mathrm{H}^1(\Omega \cap B \cap \rho^{-1}(Q_n) \cap P_t \cap W_t, \mathcal{O}) = 0.$$

Since the canonical homomorphism $\mathrm{H}^1(\mathcal{U}_t, \mathcal{O}) \to \mathrm{H}^1(E_t \cap W_t, \mathcal{O})$ is injective by Lemma L_0 of [45], the 1-cocycle $\{f_{ij,n} \circ \rho_n|_{U_{t,i} \cap U_{t,j} \cap E_t \cap W_t}; i, j \in I\}$ of the covering $\mathcal{U}_t$ is a coboundary. Hence, there exists a 0-cochain $\{f_{t,i}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_t, \mathcal{O})$ such that $f_{ij,n} \circ \rho_n = f_{t,j} - f_{t,i}$ on $U_{t,i} \cap U_{t,j} \cap E_t \cap W_t$ for any $i, j \in I$. For any $i \in I$ and any positive number t smaller than $1/2$, we put $U_{t,i,n} := U_{i,n} \cap E_{t,n}$. Then, $\mathcal{U}_{t,n} := \{U_{t,i,n}; i \in I\}$ is a pseudoconvex covering of $E_t \cap W_t$. For any $i \in I$ and any positive number t smaller than $1/2$, let $f_{t,i,n}$ be the restriction of $f_{t,i}$ to $U_{t,i,n}$. Then, the coboundary of the 0-cochain $\{f_{t,i,n}; i \in I\} \in \mathrm{C}^0(\mathcal{U}_{t,n}, \mathcal{O}_n)$ is the 1-cocycle $\{f_{t,ij}|_{U_{t,i,n} \cap E_t}\} \in \mathrm{Z}^1(\mathcal{U}_{t,n}, \mathcal{O}_n)$, which is the restriction of the 1-cocycle $\{f_{ij,n}; i, j \in I\} \in \mathrm{Z}^1(\mathcal{U}_n, \mathcal{O}_n)$.

Now, we take a sequence $\{t(\nu); \nu = 1, 2, 3, \cdots\}$ of positive numbers $t(\nu)$ smaller than $1/2$ such that $t(\nu) > t(\nu+1)$ for $\nu = 1, 2, 3, \cdots, t(\nu) \to 0$ as $\nu \to \infty$ and that the preceding open set $E_{t(\nu),n}$ is relatively compact in the rear open set $E_{t(\nu+1),n}$. Since the canonical homomorphism

$$\mathrm{H}^1(E_{0,n}, \mathcal{O}_n) \to \lim_{\nu \to \infty} \mathrm{H}^1(E_{t(\nu),n}, \mathcal{O}_n) \tag{4.10}$$

is injective by Lemma 6 of [19], the 1-cocycle $\{f_{ij,n}; i, j \in I\}$ is a coboundary. Thus, we have proved $\mathrm{H}^1(\Omega_n \cap B_n \cap Q_n, \mathcal{O}_n) = 0$. As stated above, by [18], $\Omega_n \cap B_n$ is a domain of holomorphy in $\mathbf{C}^n$. Hence, the open set Ω_n is Cartan-pseudoconvex in the finite n-dimensional complex space $\mathbf{C}^n$. And, by [37], Ω is pseudoconvex in the infinite dimensional Hilbert space S. Lastly, by [13], Ω is a domain of holomorphy in the Hilbert space S.

Finally, we prove that the cohomology vanishes. For any pseudoconvex domain P in S, the intersection $\Omega \cap P$ of the pseudoconvex domain Ω and the pseudoconve domain B is also pseudoconvex. So, we have $\mathrm{H}^1(\Omega \cap P, \mathcal{O}) = 0$ by [30].

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