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Fractional Sobolev-Poincaré Inequalities in Irregular Domains*

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Abstract This paper is devoted to the study of fractional (q,p)-Sobolev-Poincaré inequalities in irregular domains. In particular, the author establishes (essentially) sharp fractional (q,p)-Sobolev-Poincaré inequalities in s-John domains and in domains satisfying the quasihyperbolic boundary conditions. When the order of the fractional derivative tends to 1, our results tend to the results for the usual derivatives. Furthermore, the author verifies that those domains which support the fractional (q,p)-Sobolev-Poincaré inequalities together with a separation property are s-diam John domains for certain s, depending only on the associated data. An inaccurate statement in [Buckley, S. and Koskela, P., Sobolev-Poincaré implies John, *Math. Res. Lett.*, **2**(5), 1995, 577–593] is also pointed out.

Keywords Fractional Sobolev-Poincaré inequality, s-John domain, Quasihyperbolic boundary condition
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1 Introduction

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is a John domain if there is a constant C and a point $x_0 \in \Omega$ so that for each $x \in \Omega$, one can find a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$Cd(\gamma(t), \partial\Omega) \geq l(\gamma([0,t])) \tag{1.1}$$

for each $0 < t \leq 1$. F. John used this condition in his work on elasticity (see [11]) and the term was coined by Martio and Sarvas [14]. Smith and Stegenga [17] introduced the more general concept of s-John domains, $s \geq 1$, by replacing (1.1) with

$$Cd(\gamma(t), \partial\Omega) \geq l(\gamma([0,t]))^s. \tag{1.2}$$

The condition (1.1) is called a "twisted cone condition" in literature. Thus the condition (1.2) should be called a "twisted cusp condition".

In the last twenty years, s-John domains have been extensively studied in connection with Sobolev-type inequalities (see [2, 7–8, 12–13, 17]). Recall that a bounded domain $\Omega \subset \mathbb{R}^n$,

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$n \geq 2$ is said to be a (q,p)-Poincaré domain if there exists a constant $C_{q,p} = C_{q,p}(\Omega)$ such that

$$\Big(\int_\Omega |u(x) - u_\Omega|^q \mathrm{d}x\Big)^{\frac{1}{q}} \leq C_{q,p}\Big(\int_\Omega |\nabla u(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \tag{1.3}$$

for all $u \in C^\infty(\Omega) \cap L^1(\Omega)$. Here $u_\Omega = \fint_\Omega u(x)\mathrm{d}x$. When $q = p$, Ω is termed a p-Poincaré domain and when $q > p$, we say that Ω supports a (q,p)-Sobolev-Poincaré inequality. Buckley and Koskela [2] have shown that a simply connected planar domain which supports a $(\frac{np}{n-p}, p)$-Sobolev-Poincaré inequality is a 1-John domain. Smith and Stegenga shown that an s-John domain Ω is a p-Poincaré domain, provided that $1 \leq s < \frac{n}{n-1} + \frac{p-1}{n}$. In particular, if $1 \leq s < \frac{n}{n-1}$, then Ω is a p-Poincaré domain for all $1 \leq p < \infty$. These results were further generalized to the case of (q,p)-Poincaré domains in [7, 12–13].

Recently, there has been a growing interest in the study of the so-called fractional (q,p)-Sobolev-Poincaré inequalities (see for instance [3, 9] and the references therein). In this paper, we continue the study of the following fractional (q,p)-Sobolev-Poincaré inequality in a domain $\Omega \subset \mathbb{R}^n$ with finite Lebesgue measure, $n \geq 2$,

$$\int_\Omega |u(x) - u_\Omega|^q \mathrm{d}x \leq C\Big(\int_\Omega \int_{\Omega \cap B(x, \tau d(x, \partial\Omega))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\delta}} \mathrm{d}y\mathrm{d}x\Big)^{\frac{q}{p}}, \tag{1.4}$$

where $1 \leq p \leq q < \infty$, $\delta \in (0,1)$, $\tau \in (0,\infty)$ and the constant C does not depend on $u \in C(\Omega) \cap L^1(\Omega)$. If Ω supports the fractional (q,p)-Sobolev-Poincaré inequality (1.4), $q \geq p$, then we say that Ω is a fractional (q,p)-Sobolev-Poincaré domain[1].

From now on, unless otherwise specified, $\delta \in (0,1)$ and $\tau \in (0,\infty)$ will be fixed constants. Given a function $u \in C(\Omega) \cap L^1(\Omega)$, we define $g_u : \Omega \to \mathbb{R}$ as

$$g_u(x) = \int_{\Omega \cap B(x, \tau d(x, \partial\Omega))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\delta}} \mathrm{d}y \tag{1.5}$$

for $x \in \Omega$.

It is well-known, due to Maz'ya [15–16], that the validity of a (q,p)-Sobolev-Poincaré inequality in Ω is equivalent to certain capacity-type estimates in Ω. Thus one would expect that a similar equivalence result holds in the setting of fractional (q,p)-Sobolev-Poincaré inequalities as well. Our first main result confirms this expectation.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a domain with finite Lebesgue measure and $1 \leq p \leq q < \infty$. Then the following statements are equivalent:*

(i) *Ω satisfies the fractional (q,p)-Sobolev-Poincaré inequality.*

(ii) *For an arbitrary ball $B_0 \subset \Omega$, there exists a constant $C = C(\Omega, p, q, B_0, \delta, \tau)$ such that*

$$|A|^{\frac{p}{q}} \leq C \inf \int_\Omega g_u(x)\mathrm{d}x \tag{1.6}$$

for every measurable set $A \subset \Omega$ such that $\overline{A} \cap \overline{B_0} = \emptyset$. The infimum above is taken over all functions $u \in C(\Omega) \cap L^1(\Omega)$ that satisfy $u|_A \geq 1$ and $u|_{B_0} = 0$.

[1] Strictly speaking, we should also indicate the parameter δ in the definition of a fractional (q,p)-Sobolev-Poincaré domain. But since we did not emphasize it in the definition of the fractional (q,p)-Sobolev-Poincaré inequality either, we keep our current terminology.

Theorem 1.1 can be regarded as a fractional version of [7, Theorem 1] and it allows us to study the fractional (q,p)-Sobolev-Poincaré inequalities in irregular domains via capacity estimates. On the other hand, as in the usual Sobolev-Poincaré case, we have standard techniques for doing capacity estimates.

Our second main result can be regarded as an (un-weighted) fractional version of [7, Theorem 9].

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an s-John domain. If $p < \frac{n}{\delta}$, $s < \frac{n}{n-p\delta}$ and $1 \leq p \leq q < \frac{np}{s(n-p\delta)+(s-1)(p-1)}$, then Ω supports the fractional (q,p)-Sobolev-Poincaré inequality* (1.4).

The range for q in Theorem 1.2 is essentially sharp as indicated by the following example.

Example 1.1 Given $\tau, \delta \in (0,1)$, $1 \leq p < \frac{n}{\delta}$ and $s < \frac{n}{n-p\delta}$, there exists an s-John domain $\Omega \subset \mathbb{R}^n$ such that Ω does not support any fractional (q,p)-Sobolev-Poincaré inequality with $q > \frac{np}{s(n-p\delta)+(s-1)(p-1)}$.

Theorem 1.2 holds for the critical case $q = \frac{np}{s(n-p\delta)+(s-1)(p-1)}$ as well, provided that $s = 1$ or $p = 1$ (see Remark 4.2). We conjecture that Theorem 1.2 holds under the same assumptions for the critical case.

The above s-John condition on a domain Ω is very "geometric" and it provides an effective estimate for capacity. There is another well-known "metric" condition on Ω that is sufficient for our capacity estimates. The condition is termed the quasihyperbolic boundary condition in literature and it requires that the quasihyperbolic distance between each point x and a fixed point x_0 in Ω is dominated from above by (a logarithmic function of) its distance to the boundary of Ω (see Section 2 below for precise definitions). With these understood, our third main result can be regarded as a fractional version of [13, Theorems 1.4–1.5] and [10, Theorem 1].

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfy the quasihyperbolic boundary condition* (2.1) *for some $\beta \leq 1$. Then Ω is a fractional (q,p)-Sobolev-Poincaré domain provided that $p \in \left[1, \frac{n}{\delta}\right)$ and $q \in [p, \frac{2\beta}{1+\beta}\frac{np}{n-p\delta})$.*

Note that the condition $q \in [p, \frac{2\beta}{1+\beta}\frac{np}{n-p\delta})$ implies that $p > \frac{1}{\delta}(n - n\frac{2\beta}{1+\beta})$.

Example 1.2 For each $q > \frac{2\beta}{1+\beta}\frac{np}{n-p\delta}$, there exists a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfying (2.1) which is not a fractional (q,p)-Sobolev-Poincaré domain. For each $1 \leq p < \frac{1}{\delta}(n - n\frac{2\beta}{1+\beta})$, there exists a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfying (2.1), which is not a fractional (p,p)-Sobolev-Poincaré domain.

Recall that we say a domain $\Omega \subset \mathbb{R}^n$ with a distinguished point x_0 has a separation property if there exists a constant C_0 such that the following property holds: For every $x \in \Omega$, there exists a curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$, such that for each t, either

$$\gamma([0,t]) \subset B_t := B(\gamma(t), C_0 d(\gamma(t), \partial\Omega))$$

or each $y \in \gamma([0,t]) \backslash B_t$ and x_0 belongs to different components of $\Omega \backslash \partial B_t$.

Theorem 1.4 *Assume that $\Omega \subset \mathbb{R}^n$ is a domain of finite Lebesgue measure that satisfies the separation property with a distinguished point x_0. Let $1 \le p < \frac{n}{\delta}$. If Ω is a fractional (q,p)-Sobolev-Poincaré domain with $\tau = 1$ for some $q > p$, then for each $x \in \Omega$, there is a curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ such that*

$$\operatorname{diam}\gamma([0,t]) \le C\varphi(d(\gamma(t), \partial\Omega)), \tag{1.7}$$

where $\varphi(t) = t^{\frac{(n-p\delta)q}{p\delta}(\frac{1}{p}-\frac{1}{q})}$.

The assumptions in Theorem 1.4 can be further relaxed. Indeed, Theorem 1.4 holds if we only assume that the fractional (q,p)-Sobolev-Poincaré inequality (1.4) holds for all locally Lipschitz continuous functions in Ω (see Remark 3.1).

Since this paper generalizes the main results of [2–3, 7, 9, 13] to the fractional setting in a natural way, some of the arguments used in this paper are similar to ones in those papers. In particular, we benefit a lot from [7, 9, 13]. This paper is organized as follows. Section 2 contains the basic definitions and Section 3 contains some auxiliary results. We prove our main results, namely, Theorems 1.1–1.2 and Example 1.1 in Section 4. In Section 5, we prove Theorem 1.3 and give the construction of Example 1.2. In the final section, i.e., Section 6, we discuss the proof of Theorem 1.4 and point out an inaccurate statement, namely, Corollary 4.1 in [2].

2 Notations and Definitions

Recall that the quasihyperbolic metric k_Ω in a domain $\Omega \subsetneq \mathbb{R}^n$ is defined to be

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{\mathrm{d}s}{d(z,\partial\Omega)},$$

where the infimum is taken over all rectifiable curves γ in Ω which join x to y. This metric was introduced by Gehring and Palka in [5]. A curve γ joining x to y for which $k_\Omega(x,y) = \int_\gamma \frac{\mathrm{d}s}{d(z,\partial\Omega)}$ is called a quasihyperbolic geodesic. Quasihyperbolic geodesics joining any two points of a proper subdomain of $\mathbb{R}^n$ always exists (see [4, Lemma 1]).

Recall that a domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is said to satisfy a β-quasihyperbolic boundary condition, $\beta \in (0,1]$, if there exists a point $x_0 \in \Omega$ and a constant C_0 such that

$$k_\Omega(x,x_0) \le \frac{1}{\beta}\log\frac{d(x_0,\partial\Omega)}{d(x,\partial\Omega)} + C_0 \tag{2.1}$$

holds for all $x \in \Omega$.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$. Then $\mathcal{W} = \mathcal{W}(\Omega)$ denotes a Whitney decomposition of Ω, i.e., a collection of closed cubes $Q \subset \Omega$ with pairwise disjoint interiors and edges parallel to the coordinate axes, such that $\Omega = \bigcup_{Q\in\mathcal{W}} Q$, and the diameters of $Q \in \mathcal{W}$ belong to the set $\{2^{-j} : j \in \mathbb{Z}\}$ and satisfy the condition

$$\operatorname{diam}(Q) \le \operatorname{dist}(Q, \partial\Omega) \le 4\operatorname{diam}(Q).$$

For $j \in \mathbb{Z}$, we define

$$\mathcal{W}_j = \{Q \in \mathbb{W} : \operatorname{diam}(Q) = 2^{-j}\}.$$

Note that when we write $f(x) \lesssim g(x)$, we mean that $f(x) \le Cg(x)$ is satisfied for all x with some fixed constant $C \ge 1$. Similarly, the expression $f(x) \gtrsim g(x)$ means that $f(x) \ge C^{-1}g(x)$ is satisfied for all x with some fixed constant $C \ge 1$. We write $f(x) \approx g(x)$ whenever $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$.

3 Auxiliary Results

We need the following "chain lemma" from [7, Proof of Theorem 9]. Note that the condition 3 below is not stated there, however, the proof adapts to our setting and we omit the details.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be an s-John domain and $M > 1$ a fixed constant. Let $B_0 = B(x_0, \frac{d(x_0,\partial\Omega)}{4M})$, where $x_0 \in \Omega$ is the John center. There exists a constant $c > 0$, depending only on Ω, M and n, such that given $x \in \Omega$, there exists a finite "chain" of balls $B_i = B(x_i, r_i)$, $i = 0, 1, \cdots, k$ (k depends on the choice of x) that joins x_0 to x with the following properties:*

(1) $|B_i \cup B_{i+1}| \le c|B_i \cap B_{i+1}|$.

(2) $d(x, B_i) \le c r_i^{\frac{1}{s}}$.

(3) $d(B_i, \partial\Omega) \ge M r_i$.

(4) $\sum\limits_{i=0}^{k} \chi_{B_i} \le c\chi_\Omega$.

(5) $|x - x_i| \le c r_i^{\frac{1}{s}}$ *and* $B_k = B(x, \frac{d(x,\partial\Omega)}{4M})$.

(6) *For any $r > 0$, the number of balls B_i with radius $r_i > r$ is less than $c r^{\frac{1-s}{s}}$ when $s > 1$.*

Recall that for a function f, the Riesz potential I_δ, $\delta \in (0, n)$ of f is defined by

$$I_\delta(f) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\delta}} \mathrm{d}y. \tag{3.1}$$

The following estimate for the Riesz potential is well-known (see for instance [1, Theorem 3.1.4 and Corollary 3.1.5]).

Theorem 3.1 *Let $0 < \delta < n$, $1 < p < q < \infty$, and $\frac{1}{p} - \frac{1}{q} = \frac{\delta}{n}$. Then $\|I_\delta(f)\|_q \le c\|f\|_p$ for some constant c independent of $f \in L^p(\mathbb{R}^n)$. Moreover, there is a constant $c_1 = c(n, \delta) > 0$ such that the weak estimate*

$$\sup_{t>0} |\{x \in \mathbb{R}^n : |I_\delta(f)(x)| > t\}| t^{\frac{n}{n-\delta}} \le c_1 \|f\|_1^{\frac{n}{n-\delta}} \tag{3.2}$$

holds for every $f \in L^1(\mathbb{R}^n)$.

The following proposition, which can be regarded as a fractional analogy of [2, Theorem 2.1], is proved in [3, Proposition 6.2].

Proposition 3.1 *Suppose that $\Omega \subset \mathbb{R}^n$ is a domain of finite Lebesgue measure. Let $1 \le p < q < \infty$. Assume that the fractional (q, p)-Sobolev-Poincaré inequality (1.4) holds with $\tau = 1$ for every $u \in C(\Omega) \cap L^1(\Omega)$. Fix a ball $B_0 \subset \Omega$, and let $d > 0$ and $w \in \Omega$. Then there exists a constant $C > 0$ such that*

$$\operatorname{diam}(T) \le C(d + |T|^{(\frac{1}{p} - \frac{1}{q})\frac{1}{\delta}})$$

and

$$|T|^{\frac{1}{n}} \le C(d + d^{\frac{(n-p\delta)q}{np}}),$$

if T is the union of all components of $\Omega \backslash B(w,d)$ that do not intersect the ball B_0. The constant C depends only on $|B_0|$, $|\Omega|$, n, p, q, δ and the constant associated to the fractional (q,p)-Sobolev-Poincaré inequality.

Remark 3.1 As in [2], one can check that the conclusion holds whenever the fractional (q,p)-Sobolev-Poincaré inequality (1.4) with $\tau = 1$ holds for every locally Lipschitz continuous functions (see [3, Proof of Proposition 6.2]).

Fix a Whitney cube Q_0 and assume that x_0 is the center of Q_0. For each cube $Q \in \mathcal{W}$, we choose a quasihyperbolic geodesic γ joining x_0 to the center of Q and we let $P(Q)$ denote the collection of all the Whitney cubes $Q' \in \mathcal{W}$ which intersect γ. Then the shadow $S(Q)$ of the cube Q is defined to be

$$S(Q) = \bigcup_{\substack{Q_1 \in \mathcal{W} \\ Q \in P(Q_1)}} Q_1.$$

The following lemma is proved in [13, Lemma 2.6].

Lemma 3.2 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be a domain that satisfies the quasihyperbolic boundary condition* (2.1)*. Then for each $\varepsilon > 0$, there exists a constant $C = C(n, \operatorname{diam}\Omega, \varepsilon)$ such that*

$$\sup_{Q_1 \in \mathcal{W}} \sum_{Q \in P(Q_1)} |Q|^{\varepsilon} \le C. \tag{3.3}$$

We also need the following estimate of the size of the shadow of a Whitney cube Q in terms of the size of Q. The proof can be found in [10, Lemma 6]. [2]

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain that satisfies the quasihyperbolic boundary condition* (2.1)*. Then there exists a constant $C = C(n, d(x_0, \partial\Omega))$ such that*

$$\operatorname{diam} S(Q) \le C(\operatorname{diam} Q)^{\frac{2\beta}{1+\beta}}$$

for all $Q \in \mathcal{W}$. Consequently,

$$|S(Q)| \le C|Q|^{\frac{2\beta}{1+\beta}}. \tag{3.4}$$

4 Main Proofs

Proof of Theorem 1.1 We first show that the condition (ii) implies the condition (i). Fix a function $u \in C(\Omega) \cap L^1(\Omega)$. Pick a real number b such that both $|\{x \in \Omega : u(x) \ge b\}|$ and $|\{x \in \Omega : u(x) \le b\}|$ are at least $\frac{|\Omega|}{2}$. It suffices to show the fractional (q,p)-Sobolev-Poincaré inequality with $|u - u_\Omega|$ replaced by $|u - b|$, and by replacing u with $u - b$, we may assume that

[2] I would like to thank Renjin Jiang for sharing the manuscript [10] and Aapo Kauranen for pointing out Lemma 3.3 in their work in [10, Lemma 6].

$b = 0$. Write $v_+ = \max\{u, 0\}$ and $v_- = -\min\{u, 0\}$. In the sequel, v denotes either v_+ or v_-; all the statements below are valid in both cases. Without loss of generality, we may assume that $v \geq 0$.

Fix a ball B_0 such that $2B_0 \subset\subset \Omega$. We may further assume that $v|_{B_0} = 0$. In fact, let $\varphi \in C^\infty(\Omega)$ satisfy $0 \leq \varphi \leq 1$, $\mathrm{spt}(\varphi) \subset 2B_0$ and $\varphi|_{B_0} = 1$. Note that we may write

$$v = \varphi v + (1 - \varphi) v.$$

The first term $\varphi v \in C^\infty(2B_0 \backslash B_0)$ and thus the fractional (q, p)-Sobolev-Poincaré inequality holds for φv in $2B_0 \backslash B_0$. On the other hand, the second term $(1 - \varphi) v \in C(\Omega) \cap L^1(\Omega)$ and it vanishes on B_0. So if one can prove the fractional (q, p)-Sobolev-Poincaré inequality for $(1 - \varphi) v$, then a simple computation, after summing up these two estimates, will imply the fractional (q, p)-Sobolev-Poincaré inequality for v.

For each $j \in \mathbb{Z}$, we define $v_j(x) = \min\{2^j, \max\{0, v(x) - 2^j\}\}$. We next prove the following inequality:

$$2^{qj} |\{x \in \Omega : v_j(x) \geq 2^j\}| \leq C \Big(\int_\Omega g_{v_j}(x) \mathrm{d}x \Big)^{\frac{q}{p}}. \tag{4.1}$$

To see it, notice that $2^{-j} v_j|_{B_0} = 0$ and $2^{-j} v_j|_{F_j} \geq 1$, where $F_j = \{x \in \Omega : v(x) \geq 2^{j+1}\}$. So by (1.6), we obtain that

$$|F_j|^{\frac{p}{q}} \leq C \int_\Omega g_{2^{-j} v_j}(x) \mathrm{d}x.$$

Note that $g_{2^{-j} v_j} = 2^{-pj} g_{v_j}$. Thus we finally arrive at

$$2^{pj} |F_j|^{\frac{p}{q}} \leq C \int_\Omega g_{v_j}(x) \mathrm{d}x,$$

which is the desired estimate (4.1).

The fractional (q, p)-Sobolev-Poincaré inequality now follows from the weak type estimates via a standard argument. Write $B_y = B(y, \tau d(y, \partial\Omega))$ and $A_k = F_{k-1} \backslash F_k$,

$$\begin{aligned}
\int_\Omega |v(x)|^q \mathrm{d}x &\leq \sum_{k=-\infty}^{\infty} 2^{(k+1)q} |A_k| \leq C \sum_{k=-\infty}^{\infty} \Big(\int_\Omega g_{v_k}(x) \mathrm{d}x \Big)^{\frac{q}{p}} \\
&\leq C \Big(\sum_{k=-\infty}^{\infty} \int_\Omega g_{v_k}(x) \mathrm{d}x \Big)^{\frac{q}{p}} \\
&\leq C \Big(\sum_{k=-\infty}^{\infty} (I_1^k + I_2^k) \Big)^{\frac{q}{p}},
\end{aligned}$$

where

$$I_1^k = \sum_{i \leq k+1} \sum_{j \geq k+1} \int_{A_i} \int_{A_j \cap B_y} \frac{|v_k(y) - v_k(z)|^p}{|y - z|^{n+p\delta}} \mathrm{d}z \mathrm{d}y$$

and

$$I_2^k = \sum_{i \geq k+1} \sum_{j \leq k+1} \int_{A_i} \int_{A_j \cap B_y} \frac{|v_k(y) - v_k(z)|^p}{|y - z|^{n+p\delta}} \mathrm{d}z \mathrm{d}y.$$

For $y \in A_i$ and $z \in A_j$ with $j-1>i$, $|v(y)-v(z)| \ge |v(z)|-|v(y)| \ge 2^{j-2}$. Hence,

$$|v_k(y)-v_k(z)| \le 2^{k+1} \le 4\cdot 2^{k+1-j}|v(y)-v(z)|. \tag{4.2}$$

Since the estimate

$$|v_k(y)-v_k(z)| \le |v(y)-v(z)|$$

holds for every $k \in \mathbb{Z}$, (4.2) is valid whenever $i \le k \le j$ and $(y,z) \in A_i \times A_j$. It follows from (4.2) that

$$\sum_{k=-\infty}^{\infty} I_1^k \le 4^p \sum_{k=-\infty}^{\infty} \sum_{i\le k+1} \sum_{j \ge k+1} 2^{p(k+1-j)} \int_{A_i}\int_{A_j\cap B_y} \frac{|v(y)-v(z)|^p}{|y-z|^{n+p\delta}} \mathrm{d}z\mathrm{d}y.$$

Since $\sum\limits_{k=i-1}^{j-1} 2^{p(k+1-j)} \le (1-2^{-p})^{-1}$, changing the order of the summation yields that the right-hand side of the above inequality is bounded by

$$\frac{4^p}{1-2^{-p}} \int_\Omega g_v(y)\mathrm{d}y.$$

The estimate of I_2^k is similar. Thus, we have proved that

$$\int_\Omega |v(x)|^q \mathrm{d}x \le C\Big(\int_\Omega g_v(y)\mathrm{d}y\Big)^{\frac{q}{p}}.$$

The desired fractional (q,p)-Sobolev-Poincaré inequality (1.4) follows from the above inequality as we notice that $|u| = v_+ + v_-$ and $|v_\pm(y)-v_\pm(z)| \le |u(y)-u(z)|$ for all $y, z \in \Omega$.

The implication from the condition (ii) to the condition (i) is easier. To see it, fix a measurable set $A \subset \Omega$ such that $\overline{A} \cap \overline{B_0} = \emptyset$ and a function $u \in C(\Omega) \cap L^1(\Omega)$ such that $u|_A \ge 1$ and $u|_{B_0} = 0$. If $u_\Omega \le \frac{1}{2}$, then by (1.4), we have

$$\begin{aligned} 2^{-q}|A| &\le \int_A |u(x)-u_\Omega|^q \mathrm{d}x \le \int_\Omega |u(x)-u_\Omega|^q \mathrm{d}x \\ &\le C\Big(\int_\Omega g_u(y)\mathrm{d}y\Big)^{\frac{q}{p}}. \end{aligned}$$

If $u_\Omega \ge \frac{1}{2}$, then by (1.4) we have

$$\begin{aligned} 2^{-q}|A| \le 2^{-q}\frac{|\Omega|}{|B_0|}|B_0| &\le \frac{|\Omega|}{|B_0|}\int_{B_0} |u(x)-u_\Omega|^q \mathrm{d}x \\ &\le \frac{|\Omega|}{|B_0|} C\Big(\int_\Omega g_u(y)\mathrm{d}y\Big)^{\frac{q}{p}}. \end{aligned}$$

Combining the above two estimates, we conclude that

$$|A|^{\frac{p}{q}} \le C\int_\Omega g_u(x)\mathrm{d}x,$$

where $C = C(\Omega, B_0, p, q, \delta, \tau)$. Taking the infimum over all such u gives (1.6).

Remark 4.1 It is clear from the proof above that the condition (ii) of Theorem 1.1 is equivalent to the following condition: For an arbitrary cube $Q_0 \subset \Omega$, there exists a constant $C = C(\Omega, Q_0, p, q, \delta, \tau)$ such that

$$|A|^{\frac{p}{q}} \leq C \inf \int_\Omega g_u(x)\mathrm{d}x$$

for every measurable set $A \subset \Omega$ with $\overline{A} \cap \overline{Q_0} = \emptyset$. The infimum above is taken over all functions $u \in C(\Omega) \cap L^1(\Omega)$ that satisfy $u|_A \geq 1$ and $u|_{Q_0} = 0$.

Proof of Theorem 1.2 Let $B_0 = B(x_0, \frac{d(x_0,\partial\Omega)}{4M})$. Assume that $p < \frac{n}{\delta}$, $1 < s < \frac{n}{n-p\delta}$ and $1 \leq p \leq q < \frac{np}{s(n-p\delta)+(s-1)(p-1)}$. Choose $\Delta > 0$ such that

$$2\Delta = \frac{np}{q} - s(n-p\delta) - (s-1)(p-1).$$

It suffices to show, by Theorem 1.1, that there exists a constant $C = C(\Omega, B_0, p, q, \delta, \tau)$ such that for every measurable set $A \subset \Omega$ with $\overline{A} \cap \overline{B_0} = \emptyset$, we have

$$|A|^{\frac{p}{q}} \leq C \int_\Omega g_u(x)\mathrm{d}x$$

whenever $u \in C(\Omega) \cap L^1(\Omega)$ satisfies $u|_A \geq 1$ and $u|_{B_0} = 0$. Since Ω is bounded, we may further assume that $\operatorname{diam} \Omega = 1$.

For any $x \in A$, we obtain from Lemma 3.1 a finite chain of balls B_i, $i = 0, 1, \cdots, k$, satisfying conditions (1)–(6) in Lemma 3.1 with $M > \frac{2}{\tau}$. For all $i = 0, 1, \cdots, k$, we have

$$B_i \subset B(y, \tau d(y, \partial\Omega)), \quad \text{if } y \in B_i. \tag{4.3}$$

To see this, fix $y \in B_i$ and let z be any other point in B_i. Then by the condition (3) in Lemma 3.1,

$$\begin{aligned} |z-y| &\leq |y-x_i| + |x_i - z| \leq 2r_i \leq 2\frac{d(B_i,\partial\Omega)}{M} \\ &\leq \frac{2}{M} d(y,\partial\Omega) < \tau d(y, \partial\Omega). \end{aligned}$$

In order to estimate $|A|$, we divide A into the "bad" and "good" parts. Setting

$$\mathcal{G} = \Big\{x \in A \mid u_{B_x} \geq \frac{1}{2}\Big\} \quad \text{and} \quad \mathcal{B} = A \backslash \mathcal{G},$$

where $B_x = B(x, \frac{d(x,\partial\Omega)}{4M})$, we have $|A| \leq |\mathcal{G}| + |\mathcal{B}|$. We first estimate $|\mathcal{G}|$.

For $x \in \mathcal{G}$, let $\{B_i\}_{i=0}^k$ be the associated chain of balls as described before. Then $B_x = B_k$. By the condition (1) in Lemma 3.1, we have

$$\begin{aligned} \frac{1}{2} &\leq |u_{B_k} - u_{B_0}| \leq \sum_{i=0}^{k-1} |u_{B_i} - u_{B_{i+1}}| \\ &\leq \sum_{i=0}^{k-1} (|u_{B_i} - u_{B_i \cap B_{i+1}}| + |u_{B_{i+1}} - u_{B_i \cap B_{i+1}}|) \end{aligned}$$

$$\lesssim \sum_{i=0}^{k} \frac{1}{|B_i|} \int_{B_i} |u(y) - u_{B_i}| \mathrm{d}y.$$

For a ball B_i,

$$\begin{aligned}\frac{1}{|B_i|} \int_{B_i} |u(y) - u_{B_i}| \mathrm{d}y &\le \frac{1}{|B_i|} \int_{B_i} \Big(\frac{1}{|B_i|} \int_{B_i} |u(y) - u(z)|^p \mathrm{d}z \Big)^{\frac{1}{p}} \mathrm{d}y \\ &= \frac{1}{|B_i|^{1+\frac{1}{p}}} \int_{B_i} \Big(\int_{B_i} |u(y) - u(z)|^p \mathrm{d}z \Big)^{\frac{1}{p}} \mathrm{d}y \\ &\lesssim |B_i|^{\frac{\delta}{n}-1} \int_{B_i} \Big(\int_{B_i} \frac{|u(y) - u(z)|^p}{|y-z|^{n+p\delta}} \mathrm{d}z \Big)^{\frac{1}{p}} \mathrm{d}y.\end{aligned}$$

By (4.3) and the condition (2) in Lemma 3.1,

$$\begin{aligned}&\sum_{i=0}^{k} \frac{1}{|B_i|} \int_{B_i} |u(y) - u_{B_i}| \mathrm{d}y \\ &\lesssim \sum_{i=0}^{k} |B_i|^{\frac{\delta}{n}-1} \int_{B_i} \Big(\int_{B_i} \frac{|u(y) - u(z)|^p}{|y-z|^{n+p\delta}} \mathrm{d}z \Big)^{\frac{1}{p}} \mathrm{d}y \\ &\le \sum_{i=0}^{k} |B_i|^{\frac{\delta}{n}-1} \int_{B_i} \Big(\int_{B(y,\tau d(y,\partial\Omega))} \frac{|u(y) - u(z)|^p}{|y-z|^{n+p\delta}} \mathrm{d}z \Big)^{\frac{1}{p}} \mathrm{d}y \\ &\lesssim \sum_{i=0}^{k} r_i^{\delta-\frac{n}{p}} \Big(\int_{B_i} g_u(y) \mathrm{d}y \Big)^{\frac{1}{p}}.\end{aligned}$$

Thus we conclude that

$$1 \lesssim \sum_{i=0}^{k} r_i^{\delta-\frac{n}{p}} \Big(\int_{B_i} g_u(y) \mathrm{d}y \Big)^{\frac{1}{p}}.$$

Hölder's inequality implies

$$1 \lesssim \Big(\sum_{i=0}^{k} r_i^{\frac{\kappa p}{p-1}} \Big)^{\frac{p-1}{p}} \Big(\sum_{i=0}^{k} r_i^{p(-\kappa+\delta-\frac{n}{p})} \int_{B_i} g_u(y) \mathrm{d}y \Big)^{\frac{1}{p}},$$

where $\kappa = \frac{(s-1)(p-1)+\Delta}{sp}$. Using the condition (6) from Lemma 3.1, one can easily conclude

$$\sum_{i=0}^{k} r_i^{\frac{\kappa p}{p-1}} \le \sum_{i=0}^{\infty} (2^{-i})^{\frac{\kappa p}{p-1}} 2^{\frac{i(s-1)}{s}} < C.$$

Therefore,

$$\sum_{i=0}^{k} r_i^{p\left(-\kappa+\delta-\frac{n}{p}\right)} \int_{B_i} g_u(y) \mathrm{d}y \ge C, \tag{4.4}$$

where the constant C depends only on p, n, Δ and the constant from s-John condition.

By the condition (2) from Lemma 3.1, $Cr_i \ge |x-y|^s$ for $y \in B_i$, and since $p(-\kappa+\delta-\frac{n}{p}) < 0$ according to our choice $p \le \frac{n}{\delta}$, we obtain

$$r_i^{-\kappa p - n + p\delta} \lesssim |x-y|^{s(-\kappa p - n + p\delta)}$$

for $y \in B_i$. For $y \in B_i \cap (2^{j+1}B_k \backslash 2^j B_k)$, we have $|x-y| \approx 2^j r_k$ and hence for such y,

$$r_i^{-\kappa p-n+p\delta} \lesssim (2^j r_k)^{s(-\kappa p-n+p\delta)}. \tag{4.5}$$

Combining (4.4) with (4.5) leads to

$$\begin{aligned} 1 &\lesssim \sum_{i=0}^{k} r_i^{p\left(-\kappa+\delta-\frac{n}{p}\right)} \int_{B_i} g_u(y)\mathrm{d}y \lesssim (r_k)^{s(-\kappa p-n+p\delta)} \int_{B_i} g_u(y)\mathrm{d}y \\ &\quad + \sum_{j=0}^{|\log r_k|} (2^j r_k)^{s(-\kappa p-n+p\delta)} \int_{(2^{j+1}B_k \backslash 2^j B_k)\cap\Omega} g_u(y)\mathrm{d}y \\ &\lesssim \sum_{l=0}^{|\log r_k|+1} (2^l r_k)^{s(-\kappa p-n+p\delta)} \int_{2^l B_k\cap\Omega} g_u(y)\mathrm{d}y. \end{aligned}$$

On the other hand,

$$\sum_{l=0}^{|\log r_k|+1} (2^l r_k)^{\Delta} < r_k^{\Delta} \sum_{l=-\infty}^{|\log r_k|+1} 2^{l\Delta} < C.$$

Comparing the above two estimates, we conclude that there exists an l (depending on Δ) such that

$$(2^l r_k)^{\Delta} \lesssim (2^l r_k)^{s(-\kappa p-n+p\delta)} \int_{2^l B_k\cap\Omega} g_u(y)\mathrm{d}y.$$

It follows that

$$\int_{\Omega\cap 2^l B_k} g_u(y)\mathrm{d}y \gtrsim (2^l r_k)^{s(n+\kappa p-p\delta)+\Delta} = (2^l r_k)^{s(n-p\delta)+(s-1)(p-1)+2\Delta}.$$

In other words, there exists an $R_x \geq \frac{d(x,\partial\Omega)}{2}$ with

$$\Big(\int_{\Omega\cap B(x,R_x)} g_u(y)\mathrm{d}y\Big)^{\frac{np}{q[s(n-p\delta)+(s-1)(p-1)+2\Delta]}} \gtrsim (R_x^n)^{\frac{p}{q}}.$$

Note that according to our choice of Δ, the above estimate reduces to the following form:

$$\int_{\Omega\cap B(x,R_x)} g_u(y)\mathrm{d}y \gtrsim |B(x,R_x)|^{\frac{p}{q}}.$$

Applying the Vitali covering lemma to the covering $\{B(x,R_x)\}_{x\in E}$ of the set $\mathcal{B}$, we can select pairwise disjoint balls $B_1,\cdots,B_k$, such that $\mathcal{B} \subset \bigcup_{i=1}^{\infty} 5B_i$. Let r_i denote the radius of the ball B_i. Then

$$\begin{aligned} |\mathcal{G}| &\leq \sum_{i=1}^{\infty} |5B_i| = 5^n \sum_{i=1}^{\infty} |B_i| \lesssim \sum_{i=1}^{\infty} \Big(\int_{\Omega\cap B_i} g_u(y)\mathrm{d}y\Big)^{\frac{q}{p}} \\ &\lesssim \Big(\sum_{i=1}^{\infty} \int_{\Omega\cap B_i} g_u(y)\mathrm{d}y\Big)^{\frac{q}{p}} \lesssim \Big(\int_{\Omega} g_u(y)\mathrm{d}y\Big)^{\frac{q}{p}}. \end{aligned}$$

We next estimate $|\mathcal{B}|$. Note that $\mathcal{B} \subset \bigcup_{x\in\mathcal{B}} B_x$. We may use the Besicovitch covering theorem to select a subcovering $\{B_{x_i}\}_{i\in\mathbb{N}}$ with bounded overlap. Since $u \geq 1$ on A and $u_{B_{x_i}} \leq \frac{1}{2}$, we obtain that

$$|u(y) - u_{B_{x_i}}|^q \geq 2^{-q}$$

for $y \in A \cap B_{x_i}$. By the fractional (q,p)-Sobolev-Poincaré inequality for balls (see for instance [9]), we get

$$\begin{aligned} |A \cap B_{x_i}| &\leq C \int_{A\cap B_{x_i}} |u(y) - u_{B_{x_i}}|^q \mathrm{d}y \\ &\leq C\Big(\int_{B_{x_i}} g_u(y)\mathrm{d}y\Big)^{\frac{q}{p}}. \end{aligned}$$

Summing over all balls B_{x_i}, we obtain that

$$|\mathcal{B}|^{\frac{p}{q}} \leq C \int_\Omega g_u(y)\mathrm{d}y.$$

The proof of Theorem 1.2 is now complete.

Remark 4.2 In Theorem 1.2, q is assumed to be strictly less than $\frac{np}{s(n-p\delta)+(s-1)(p-1)}$. However, one can easily adapt the proof of Theorem 1.2 to show that when $s = 1$ or $p = 1$, q can reach the critical value (the case $s = 1$ has already been proved in [3]). Indeed, we only need to use a variant of Lemma 3.1. Namely, for each $x \in \Omega$, we may join x to x_0 via an infinite chain of balls $\{B_i\}_{i\in\mathbb{N}}$ with all the properties listed in Lemma 3.1 except the condition (5) in Lemma 3.1 replaced by

$$|x - x_i| \leq c r_i^{\frac{1}{s}} \to 0$$

as $i \to \infty$. Then following the proof of Theorem 1.2, we easily deduce the following Riesz-potential-type estimate:

$$|u(x) - u_{B_0}| \lesssim \sum_{i=1}^{\infty} r_i^{\delta-n} \int_{B_i} g(y)\mathrm{d}y \lesssim \int_\Omega \frac{g(y)}{|x-y|^{s(n-\delta)}}\mathrm{d}y.$$

Note that

$$\int_\Omega \frac{g(y)}{|x-y|^{s(n-\delta)}}\mathrm{d}y = I_\beta(\chi_\Omega g)(x),$$

where $\beta = s\delta - (s-1)n$. Thus we conclude that

$$|u(x) - u_{B_0}| \lesssim I_\beta(\chi_\Omega g)(x).$$

For $s = 1$ and $p > 1$, the claim follows from the strong-type estimate in Theorem 3.1. For $p = 1$, the claim follows from the weak-type estimate (3.2) and the weak-to-strong principle for fractional Sobolev-Poincaré inequalities (see [9, Theorem 4.1]).

Proof of Example 1.1 We will use the mushroom-like domain as used in [7]. The mushroom-like domain $\Omega \subset \mathbb{R}^n$ consists of a cube Q and an attached infinite sequence of mushrooms $F_1, F_2, \cdots$ growing on the "top" of the cube. By a mushroom F of size r, we mean a cap $\mathcal{C}$, which is a ball of radius r, and an attached cylindrical stem $\mathcal{P}$ of height r and radius r^s. The mushrooms are disjoint, and the corresponding cylinders are perpendicular to the side of the cube that we have selected as the top of the cube. We can make the mushrooms pairwise disjoint if the number r_i associated with F_i converges to 0 sufficiently fast as $i \to \infty$. We further write $\mathcal{P} = \mathcal{T} \cup \mathcal{M} \cup \mathcal{D}$, where $\mathcal{T}$ is the top $\frac{3}{8}$-part of $\mathcal{P}$, $\mathcal{M}$ is the middle $\frac{1}{4}$-part of $\mathcal{P}$, and $\mathcal{D}$ is the bottom $\frac{3}{8}$-part of $\mathcal{P}$.

Let u_i be a piecewise linear function on Ω such that $u_i = 1$ on the cap $\mathcal{C}_i \cup \mathcal{T}_i$, u_i is linear on $\mathcal{M}_i$ and $u_i = 0$ elsewhere. Assume that $1 \le s < \frac{n}{n-p\delta}$, and that one can prove the fractional (q,p)-Sobolev-Poincaré inequality with $q > \frac{np}{s(n-p\delta)+(s-1)(p-1)}$.

Note that

$$\Big(\int_\Omega |u(x) - u_\Omega|^q \mathrm{d}x\Big)^{\frac{1}{q}} \gtrsim r_i^{\frac{n}{q}}.$$

On the other hand,

$$\begin{aligned}
&\Big(\int_\Omega \int_{\Omega \cap B(x,\tau d(x,\partial\Omega))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\delta}} \mathrm{d}x\Big)^{\frac{1}{p}} \\
&= \Big(\int_{\mathcal{P}_i} \int_{\mathcal{P}_i \cap B(x,\tau d(x,\partial\Omega))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\delta}} \mathrm{d}x\Big)^{\frac{1}{p}} \\
&\lesssim \Big(r_i^{-p} \int_{\mathcal{P}_i} \mathrm{d}(x,\partial\Omega)^{p(1-\delta)} \mathrm{d}x\Big)^{\frac{1}{p}} \\
&\lesssim (r_i^{s(n-p\delta)+(s-1)(p-1)})^{\frac{1}{p}}.
\end{aligned}$$

Thus we obtain that for all $i \in \mathbb{N}$,

$$r_i^{\frac{n}{q}} \lesssim r_i^{\frac{s(n-p\delta)+(s-1)(p-1)}{p}},$$

which is impossible if $q > \frac{np}{s(n-p\delta)+(s-1)(p-1)}$.

5 Fractional (q,p)-Sobolev-Poincaré Inequalities in Domains with Quasihyperbolic Boundary Conditions

Lemma 5.1 *Fix p and q as in Theorem 1.3. Then there exists a constant $C = C(n,p,q,\beta)$ such that*

$$\sum_{Q \in \mathcal{W}} |S(Q) \cap E|^{\frac{p}{p-1}} |Q|^{-\frac{n-p\delta}{n(p-1)}} \le C|E|^{\frac{p}{p-1}\frac{q-1}{q}}$$

whenever $E \subset \Omega$.

Proof For simplicity, we write $p^{*,\delta}=\frac{np}{n-p\delta}$, $\kappa=\frac{p}{p-1}$ and $\lambda=\frac{q}{q-1}$. Then $\frac{n-p\delta}{n(p-1)}=\frac{\kappa}{p^{*,\delta}}$. Thus

$$\begin{aligned}\sum_{Q\in\mathcal{W}}|S(Q)\cap E|^{\kappa}|Q|^{-\frac{\kappa}{p^{*,\delta}}}&\le|E|^{\frac{\kappa}{p}-\frac{\kappa}{q}}\sum_{Q\in\mathcal{W}}\sum_{Q_1\in S(Q)}|Q_1\cap E|\Big(\frac{|S(Q)|^{\frac{1}{q}}}{|Q|^{\frac{1}{p^{*,\delta}}}}\Big)^{\kappa}\\&=|E|^{\frac{\kappa}{p}-\frac{\kappa}{q}}\sum_{Q_1\in\mathcal{W}}|Q_1\cap E|\sum_{Q\in P(Q_1)}\Big(\frac{|S(Q)|^{\frac{1}{q}}}{|Q|^{\frac{1}{p^{*,\delta}}}}\Big)^{\kappa}\\&\lesssim|E|^{\frac{\kappa}{p}-\frac{\kappa}{q}}\sum_{Q_1\in\mathcal{W}}|Q_1\cap E|\sum_{Q\in P(Q_1)}|Q|^{(\frac{2\beta}{1+\beta}\frac{1}{q}-\frac{1}{p^{*,\delta}})\kappa}\\&\lesssim|E|^{\frac{\kappa}{p}-\frac{\kappa}{q}}\sum_{Q_1\in\mathcal{W}}|Q_1\cap E|=|E|^{\frac{\kappa}{\lambda}},\end{aligned}$$

where we have used (3.3)–(3.4) with $\varepsilon=(\frac{2\beta}{(1+\beta)q}-\frac{1}{p^{*,\delta}})\kappa>0$.

The proof of Theorem 1.3 is again based on Theorem 1.1.

Proof of Theorem 1.3 Fix $Q_0\subset\Omega$ to be the central Whitney cube containing x_0. For each measurable set $A\subset\Omega$ with $\overline{A}\cap\overline{Q_0}=\emptyset$, let $u\in C(\Omega)\cap L^1(\Omega)$ satisfy $u|_A\ge 1$ and $u|_{Q_0}=0$. As in the proof of Theorem 1.2, we divide A into "good" and "bad" parts. Set

$$\mathcal{G}=\Big\{x\in A\mid u_Q\ge\frac{1}{2}\text{ for some Whitney cube }Q\ni x\Big\}\quad\text{and}\quad\mathcal{B}=A\backslash\mathcal{G}.$$

We have $|A|\le|\mathcal{G}|+|\mathcal{B}|$ and we first estimate $|\mathcal{B}|$.

For points $x\in\mathcal{B}$, the standard fractional $(p^{*,\delta},p)$-Sobolev-Poincaré inequality on cubes provides a trivial estimate

$$|A\cap Q|^{\frac{1}{p^{*,\delta}}}\le C\Big(\int_Q|u-u_Q|^{p^{*,\delta}}\mathrm{d}y\Big)^{\frac{1}{p^{*,\delta}}}\le C\Big(\int_Q g_u(y)\mathrm{d}y\Big)^{\frac{1}{p}}$$

on Whitney cube Q containing x, where $p^{*,\delta}=\frac{np}{n-p\delta}$. Since $q<p^{*,\delta}$, this yields

$$\int_Q g_u(y)\mathrm{d}y\ge\frac{1}{C}|A\cap Q|^{\frac{p}{q}},$$

and by summing over all such Whitney cubes, we deduce that

$$\int_\Omega g_u(y)\mathrm{d}y\ge\frac{1}{C}|\mathcal{B}|^{\frac{p}{q}}.\tag{5.1}$$

We next estimate $|\mathcal{G}|$ and our aim is to show that

$$\int_\Omega g_u(y)\mathrm{d}y\ge\frac{1}{C}|\mathcal{G}|^{\frac{p}{q}},\tag{5.2}$$

so then the conclusion follows from Theorem 1.1.

For each $x\in\mathcal{G}$, let $Q(x)$ be the Whitney cube containing x, for which $u_{Q(x)}\ge\frac{1}{2}$. Then the chaining argument used in the proof of Theorem 1.2 gives us the estimate

$$1\lesssim\sum_{Q\in P(Q(x))}(\operatorname{diam}Q)^{\delta-\frac{n}{p}}\Big(\int_Q g_u(y)\mathrm{d}y\Big)^{\frac{1}{p}}.\tag{5.3}$$

Recall that $P(Q(x))$ consists of the collection of all the Whitney cubes which intersect the quasi-hyperbolic geodesic joining x_0 to the center of $Q(x)$. Strictly speaking, on the right-hand side of (5.3), one should replace Q with λQ, where $1 < \lambda < \frac{11}{10}$ is a fixed constant, when applying the chaining argument. But a simple maximal function argument would imply that the two quantities are comparable. We leave the details to the interested readers.

Integrating (5.3) with respect to the Lebesgue measure and interchanging the order of summation and integration yield

$$\begin{aligned}|\mathcal{G}| &\lesssim \int_{\mathcal{G}} \sum_{Q\in P(Q(x))} (\operatorname{diam} Q)^{\delta-\frac{n}{p}} \Big(\int_Q g_u(y)\mathrm{d}y\Big)^{\frac{1}{p}} \mathrm{d}x \\ &= \sum_{Q\in\mathcal{W}} |S(Q)\cap\mathcal{G}|(\operatorname{diam} Q)^{\delta-\frac{n}{p}} \Big(\int_Q g_u(y)\mathrm{d}y\Big)^{\frac{1}{p}}. \end{aligned} \tag{5.4}$$

Applying Hölder's inequality leads to

$$\begin{aligned}|\mathcal{G}| &\lesssim \Big(\sum_{Q\in\mathcal{W}} |S(Q)\cap\mathcal{G}|^{\frac{p}{p-1}}|Q|^{-\frac{n-p\delta}{n(p-1)}}\Big)^{\frac{p-1}{p}} \Big(\sum_{Q\in\mathcal{W}}\int_Q g_u(y)\mathrm{d}y\Big)^{\frac{1}{p}} \\ &\le \Big(\sum_{Q\in\mathcal{W}} |S(Q)\cap\mathcal{G}|^{\frac{p}{p-1}}|Q|^{-\frac{n-p\delta}{n(p-1)}}\Big)^{\frac{p-1}{p}} \Big(\int_\Omega g_u(y)\mathrm{d}y\Big)^{\frac{1}{p}}.\end{aligned}$$

Applying Lemma 5.1, we find that

$$|\mathcal{G}| \lesssim |\mathcal{G}|^{\frac{q-1}{q}} \Big(\int_\Omega g_u(y)\mathrm{d}y\Big)^{\frac{1}{p}},$$

which proves (5.2).

Proof of Example 1.2 The construction here is similar to that used in the proof of Example 1.1 and thus we only point out the difference. The mushroom-like domain $\Omega \subset \mathbb{R}^n$ consists of a cube Q and an attached infinite sequence of mushrooms $F_1, F_2, \cdots$ growing on the "top" of the cube as in Example 1.1. Now, by a mushroom F of size r, we mean a cap $\mathcal{C}$, which is a ball of radius r, and an attached cylindrical stem $\mathcal{P}$ of height r^τ and radius r^σ. The mushrooms are disjoint, and the corresponding cylinders are perpendicular to the side of the cube that we have selected as the top of the cube. We can make the mushrooms pairwise disjoint if the number r_i associated with F_i converges to 0 sufficiently fast as $i \to \infty$. We further write $\mathcal{P} = \mathcal{T} \cup \mathcal{M} \cup \mathcal{D}$, where $\mathcal{T}$ is the top $\frac{3}{8}$-part of $\mathcal{P}$, $\mathcal{M}$ is the middle $\frac{1}{4}$-part of $\mathcal{P}$, and $\mathcal{D}$ is the bottom $\frac{3}{8}$-part of $\mathcal{P}$.

It is easy to show that Ω satisfies the β-quasihyperbolic boundary condition (2.1) if $\sigma = \frac{1+\beta}{2\beta} \le \tau$ (see for instance [13, Example 5.5]). We next show that Ω is not a fractional (q,p)-Sobolev-Poincaré domain if

$$q > \frac{np}{\sigma(n-p\delta)+(p-1)(\sigma-\tau)}. \tag{5.5}$$

When $\tau = \sigma = \frac{1+\beta}{2\beta}$, (5.5) implies that Ω is a β-quasihyperbolic boundary condition which does not support a fractional (q,p)-Sobolev-Poincaré inequality. This verifies Example 1.2.

Let u_i be a piecewise linear function on Ω such that $u_i = 1$ on the cap $\mathcal{C}_i \cup \mathcal{T}_i$, and u_i be linear on $\mathcal{M}_i$ and $u_i = 0$ elsewhere. Assume that the fractional (q,p)-Sobolev-Poincaré inequality holds on Ω.

Note that

$$\Big(\int_\Omega |u(x)-u_\Omega|^q \mathrm{d}x\Big)^{\frac{1}{q}} \gtrsim r_i^{\frac{n}{q}}.$$

On the other hand,

$$\begin{aligned}
&\Big(\int_\Omega \int_{\Omega\cap B(x,d(x,\partial\Omega))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\delta}}\mathrm{d}x\Big)^{\frac{1}{p}}\\
&=\Big(\int_{\mathcal{P}_i} \int_{\mathcal{P}_i\cap B(x,d(x,\partial\Omega))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\delta}}\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\lesssim \Big(r_i^{-\tau p}\int_{\mathcal{P}_i} \mathrm{d}(x,\partial\Omega)^{p(1-\delta)}\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\lesssim (r_i^{\sigma(n-p\delta)+(p-1)(\sigma-\tau)})^{\frac{1}{p}}.
\end{aligned}$$

Thus we obtain that for all $i\in\mathbb{N}$,

$$r_i^{\frac{n}{q}} \lesssim r_i^{\frac{\sigma(n-p\delta)+(p-1)(\sigma-\tau)}{p}},$$

which is impossible if $q > \frac{np}{\sigma(n-p\delta)+(\sigma-\tau)(p-1)}$.

6 Necessary Conditions for the Fractional (q,p)-Sobolev-Poincaré Domains

Proof of Theorem 1.4 Fix $x\in\Omega$. Pick a curve $\gamma:[0,1]\to\Omega$ with $\gamma(0)=x$ and $\gamma(1)=x_0$ as in the definition of separation property.

Let $0<t<1$ and $\delta(t)=d(\gamma(t),C\delta(t))$. If $\gamma([0,t])\subset B(\gamma(t),C\delta(t))$, then there is nothing to prove. Otherwise, the separation property implies that $\partial B=\partial B(\gamma(t),C\delta(t))$ separates $\gamma([0,t])\backslash B$ from x_0. If the component of $\Omega\backslash\partial B$ containing x_0 does not contain a ball centered at x_0 of a radius $\frac{\delta(1)}{2}$, then B must have a radius at least $\frac{\delta(1)}{4}$ since it intersects both $B(x_0,\frac{\delta(1)}{2})$ and $\partial\Omega$. In this case, $B'=4B$ contains $B(x_0,\frac{\delta(1)}{4})$ and we may assume that B' does not contain $\gamma([0,t])$ (since otherwise we are done). Thus either $\Omega\backslash\partial B$ or B' contains a ball centered at x_0 of a radius comparable to $\delta(1)$. In either case, we conclude from Proposition 3.1 that

$$\operatorname{diam}\gamma([0,t])\le C\varphi(d(\gamma(t),\partial\Omega)),$$

where $\varphi(t)=t^{\frac{(n-p\delta)q}{p\delta}(\frac{1}{p}-\frac{1}{q})}$.

A bounded domain $\Omega\subset\mathbb{R}^n$ with a distinguished point x_0 satisfying (1.7) with $\varphi(t)=t^{\frac{1}{s}}$ is termed s-diam John in [6]. It was proved in [6] that, for $s>1$, s-diam John domains are not necessarily s-John.

In [2, Corollary 4.1], it was stated that if a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies a separation property and supports a (q,p)-Sobolev-Poincaré inequality (1.3) with $q > p$, then Ω is s-John with $s = \frac{p^2}{(n-p)(q-p)}$. One could immediately check that the proof given there is only sufficient to deduce that Ω is s-diam John with $s = \frac{p^2}{(n-p)(q-p)}$. In fact, combining [6, Example 5.1] and [2, Section 4], one can produce an s-diam John domain $\Omega \subset \mathbb{R}^n$ with $s = \frac{p^2}{(n-p)(q-p)}$ such that Ω supports a (q,p)-Sobolev-Poincaré inequality. Moreover, Ω is not s'-diam John whenever $s' < s$ and Ω is not s-John.

We next briefly discuss how to construct such an example in the plane (it works in higher dimensions as well). Set

$$C(r;\alpha,\beta) = C(r) = \{(x_1,x) : 0 < x_1 < r^\alpha,\ |x'| < r^\beta\},$$

where $0 < \alpha < \beta \le 1$ will be specified later. The idea is very simple: We first use the mushroom-like domain $\Omega' \subset \mathbb{R}^2$ as is constructed in [2] (with different choices of parameters) and then modify Ω' to be a spiral domain Ω as in [6, Example 5.1].

The mushroom-like domain $\Omega' \subset \mathbb{R}^2$ consists of a cube Q and an attached infinite sequence of mushrooms $F_1, F_2, \cdots$ growing on the "top" of the cube as in Example 1.1. Now, by a mushroom F of size r, we mean a cap $\mathcal{C}$, which is a ball of radius r, and an attached cylindrical stem $C(r)$. The mushrooms are disjoint, and the corresponding cylinders are perpendicular to the side of the cube that we have selected as the top of the cube. We can make the mushrooms pairwise disjoint if the numbers r_i associated with F_i converge to 0 sufficiently fast as $i \to \infty$.

Note first that if $\beta = \alpha\frac{p+(p-1)q}{(n-1)(q-p)}$ with $n = 2$, then $C(r)$ satisfies the (q,p)-Sobolev-Poincaré inequality uniformly in r (see [2]). Let $\mu = s\beta = \frac{p^2}{(2-p)(q-p)}\beta$ and $p^* = \frac{np}{n-p}$. One can show that Ω' is a (q,p)-Sobolev-Poincaré domain if

$$\alpha + \beta(n-1) - \frac{nq}{p^*} > 0 \tag{6.1}$$

holds with $n = 2$ (see [2]). Note also that Ω is $\frac{1}{\alpha}$-John.

We next bend each mushroom F_i to make it spiral so that the resulting domain Ω is an s-diam John domain. According to our choice, $s = \frac{\mu}{\beta}$. One can check that if $\beta = \alpha\frac{p+(p-1)q}{q-p}$, then (6.1) reduces to

$$\frac{1}{\beta} < \frac{p^2}{(2-p)[p+(p-1)q]}. \tag{6.2}$$

Since $p < q < p^*$, $\frac{p^2}{(2-p)[p+(p-1)q]} > 1$. For any β satisfying (6.2) and $\beta = \alpha\frac{p+(p-1)q}{q-p}$, it is easy to check that $\frac{1}{\alpha} > \frac{\mu}{\beta} = s$. It is clear that Ω' and Ω are bi-Lipschitz equivalent, so the (q,p)-Sobolev-Poincaré inequality holds in Ω as well. Moreover, Ω satisfies all the required properties.

One could also modify the above example to the fractional (q,p)-Sobolev-Poincaré case, but the computations will be too complicated, so we omit them in the present paper.

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