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Continuity of Weak Solutions for Quasilinear Parabolic Equations with Strong Degeneracy**

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Abstract The aim of this paper is to study the continuity of weak solutions for quasilinear degenerate parabolic equations of the form

$$u_t - \Delta\phi(u) = 0,$$

where $\phi \in C^1(\mathbb{R}^1)$ is a strictly monotone increasing function. Clearly, the above equation has strong degeneracy, i.e., the set of zero points of $\phi'(\,\cdot\,)$ is permitted to have zero measure. This is an answer to an open problem in [13, p. 288].

Keywords Continuity of weak solutions, Quasilinear degenerate parabolic equation
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1 Introduction

In this paper we study the continuity of weak solutions for quasilinear degenerate parabolic equations

$$\frac{\partial u}{\partial t} - \Delta\phi(u) = 0, \tag{1.1}$$

where ϕ satisfies the following conditions (H1)–(H2):

(H1) ϕ is a strictly monotone increasing function, i.e.,

$$\phi(s_1) > \phi(s_2) \Longleftrightarrow s_1 > s_2,$$

and $\phi(0) = 0$.

(H2) ϕ is locally Lipshitz continuous, i.e., for any $a \in (0, +\infty)$, there exists a positive number $A \equiv A(a)$ such that

$$|\phi(s_1) - \phi(s_2)| \leq A|s_1 - s_2|$$

for all $s_1 \in [-a, a]$ and all $s_2 \in [-a, a]$.

Clearly, the equation (1.1) has strong degeneracy, i.e., the set of zero points of $\phi'(\,\cdot\,)$ is permitted to have zero measure.

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The equation (1.1) has been suggested as a mathematical model for a variety of physical problems. We shall not recall them here but refer to [1], where a very extensive literature about the porous medium equation and some of its generalizations have been summarized.

For the regularity of solutions of the quasilinear degenerate parabolic equations, results are obtained in one-dimensional case by a number of authors, for example, D. G. Aronson [2–4], D. G. Aronson and J. L. Vazquez [5].

In multi-dimensional case, the Hölder continuity of solutions of the porous medium equation

$$\frac{\partial u}{\partial t} - \Delta(|u|^{m-1}u) = 0, \quad m > 1$$

is obtained first by L. A. Caffarelli and A. Friedman [6] by means of the inequality

$$\frac{\partial u}{\partial t} \geq -\frac{ku}{t}, \quad k = \frac{1}{m-1-\frac{2}{N}}$$

estiblished by D. G. Aronson and P. Benilan in [7]. The work in [6] is highly important for the nonlinear degenerate parabolic equations. In addition, such results for general nonlinear degenerate parabolic equations have been obtained by a number of authors, for example, L. A. Caffarelli, J. L. Vazquez and N. I. Wolanski [8], L. A. Caffarelli and N. I. Wolanski [9], Chen Yazhe [10], E. DiBenedetto [11] and E. DiBenedetto and A. Friedman [12].

But, the continuity of solutions for the equation (1.1) is still an open problem (for example, see [13, p. 288]). For this problem, we shall give an answer as follows.

Let us consider the Cauchy problem of (1.1) with the following initial condition

$$u(x,0) = u_0(x) \quad \text{for all } x \in \mathbb{R}^N, \tag{1.2}$$

where $u_0 \in L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ and $Q_T \equiv \mathbb{R}^N \times (0,T)$ with $T > 0$.

The definition of weak solutions of (1.1)–(1.2) is given by

Definition 1.1 *A function $u \in L^\infty(Q_T)$ is said to be a solution of* (1.1)–(1.2) *in Q_T, if u satisfies following conditions* (i) *and* (ii):

(i) *We have*

$$u \in C(0,T;L^2(\Omega)), \quad |\nabla v| \in L^\infty(0,T;L^2_{\text{loc}}(\mathbb{R}^N)),$$

where $v \equiv \phi(u)$.

(ii) *For any $\xi \in C^2(0,T;C_0^2(\mathbb{R}^N))$ with $\xi(x,T) = 0$ for $x \in \mathbb{R}^N$, we have*

$$\iint_{Q_T} [u\xi_t - \nabla\phi(u)\nabla\xi]dxdt + \int_{\mathbb{R}^N} u_0(x)\xi(x,0)dx = 0.$$

Our main result is the following theorem.

Theorem 1.1 *Assume that u is a solution of* (1.1)–(1.2)*, and u_0 satisfies*

$$\Delta\phi(u_0) \geq 0 \quad (\text{or } \Delta\phi(u_0) \leq 0) \tag{1.3}$$

in the sense of distributions. Then we have $u \in C(Q_T)$.

Our proof of the main theorem is very interesting, which is based on some new ideas.

The proof of Theorem 1.1 is completed in Section 6. In the proving process of these theorems, we need some results in Sections 2–5.

Without loss of generality, we assume that $N \geq 3$ in this paper.

2 Some Known Lemmas

In order to discuss our main result, we need following lemmas.

Lemma 2.1 *Let y_n $(n = 0, 1, 2, \cdots)$ be a sequence of real numbers satisfying the following inequalities $0 \leq y_{n+1} \leq cb^n y_n^{1+\sigma}$ for $n = 0, 1, 2, \cdots$, where $c > 0$, $\sigma > 0$ and $b > 1$. Then*

$$y_n \leq c^{[(1+\sigma)^n - 1]/\sigma} b^{[(1+\sigma)^n - 1 - n\sigma]/\sigma^2} y_0^{(1+\sigma)^n} \quad \text{for } n = 0, 1, 2, \cdots .$$

In particular, we have the following conclusions:

(i) *The following inequality holds:*

$$\overline{\lim_{n \to +\infty}}\, y_n^{1/(1+\sigma)^n} \leq c^{1/\sigma} b^{1/\sigma^2} y_0;$$

(ii) *If $y_0 < c^{-1/\sigma} b^{-1/\sigma^2}$, then*

$$\lim_{n \to +\infty} y_n = 0.$$

The proof can be found in [15].

Lemma 2.2 *Assume that $p \geq 1$, $\sigma \geq 1$, $u \in W_0^{1,p}(\Omega)$, and Ω is a bounded and smooth domain in $\mathbb{R}^N$. Then we have*

(i) *If $p < N$, then*

$$\|u\|_{L^{Np/(N-p)}(\Omega)} \leq C_1 \|\nabla u\|_{L^p(\Omega)},$$

where C_1 is a positive constant depending only on p and N.

(ii) *If $p \geq N$, then*

$$\|u\|_{L^\gamma(\Omega)} \leq C_2 \|\nabla u\|_{L^p(\Omega)}^{\Theta} \|u\|_{L^\sigma(\Omega)}^{1-\Theta} \quad \text{for } \gamma > \sigma,$$

where $C_2 = \max\big\{\frac{\gamma(N-1)}{N}, 1 + \frac{(p-1)\sigma}{N}\big\}^{\Theta}$, and

$$\frac{1}{\gamma} = \frac{1}{\sigma} - \Theta\Big(\frac{1}{\sigma} - \frac{1}{p} + \frac{1}{N}\Big).$$

The proof can be found in [15].

Lemma 2.3 *Assume that A is a smooth bounded domain in $\mathbb{R}^N$ and*

$$u \in L^\infty(A \times (0, T)), \quad v \in L^2(0, T; W_0^{1,2}(A)).$$

Then we have

$$\iint_{A_T} u^{2/q} v^2 dxdt \leq C(N)\Big(\sup_{0<t<T} \int_A u(x,t)dx\Big)^{2/q} \iint_{A_T} |\nabla v|^2 dxdt,$$

where $q \geq N$ for $N \geq 3$; $q > 2$ for $N = 2$; and $q \geq 2$ for $N = 1$.

The proof can be found in [16].

Lemma 2.4 *Assume that $u \in W^{1,1}(B_R(X))$ with $X \in \mathbb{R}^N$ and $l, k \in \mathbb{R}^1$ with $l > k$. Then*

$$(l-k)|A^+_{l,R}| \leq \frac{CR^{N+1}}{|B_R(X) \setminus A^+_{k,R}|} \int_{A^+_{k,R} \setminus A^+_{l,R}} |\nabla u| dx,$$

where $A^+_{k,R} = \{x \in B_R(X) : u(x) > k\}$, and C is a positive constant depending only on N.

The proof can be found in [15] or [16].

3 Some Properties of Weak Solutions

Let us consider the regularized equations

$$u_{\epsilon t} - \Delta \phi_{0\epsilon}(u_\epsilon) = 0 \tag{3.1}$$

in Q_T with initial data

$$u_\epsilon(x, 0) = u_{0\epsilon}(x) \quad \text{for all } x \in \mathbb{R}^N, \tag{3.2}$$

where

$$\phi_{0\epsilon}(s) = (j_\epsilon * \phi_0)(s) + \epsilon s \tag{3.3}$$

for all $\epsilon \in (0, 1)$ and all $s \in \mathbb{R}^1$, and

$$u_{0\epsilon}(x) = \phi^{-1}_{0\epsilon}((J_\epsilon * (\phi_0(u_0)))(x)) \tag{3.4}$$

for all $\epsilon \in (0, 1)$ and all $x \in \mathbb{R}^N$, $\phi_0 \in C(\mathbb{R}^1)$, $j_\epsilon(s) = \frac{1}{\epsilon} j(\frac{s}{\epsilon})$ and $J_\epsilon(x) = \frac{1}{\epsilon^N} j(\frac{x}{\epsilon^N})$ are defined as follows:

$$\phi_0(s) = \begin{cases} s + \phi(B_0 + 1) - B_0 - 1, & s > B_0 + 1, \\ \phi(s), & |s| \leq B_0 + 1, \\ s + \phi(-B_0 - 1) + B_0 + 1, & s < -B_0 - 1, \end{cases} \tag{3.5}$$

with

$$B_0 = \sup_{x \in \mathbb{R}^N} |u_0(x)| < +\infty, \tag{3.6}$$

and

$$0 \leq j \in C_0^\infty(\mathbb{R}^1), \quad j(-s) = j(s), \quad \operatorname{supp} j \subset (-1, 1), \quad \int_{\mathbb{R}^1} j(s) ds = 1; \tag{3.7}$$

$$0 \leq J \in C_0^\infty(\mathbb{R}^N), \quad J(-x) = J(x), \quad \operatorname{supp} J \subset B_1(0), \quad \int_{\mathbb{R}^N} J(x) dx = 1. \tag{3.8}$$

Clearly, the Cauchy problem (3.1)–(3.2) has a unique classical solution $u_\epsilon \in L^\infty(Q_T) \cap C^2(\overline{Q_T})$.

Applying some proofs in [13, pp. 348–349], we can find a subsequence $\{\epsilon_n\}$ of $\{\epsilon\}$ and a function $u \in L^\infty(Q_T)$ such that

$$u_{\epsilon_n} \to u, \quad \text{a.e. in } Q_T, \tag{3.9}$$

$$v_{\epsilon_n} \to v, \quad \text{a.e. in } Q_T, \tag{3.10}$$

$$\nabla v_{\epsilon_n} \rightharpoonup \nabla v, \quad \text{in } L^2_{\text{loc}}(Q_T), \tag{3.11}$$

as $\epsilon_n \to 0^+$, where

$$v_\epsilon(x,t) = \phi_{0\epsilon}(u_\epsilon(x,t)), \quad v(x,t) = \phi(u(x,t)). \tag{3.12}$$

In addition, we conclude that the function u is the unique weak solution u of the Cauchy problem (1.1)–(1.2).

Lemma 3.1 *If*

$$\Delta\phi(u_0) \geq 0 \tag{3.13}$$

in the sense of distributions, then we have

$$u_{\epsilon t} = \Delta\phi_{0\epsilon}(u_\epsilon) \geq 0 \quad in\ Q_T$$

for all $\epsilon \in (0,1)$.

Proof By (3.1), we have

$$(u_{\epsilon t})_t - \Delta[\phi_{0\epsilon}(u_\epsilon)]_t = 0,$$

which implies

$$w_t - \phi'_{0\epsilon}(u_\epsilon)\Delta w - 2\nabla\phi'_{0\epsilon}(u_\epsilon)\nabla w - [\Delta\phi'_{0\epsilon}(u_\epsilon)]w = 0, \tag{3.14}$$

where $w = u_{\epsilon t}$. On the other hand, by (3.1) and (3.4) with (3.13), we compute

$$\begin{aligned} w(x,0) = u_{\epsilon t}(x,0) &= \Delta\phi_{0\epsilon}(u_{0\epsilon}(x)) \\ &= \Delta\phi_{0\epsilon}(\phi_{0\epsilon}^{-1}((J_\epsilon * (\phi_0(u_0)))(x))) \\ &= \Delta((J_\epsilon * (\phi_0(u_0)))(x)) \geq 0. \end{aligned} \tag{3.15}$$

Applying the comparison principle and using (3.15)–(3.16), we conclude that

$$u_{\epsilon t}(x,t) = w(x,t) \geq 0 \quad \text{for all } (x,t) \in Q_T.$$

Thus the proof is completed.

Lemma 3.2 *If*

$$\Delta\phi(u_0) \leq 0$$

in the sense of distributions, then we have

$$u_{\epsilon t} = \Delta\phi_{0\epsilon}(u_\epsilon) \leq 0 \quad in\ Q_T$$

for all $\epsilon \in (0,1)$.

The proof is similar to the proof of Lemma 3.2, so the details are omitted here.

Lemma 3.3 *If* (3.13) *holds then we have*

$$\int_{\mathbb{R}^N} \xi^2|\nabla(v_\epsilon(x,t)-k)^+|^2dx \leq 16\int_{\mathbb{R}^N} |\nabla\xi|^2[(v_\epsilon(x,t)-k)^+]^2dx, \tag{3.16}$$

$$\int_{\mathbb{R}^N} |\nabla[\xi(v_\epsilon(x,t)-k)^+]|^2dx \leq 36\int_{\mathbb{R}^N} |\nabla\xi|^2[(v_\epsilon(x,t)-k)^+]^2dx \tag{3.17}$$

for all $t \in (0,T)$, *and all* $k \in \mathbb{R}^1$ *and all* $\xi \in C_0^\infty(\mathbb{R}^N)$ *such taht* $0 \leq \xi(x) \leq 1$.

Proof For $\xi \in C_0^\infty(\mathbb{R}^N)$ with $0 \le \xi \le 1$, we multiply (3.1) by $\xi^2(x)(v_\epsilon(x,t)-k)^+$ and integrate over $\mathbb{R}^N$ to obtain

$$\int_{\mathbb{R}^N} [u_{\epsilon t} \cdot \xi^2(v_\epsilon(x,t)-k)^+ + \nabla v_\epsilon(x,t)\nabla(\xi^2(v_\epsilon(x,t)-k)^+)]dx = 0 \tag{3.18}$$

for $t \in (0,T)$. Applying the Young's inequality, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla v_\epsilon \nabla(\xi^2(v_\epsilon(x,t)-k)^+)dx \\
&= \int_{\mathbb{R}^N} \xi^2|\nabla(v_\epsilon(x,t)-k)^+|^2 dx + \int_{\mathbb{R}^N} 2\xi(v_\epsilon(x,t)-k)^+\nabla v_\epsilon(x,t)\nabla\xi dx \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} \xi^2|\nabla(v_\epsilon(x,t)-k)^+|^2 dx - 8\int_{\mathbb{R}^N} |\nabla\xi|^2[(v_\epsilon(x,t)-k)^+]^2 dx.
\end{aligned} \tag{3.19}$$

Combining (3.18)–(3.19) and applying Lemma 3.1, we get

$$\int_{\mathbb{R}^N} \xi^2|\nabla(v_\epsilon(x,t)-k)^+|^2 dx \le 16\int_{\mathbb{R}^N} |\nabla\xi|^2[(v_\epsilon(x,t)-k)^+]^2 dx.$$

Therefore, (3.16) is proved.

In addition, by (3.16), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla[\xi(v_\epsilon(x,t)-k)^+]|^2 dx \\
&= \int_{\mathbb{R}^N} |(v_\epsilon(x,t)-k)^+\nabla\xi + \xi\nabla(v_\epsilon(x,t)-k)^+|^2 dx \\
&\le 2\int_{\mathbb{R}^N} |\nabla\xi|^2[(v_\epsilon(x,t)-k)^+]^2 dx + 2\int_{\mathbb{R}^N} \xi^2|\nabla(v_\epsilon(x,t)-k)^+|^2 dx \\
&\le 36\int_{\mathbb{R}^N} |\nabla\xi|^2[(v_\epsilon(x,t)-k)^+]^2 dx,
\end{aligned}$$

which implies that (3.17). Thus the proof is completed.

Lemma 3.4 *For any $\epsilon \in (0,1)$, we have*

$$\epsilon \le \phi'_{0\epsilon}(s) \le C, \quad (\phi_{0\epsilon}^{-1})'(s) \ge \frac{1}{C} \quad \textit{for all } s \in \mathbb{R}^1.$$

Proof By (3.3), we compute

$$\frac{\phi_{0\epsilon}(s+h)-\phi_{0\epsilon}(s)}{h} = \int_{\mathbb{R}^1} j(z)\frac{\phi_0(s+h+\epsilon z)-\phi_0(s+\epsilon z)}{h}dz + \epsilon \quad \text{for all } h \in (0,1).$$

Applying (H2), we have

$$\epsilon \le \frac{\phi_{0\epsilon}(s+h)-\phi_{0\epsilon}(s)}{h} \le C \quad \text{for all } h \in (0,1).$$

Letting $h \to 0^+$, we get

$$\epsilon \le \phi'_{0\epsilon}(s) \le A_0 + 1, \quad \forall\, s \in \mathbb{R}^1,\ \forall\, \epsilon \in (0,1).$$

Therefore, we get

$$(\phi_{0\epsilon}^{-1})'(s) = \frac{1}{\phi_{0\epsilon}'[\phi_{0\epsilon}^{-1}(s)]} \geq \frac{1}{A_0+1} \quad \text{for all } \epsilon \in (0,1).$$

Thus the proof is completed.

Lemma 3.5 *For any $\epsilon \in (0,1)$, $\phi_{0\epsilon} \in C^\infty(\mathbb{R}^1)$ and $\phi_{0\epsilon}^{-1} \in C^\infty(\mathbb{R}^1)$ satisfy*

$$|\phi_{0\epsilon}(s)| \leq \Lambda + 2|s|, \quad |\phi_{0\epsilon}^{-1}(s)| \leq \Lambda + |s| \quad \textit{for all } s \in \mathbb{R}^1,$$

where $\Lambda = |\phi(B_0+1)| + |\phi(-B_0-1)| + B_0 + 2$.

Proof By (3.3) and (3.5), we compute

$$\begin{aligned}
\phi_{0\epsilon}(s) &= \int_{\mathbb{R}^1} j_\epsilon(s-x)\phi_0(x)dx + \epsilon s \\
&= \frac{1}{\epsilon}\int_{\mathbb{R}^1} j\Big(\frac{s-x}{\epsilon}\Big)\phi_0(x)dx + \epsilon s \\
&= \int_{\mathbb{R}^1} j(-z)\phi_0(s+\epsilon z)dz + \epsilon s \\
&= \int_{\mathbb{R}^1} j(z)[s+\epsilon z + \phi(B_0+1) - B_0 - 1]dz + \epsilon s \\
&= (1+\epsilon)s + \phi(B_0+1) - B_0 - 1
\end{aligned}$$

for all $s > B_0+2$. Similarly, we also have

$$\phi_{0\epsilon}(s) = (1+\epsilon)s + \phi(-B_0-1) + B_0 + 1 \quad \text{for all } s < -B_0-2.$$

In addition, we have

$$\begin{aligned}
|\phi_{0\epsilon}(s)| &= \Big|\int_{\mathbb{R}^1} j_\epsilon(s-x)\phi_0(x)dx + \epsilon s\Big| \\
&\leq \Big|\frac{1}{\epsilon}\int_{\mathbb{R}^1} j\Big(\frac{s-x}{\epsilon}\Big)\phi_0(x)dx\Big| + \epsilon|s| \\
&\leq \int_{\mathbb{R}^1} j(-z)|\phi_0(s+\epsilon z)|dz + \epsilon|s| \\
&\leq \max\{|\phi(-B_0-1)|+1, |\phi(B_0+1)|+1\} + \epsilon|s|
\end{aligned}$$

for all $s \in (-B_0-2, B_0+2)$. By the above computation, we have

$$|\phi_{0\epsilon}(s)| \leq \Lambda + 2|s|.$$

We now prove

$$|\phi_{0\epsilon}^{-1}(s)| \leq \Lambda + |s| \quad \text{for all } s \in \mathbb{R}^1.$$

In fact, we compute

$$|\phi_{0\epsilon}^{-1}(s)| \begin{cases} = \dfrac{|s - \phi(B_0+1) + B_0 + 1|}{1+\epsilon}, & \text{if } \phi_{0\epsilon}^{-1}(s) > B_0+2, \\ \leq B_0 + 2, & \text{if } |\phi_{0\epsilon}^{-1}(s)| \leq B_0+2, \\ = \dfrac{|s - \phi(-B_0-1) - B_0 - 1|}{1+\epsilon}, & \text{if } \phi_{0\epsilon}^{-1}(s) < -B_0-2. \end{cases}$$

Thus the proof is completed.

Theorem 3.1 *For any* $(x_0, t_0) \in Q_T$, *we have*

$$\sup_{t_0-\rho_1\le t\le t_0+\rho_2}\int_{B_R(x_0)}\psi^2(H^\pm,(u-k)^\pm,\nu)\xi^2(x)dx$$

$$\le \gamma_1\int_{t_0-\rho_1}^{t_0+\rho_2}\int_{B_R(x_0)}\psi(H^\pm,(u-k)^\pm,\nu)|\nabla\xi|^2dxdt$$

$$+\int_{B_R(x_0)}\psi^2(H^\pm,(u(x,t_0-\rho_1)-k)^\pm,\nu)\xi^2(x)dx$$

for all $k \in [-\Lambda, \Lambda]$, *all* $0 \le t_0 - \rho_1 < t_0 + \rho_2$, *and all* $\xi \in C^\infty(B_R(x_0))$ *such that* $0 \le \xi(x) \le 1$ *for* $x \in B_R(x_0)$ *and* $\xi(x,t) = 0$ *for* $x \in \partial B_R(x_0)$, *where* $H^\pm \in (0, \Lambda]$, *and for* $(u-k)^{pm} \le H^\pm$,

$$\psi(H^\pm,(u-k)^\pm,\nu)\equiv \ln^+\Big\{\frac{H^\pm}{H^\pm-(u-k)^\pm+\nu}\Big\},\quad \nu<\min\{H^\pm,1\},$$

and γ_2 *is a positive constant depending only on* N *and* Λ.

Proof For simplicity, we let

$$\psi(H^\pm,(u_\epsilon-k)^\pm,\nu)\equiv\psi((u_\epsilon-k)^\pm).$$

For $\xi \in C^\infty(B_R(x_0))$ such taht $0 \le \xi(x) \le 1$ for $x \in B_R(x_0)$ and $\xi(x,t) = 0$ for $x \in \partial B_R(x_0)$, we multiply (3.1) by $\xi^2[(\psi^2)'((u_\epsilon - k)^+)]$ and integrate over $B_R(x_0) \times (t_0 - \rho_1, s)$ to obtain

$$\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2(\psi^2)'((u_\epsilon-k)^+)(u_{\epsilon t}-\Delta\phi_{0\epsilon}(u_\epsilon))dxdt=0 \tag{3.20}$$

for all $s \in (t_0 - \rho_1, t_0 + \rho_2)$. Using the Young's inequality, we compute

$$\begin{aligned}&\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}[-\xi^2(\psi^2)'((u_\epsilon-k)^+)]\Delta\phi_{0\epsilon}(u_\epsilon)dx\\
&=\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\nabla[\xi^2(\psi^2)'((u_\epsilon-k)^+)]\nabla\phi_{0\epsilon}(u_\epsilon)dx\\
&=2\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2(1+\psi)(\psi')^2\phi'_{0\epsilon}(u_\epsilon)|\nabla((u_\epsilon-k)^+)|^2dx\\
&\quad+2\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi\psi\psi'\phi'_{0\epsilon}(u_\epsilon)\nabla\xi\nabla u_\epsilon dx\\
&\ge\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2(1+\psi)(\psi')^2\phi'_{0\epsilon}(u_\epsilon)|\nabla((u_\epsilon-k)^+)|^2dx\\
&\quad-4\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\psi((u_\epsilon-k)^+)\phi'_{0\epsilon}(u_\epsilon)|\nabla\xi|^2dx\\
&\ge -C\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\psi((u_\epsilon-k)^+)|\nabla\xi|^2dx.\end{aligned} \tag{3.21}$$

Combining (3.20) with (3.21), we obtain

$$\int_{B_R(x_0)}\xi^2\psi^2((u_\epsilon(x,s)-k)^+)dx\le C\int_{t_0-\rho_1}^{t_0+\rho_2}\int_{B_R(x_0)}\psi((u_\epsilon-k)^+)|\nabla\xi|^2dxdt$$

$$+\int_{B_R(x_0)}\xi^2\psi^2((u_\epsilon(x,t_0-\rho_1)-k)^+)dx$$

for all $s \in (t_0 - \rho_1, t_0 + \rho_2)$, where C is a positive constant depending only on Λ. Letting $\epsilon = \epsilon_n \to 0^+$ and using (3.9), we conclude that

$$\int_{B_R(x_0)} \xi^2 \psi^2((u(x,s)-k)^+)dx \le C \int_{t_0-\rho_1}^{t_0+\rho_2} \int_{B_R(x_0)} \psi((u-k)^+)|\nabla \xi|^2 dxdt + \int_{B_R(x_0)} \xi^2 \psi^2((u(x,t_0-\rho_1)-k)^+)dx$$

for $s \in (t_0 - \rho_1, t_0 + \rho_2)$. Similarly, we also have

$$\int_{B_R(x_0)} \xi^2 \psi^2((u(x,s)-k)^-)dx \le C \int_{t_0-\rho_1}^{t_0+\rho_2} \int_{B_R(x_0)} \psi((u-k)^-)|\nabla \xi|^2 dxdt + \int_{B_R(x_0)} \xi^2 \psi^2((u(x,t_0-\rho_1)-k)^-)dx$$

for $s \in (t_0 - \rho_1, t_0 + \rho_2)$. Thus the proof is completed.

Theorem 3.2 *For any $(x_0, t_0) \in Q_T$, we have*

$$\begin{aligned}&\sup_{s\in(t_0-\rho_1,t_0+\rho_2)} \int_{B_R(x_0)} \xi^2[(v(x,s)-k)^\pm]^2 dx + \iint_{Q_{\rho_1}^{\rho_2}(R)} |\nabla[\xi(v_\epsilon - k)^\pm]|^2 dxdt \\ &\le \gamma_2 \Big\{ \iint_{Q_{\rho_1}^{\rho_2}(R)} [(v-k)^\pm]^2 |\nabla \xi|^2 dxdt + \iint_{Q_{\rho_1}^{\rho_2}(R)\cap\{(v-k)^\pm>0\}} |\xi_t| dxdt \\ &\quad + \int_{B_R(x_0)\cap\{x:(v(x,t_0-\rho_1)-k)^\pm>0\}} \xi^2(x,t_0-\rho_1)dx \Big\}\end{aligned}$$

for all $k \in [-\Lambda, \Lambda]$, all $Q_{\rho_1}^{\rho_2}(R) \subset Q_T$ and all $\xi \in C^\infty(Q_{\rho_1}^{\rho_2}(R))$ such that $0 \le \xi(x,t) \le 1$ for $(x,t) \in Q_{\rho_1}^{\rho_2}(R)$ and $\xi(x,t) = 0$ for $(x,t) \in \partial B_R(x_0) \times (t_0 - \rho_1, t_0 + \rho_2)$, where $Q_{\rho_1}^{\rho_2}(R) \equiv B_R(x_0) \times (t_0 - \rho_1, t_0 + \rho_2)$, and γ_2 is a positive constant depending only on Λ and N.

Proof For $\xi \in C^\infty(Q_{\rho_1}^{\rho_2}(R))$ such taht $0 \le \xi(x,t) \le 1$ for $(x,t) \in Q_{\rho_1}^{\rho_2}(R)$ and $\xi(x,t) = 0$ for $(x,t) \in \partial B_R(x_0) \times (t_0 - \rho_1, t_0 + \rho_2)$, we multiply (3.1) by $\xi^2(v_\epsilon - k)^+$ and integrate over $B_R(x_0) \times (t_0 - \rho_1, s)$ to obtain

$$\int_{t_0-\rho_1}^{s} \int_{B_R(x_0)} [\xi^2(v_\epsilon - k)^+ u_{\epsilon t} - \xi^2 (v_\epsilon - k)^+ \Delta v_\epsilon] dxdt = 0 \tag{3.22}$$

for all $s \in (t_0 - \rho_1, t_0 + \rho_2)$. Using the Young's inequality, we compute

$$\begin{aligned}&\int_{t_0-\rho_1}^{s} \int_{B_R(x_0)} [-\xi^2(v_\epsilon - k)^+ \Delta v_\epsilon] dxdt \\ &= \int_{t_0-\rho_1}^{s} \int_{B_R(x_0)} \nabla[\xi^2(v_\epsilon - k)^+] \nabla v_\epsilon dxdt \\ &= \int_{t_0-\rho_1}^{s} \int_{B_R(x_0)} \xi^2 |\nabla (v_\epsilon - k)^+|^2 dxdt + \int_{t_0-\rho_1}^{s} \int_{B_R(x_0)} 2\xi(v_\epsilon - k)^+ \nabla\xi \nabla v_\epsilon dxdt \\ &\ge \frac{1}{2} \int_{t_0-\rho_1}^{s} \int_{B_R(x_0)} \xi^2 |\nabla (v_\epsilon - k)^+|^2 dxdt - C \int_{t_0-\rho_1}^{s} \int_{B_R(x_0)} [(v_\epsilon - k)^+]^2 |\nabla \xi|^2 dxdt,\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2(v_\epsilon-k)^+u_{\epsilon t}dxdt\\
&=\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2(v_\epsilon-k)^+(\phi_{0\epsilon}^{-1})'(v_\epsilon)v_{\epsilon t}dxdt\\
&=\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2\frac{\partial}{\partial t}\Big\{\int_0^{(v_\epsilon(x,t)-k)^+}\tau(\phi_{0\epsilon}^{-1})'(\tau+k)d\tau\Big\}dxdt\\
&=\int_{B_R(x_0)}\xi^2\Big\{\int_0^{(v_\epsilon(x,s)-k)^+}\tau(\phi_{0\epsilon}^{-1})'(\tau+k)d\tau\Big\}dx\\
&\quad-\int_{B_R(x_0)}\xi^2\Big\{\int_0^{(v_\epsilon(x,t_0-\rho_1)-k)^+}\tau(\phi_{0\epsilon}^{-1})'(\tau+k)d\tau\Big\}dx\\
&\quad-\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}2\xi\xi_t\Big\{\int_0^{(v_\epsilon(x,t)-k)^+}\tau(\phi_{0\epsilon}^{-1})'(\tau+k)d\tau\Big\}dxdt. \qquad (3.24)
\end{aligned}$$

Using Lemma 3.4, we have

$$\begin{aligned}
\int_{B_R(x_0)}\xi^2\Big\{\int_0^{(v_\epsilon(x,s)-k)^+}\tau(\phi_{0\epsilon}^{-1})'(\tau+k)d\tau\Big\}dx &\geq \int_{B_R(x_0)}\xi^2\Big\{\int_0^{(v_\epsilon(x,s)-k)^+}\frac{\tau}{C}d\tau\Big\}dx\\
&=\frac{1}{2C}\int_{B_R(x_0)}\xi^2[(v_\epsilon(x,s)-k)^+]^2dx. \qquad (3.25)
\end{aligned}$$

We have

$$|v_\epsilon(x,t)|=|\phi_{0\epsilon}(u_\epsilon(x,t))|\leq C \qquad (3.26)$$

for all $\epsilon\in(0,\epsilon_0)$ and $(x,t)\in Q_T$. Applying Lemma 3.5 and using (3.26), we have

$$\begin{aligned}
&\int_0^{(v_\epsilon(x,s)-k)^+}\tau(\phi_{0\epsilon}^{-1})'(\tau+k)d\tau\\
&=(v_\epsilon(x,s)-k)^+(\phi_{0\epsilon}^{-1})((v_\epsilon(x,s)-k)^+)-\int_0^{(v_\epsilon(x,s)-k)^+}\phi_{0\epsilon}^{-1}(\tau+k)d\tau\\
&\leq(v_\epsilon(x,s)-k)^+(\phi_{0\epsilon}^{-1})((v_\epsilon(x,s)-k)^+)-(v_\epsilon(x,s)-k)^+\phi_{0\epsilon}^{-1}(k)\\
&\leq(v_\epsilon(x,s)-k)^+[(\Lambda+|(v_\epsilon(x,s)-k)^+|)+(\Lambda+|k|)]\\
&\leq C \qquad (3.27)
\end{aligned}$$

for $k\in[-\Lambda,\Lambda]$, where C is a positive constant depending only on Λ. Combining (3.24)–(3.25) with (3.27), we obtain

$$\begin{aligned}
\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2(v_\epsilon-k)^+u_{\epsilon t}dxdt \geq{}& \frac{1}{2C}\int_{B_R(x_0)}\xi^2[(v_\epsilon(x,s)-k)^+]^2dx\\
&-C\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t_0-\rho_1)-k)^+>0\}}\xi^2(x,t_0-\rho_1)dx\\
&-C\int_{t_0-\rho_1}^{t_0+\rho_2}\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t)-k)^+>0\}}|\xi_t|dxdt, \qquad (3.28)
\end{aligned}$$

where C is a positive constant depending only on Λ. Using (3.22), (3.23) and (3.28), we get

$$\begin{aligned}&\frac{1}{2C}\int_{B_R(x_0)}\xi^2[(v_\epsilon(x,s)-k)^+]^2dx+\frac{1}{2}\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}\xi^2|\nabla(v_\epsilon-k)^+|^2dxdt\\&\le C\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}[(v_\epsilon-k)^+]^2|\nabla\xi|^2dxdt+C\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t_0-\rho_1)-k)^+>0\}}\xi^2(x,t_0-\rho_1)dx\\&\quad+C\int_{t_0-\rho_1}^{t_0+\rho_2}\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t)-k)^+>0\}}|\xi_t|dxdt\end{aligned}$$

for all $s\in(t_0-\rho_1,t_0+\rho_2)$. This implies that

$$\begin{aligned}&\sup_{s\in(t_0-\rho_1,t_0+\rho_2)}\int_{B_R(x_0)}\xi^2[(v_\epsilon(x,s)-k)^+]^2dx+\iint_{Q_{\rho_1}^{\rho_2}(R)}\xi^2|\nabla(v_\epsilon-k)^+|^2dxdt\\&\le C\int_{t_0-\rho_1}^{t_0-\rho_1}\int_{B_R(x_0)}[(v_\epsilon-k)^+]^2|\nabla\xi|^2dxdt+C\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t_0-\rho_1)-k)^+>0\}}\xi^2(x,t_0-\rho_1)dx\\&\quad+C\int_{t_0-\rho_1}^{t_0+\rho_2}\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t)-k)^+>0\}}|\xi_t|dxdt.\end{aligned}\tag{3.29}$$

In addition, we compute

$$\begin{aligned}\iint_{Q_{\rho_1}^{\rho_2}(R)}|\nabla[\xi(v_\epsilon-k)^+]|^2dxdt\le{}&2\iint_{Q_{\rho_1}^{\rho_2}(R)}\xi^2|\nabla(v_\epsilon-k)^+|^2dxdt\\&+2\iint_{Q_{\rho_1}^{\rho_2}(R)}|\nabla\xi|^2[(v_\epsilon-k)^+]^2dxdt.\end{aligned}\tag{3.30}$$

From (3.29) and (3.30), it follows that

$$\begin{aligned}&\sup_{s\in(t_0-\rho_1,t_0+\rho_2)}\int_{B_R(x_0)}\xi^2[(v_\epsilon(x,s)-k)^+]^2dx+\iint_{Q_{\rho_1}^{\rho_2}(R)}|\nabla[\xi(v_\epsilon-k)^+]|^2dxdt\\&\le C\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}[(v_\epsilon-k)^+]^2|\nabla\xi|^2dxdt-C\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t_0-\rho_1)-k)^+>0\}}\xi^2(x,t_0-\rho_1)dx\\&\quad-C\int_{t_0-\rho_1}^{t_0+\rho_2}\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t)-k)^+>0\}}|\xi_t|dxdt.\end{aligned}\tag{3.31}$$

Similarly, we also have

$$\begin{aligned}&\sup_{s\in(t_0-\rho_1,t_0+\rho_2)}\int_{B_R(x_0)}\xi^2[(v_\epsilon(x,s)-k)^-]^2dx+\iint_{Q_{\rho_1}^{\rho_2}(R)}|\nabla[\xi(v_\epsilon-k)^-]|^2dxdt\\&\le C\int_{t_0-\rho_1}^{s}\int_{B_R(x_0)}[(v_\epsilon-k)^-]^2|\nabla\xi|^2dxdt+C\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t_0-\rho_1)-k)^->0\}}\xi^2(x,t_0-\rho_1)dx\\&\quad+C\int_{t_0-\rho_1}^{t_0+\rho_2}\int_{B_R(x_0)\cap\{x:(v_\epsilon(x,t_0-\rho_1)-k)^->0\}}|\xi_t|dxdt.\end{aligned}$$

Applying (3.9)–(3.10) and using (3.31), we get the conclusion of Theorem 3.2. Thus the proof is completed.

4 Some Estimates on Lower Bound

Let the point $(x_0, t_0) \in \Omega_T$ be fixed throughout, and consider the cylinder

$$Q(R) \equiv B_R(x_0) \times (t_0 - (2R)^2, t_0 + (2R)^2) \subset \Omega_T \tag{4.1}$$

for some $R > 0$. In addition, we take

$$Q_R^\tau \equiv B_R(x_0) \times (\tau - R^2, \tau), \tag{4.2}$$

where $\tau = t_0 - R^2$. Clearly, we have

$$Q_R^\tau \subset Q(R). \tag{4.3}$$

Denote

$$\omega \equiv \omega(R) = M - m, \quad M = \operatorname*{ess\,sup}_{Q(R)} u, \quad m = \operatorname*{ess\,inf}_{Q(R)} u, \tag{4.4}$$

$$\omega_v \equiv \omega_v(R) = M_v - m_v, \quad M_v = \operatorname*{ess\,sup}_{Q(R)} v, \quad m = \operatorname*{ess\,inf}_{Q(R)} v, \tag{4.5}$$

$$\theta(r) \equiv \inf_{s\in[-\Lambda,\Lambda]}[\phi(s+r) - \phi(s)], \quad \vartheta(r) \equiv \inf_{s\in[-\Lambda,\Lambda]}[\varphi(s+r) - \varphi(s)] \tag{4.6}$$

for all $r \in (0, \Lambda)$, where $\varphi(s) \equiv \phi^{-1}(s)$ for all $s \in \mathbb{R}^1$.

In this section, we assume that

$$\omega \equiv \omega(R) \geq \lambda, \quad \forall\, R \in (0, R_0) \tag{4.7}$$

for some $R_0 > 0$ and some $\lambda > 0$ independent of R. Clearly, if (4.7) holds then

$$\omega_v = \phi(M) - \phi(m) \geq \theta(\lambda) \equiv \lambda_0 > 0. \tag{4.8}$$

Clearly, λ_0 is also some positive constant independent of R.

First, we have

Lemma 4.1 *Assume that* (4.7) *holds. There exists a positive number $\alpha \in (0,1)$ depending only on N, λ_0, γ_2 and Λ such that, if*

$$\left|\left\{(x,t) \in Q_R^\tau : v(x,t) < m_v + \frac{\omega_v}{2}\right\}\right| \leq \alpha |Q_R^\tau| \tag{4.9}$$

with $\tau = t_0 - R^2$, we have

$$v(x,t) \geq m_v + \frac{\omega_v}{4} \quad \textit{for } (x,t) \in Q_{R/2}^\tau.$$

Proof Denote

$$R_n = \frac{R}{2} + \frac{R}{2^n}, \quad Q_{R_n}^\tau(x_0) = B_{R_n}(x_0) \times (\tau - R_n^2, \tau), \quad n = 1, 2, \cdots$$

and $\xi_n \in C^\infty(Q_R^\tau)$ such that

$$\begin{cases} \xi_n(x,t) = 1, \quad \forall\,(x,t) \in Q_{R_{n+1}}^\tau(x_0), \\ \xi_n(x,t) = 0, \quad \forall\,(x,t) \in \partial B_{R_n}(x_0) \times (\tau - R_n^2, \tau), \\ \xi_n(x, \tau - R_n^2) = 0, \quad \forall x \in B_{R_n}(x_0), \\ |\nabla \xi_n| \leq C 2^n R^{-1}, \quad 0 \leq \xi_{nt} \leq C 2^n R^{-2}, \quad \text{in } Q_{R_n}^\tau(x_0) \end{cases}$$

and let

$$k_n = m_v + \frac{\omega_v}{4} + \frac{\omega_v}{2^{n+1}}, \quad n = 1, 2, \cdots.$$

Using Theorem 3.2 and choosing $k = k_n$ and $\xi = \xi_n$, we get

$$\sup_{\tau - R_n^2 \le t \le \tau} \int_{B_{R_n}(x_0)} [(v - k_n)^-]^2 \xi_n^2(x,t)dx + \iint_{Q^\tau_{R_n}(x_0)} |\nabla[\xi_n(v - k_n)^-]|^2 dxdt$$
$$\le \gamma_3 \Big\{ \iint_{Q^\tau_{R_n}(x_0)} [(v - k_n)^-]^2 |\nabla \xi_n|^2 dxdt + \iint_{Q^\tau_{R_n}(x_0) \cap \{v < k_n\}} |\xi_{nt}| dxdt \Big\}.$$

Using Lemma 2.3, we obtain

$$\iint_{Q^\tau_{R_n}(x_0)} [\xi_n(v - k_n)^-]^{2\mu} dxdt \le C \Big\{ \iint_{Q^\tau_{R_n}(x_0)} [(v - k_n)^-]^2 |\nabla \xi_n|^2 dxdt + \iint_{Q^\tau_{R_n}(x_0) \cap \{v < k_n\}} |\xi_{nt}| dxdt \Big\}^\mu. \tag{4.10}$$

We have

$$|v(x,t)| \le \Lambda_2. \tag{4.11}$$

On the other hand, by (4.8), we have

$$\iint_{Q^\tau_{R_n}(x_0)} \{[(v - k_n)^-]^2 \xi_n^2\}^\mu dxdt \ge (k_n - k_{n+1})^{2\mu} A_{n+1} \ge C 2^{-2n\mu} A_{n+1}, \tag{4.12}$$

where $A_n = |\{(x,t) \in Q^\tau_{R_n}(x_0) : v(x,t) < k_n\}|$, and C is a positive constant depending only on λ_0. Combining (4.10)–(4.11) with (4.12), we get

$$A_{n+1} \le C R^{-2\mu} 2^{4\mu n} A_n^\mu,$$

where C is a positive constant depending only on N,λ_0,γ_2 and Λ. Applying Lemma 2.1, we obtain that, if

$$A_1 \le (CR^{-2\mu})^{-1/(\mu-1)} 2^{-4\mu/(\mu-1)^2}, \tag{4.13}$$

then

$$\lim_{n \to +\infty} A_n = 0. \tag{4.14}$$

Choose $\alpha \in (0,1)$ such that

$$\alpha |Q^\tau_R| = 2^{-1} \cdot (CR^{-2\mu})^{-1/(\mu-1)} 2^{-4\mu/(\mu-1)^2}. \tag{4.15}$$

Therefore, by (4.13)–(4.15), we conclude that, if $|A_{k_1,R_1}| \le \alpha |Q^\tau_R|$,

$$v(x,t) \ge m_v + \frac{\omega_v}{4} \quad \text{for } (x,t) \in Q^\tau_{R/2}.$$

Thus the proof is completed.

Denote

$$l = \varphi\Big(m_v + \frac{\omega_v}{4}\Big) - \varphi(m_v) = \varphi\Big(m_v + \frac{\omega_v}{4}\Big) - m. \tag{4.16}$$

We have

$$l \geq \vartheta\Big(\frac{\lambda_0}{4}\Big) > 0, \tag{4.17}$$

$$v(x,t) \geq m_v + \frac{\omega_v}{4} \Longleftrightarrow u(x,t) \geq m + l. \tag{4.18}$$

Lemma 4.2 *Assume that* (4.9) *holds, and*

$$H^- = \|(u - (m+l))^-\|_{L^\infty(B_{R/2}(x_0)\times(t_0 - R^2/2, t_0+R^2))} > \frac{l}{2}. \tag{4.19}$$

Then for any $\alpha_1 \in (0,1)$, there exists a positive integer $q_1 \equiv q_1(\alpha_1)$, such that

$$\Big|\Big\{x \in B_{R/4}(x_0) : u(x,t) < m + \frac{l}{2^{q_1}}\Big\}\Big| \leq \alpha_1 |B_{R/4}(x_0)|$$

for $t \in \big(t_0 - \frac{R^2}{2}, t_0 + R^2\big)$, where C_0 is some positive constant depending only on γ_3.

Proof Denote a nonnegative function $\xi \in C_0^\infty(B_{R/2}(x_0))$ such that

$$\xi(x) = 1, \quad \forall\, x \in B_{R/4}(x_0); \quad |\nabla\xi(x)| \leq CR^{-1}, \quad \forall\, x \in B_{R/2}(x_0).$$

From Theorem 3.1, it follows that

$$\begin{aligned}
&\int_{B_{R/2}(x_0)} \xi^2 \psi^2\Big(H^-, (u(x,t) - (m+l))^-, \frac{l}{2^n}\Big)dx \\
&\leq \gamma_1\Big\{\int_{t_0 - R^2/2}^{t_0+R^2}\int_{B_{R/2}(x_0)} |\nabla\xi|^2 \psi\Big(H^-, (u(x,t) - (m+l))^-, \frac{l}{2^n}\Big)dxdt \\
&\quad + \int_{B_{R/2}(x_0)} \xi^2\psi^2\Big(H^-, \Big(u\Big(x, t_0 - \frac{R^2}{2}\Big) - (m+l)\Big)^-, \frac{l}{2^n}\Big)dx\Big\}.
\end{aligned} \tag{4.20}$$

Using Lemma 4.1 and (4.18), we have

$$\int_{B_{R/2}(x_0)} \xi^2\psi^2\Big(H^-, \Big(u\Big(x, t_0 - \frac{R^2}{2}\Big) - (m+l)\Big)^-, \frac{l}{2}^n\Big)dx = 0 \quad \text{for } n \geq 1. \tag{4.21}$$

By (4.19), we get

$$\psi\Big(H^-, (u(x,t) - (m+l))^-, \frac{l}{2^n}\Big)dx \leq n \ln 2. \tag{4.22}$$

We bound the integral on the left hand of (4.20) from below by extending the integration to the small set

$$\Big\{x \in B_{R/4}(x_0) : u(x,t) < m + \frac{l}{2^n}\Big\}.$$

On such set, by (4.19), we have

$$\psi\Big(H^-, (u(x,t) - (m+l))^-, \frac{l}{2^n}\Big) \geq (n-2)\ln 2. \tag{4.23}$$

Combining (4.21)–(4.23) with (4.20), we conclude that

$$\Big|\Big\{x \in B_{R/4} : u(x,t) < m + \frac{l}{2^n}\Big\}\Big| \leq \frac{Cn}{(n-2)^2}|B_{R/4}(x_0)|$$

for $t \in \big(t_0 - \frac{R^2}{2}, t_0 + R^2\big)$, where C and C_0 are some positive constants depending only on γ_1. We have only to choose $q_1 = n$ such that

$$\frac{Cn}{(n-2)^2} \le \frac{\alpha_1}{2}$$

and then obtain the conclusion of Lemma 4.2 for $R \in (0, \alpha_1 C_0^{-1} 2^{-(2q_1+1)})$. Thus the proof is completed.

Denote

$$l_1(q_1) \equiv \phi\Big(m + \frac{l}{2^{q_1}}\Big) - m_v \ge \theta\Big(2^{-q_1}\vartheta\Big(\frac{\lambda_0}{4}\Big)\Big). \tag{4.24}$$

Clearly, we have

$$v(x,t) < m_v + l_1(q_1) \Longleftrightarrow u(x,t) < m + \frac{l}{2^{q_1}}. \tag{4.25}$$

Lemma 4.3 *Assume that* (4.9) *and* (4.19) *hold. Then we have*

$$v(x,t) \ge m_v + \frac{l_1(q_2)}{2}$$

for $(x,t) \in B_{R/8}(x_0) \times \big(t_0 - \frac{R^2}{2}, t_0 + R^2\big)$ *and* $R \in (0, R_0)$, *where* q_2 *and* R_0 *are some positive constants independent of* R.

Proof Denote

$$R_n = \frac{R}{8} + \frac{R}{2^{n+2}}, \quad k_n = m_v + \frac{l_1(q_1)}{2} + \frac{l_1(q_1)}{2^n}$$

for $n = 1, 2, \cdots$, and choose nonnegative functions $\xi_n \in C_0^\infty(B_{R_n}(x_0))$ such that

$$\xi_n(x) = 1, \quad \forall\, x \in B_{R_{n+1}}(x_0); \quad |\nabla \xi_n(x)| \le C2^n R^{-1}, \quad \forall\, x \in B_{R_n}(x_0).$$

From Theorem 3.2, it follows that

$$\sup_{t_0 - R^2/2 \le t \le t_0 + R^2} \int_{B_{R_n}(x_0)} [(v-k_n)^-]^2 \xi_n^2(x,t)dx + \iint_{Q_n} |\nabla[\xi_n(v-k_n)^-]|^2 dxdt$$

$$\le \gamma_3 \Big\{ \iint_{Q_n} [(v-k_n)^-]^2 |\nabla \xi_n|^2 dxdt + \int_{B_{R_n}(x_0) \cap \{x: (v(x,0)-k_n)^- > 0\}} \xi_n^2(x)dx \Big\},$$

where $Q_n = B_{R_n}(x_0) \times (t_0 - \frac{R^2}{2}, t_0 + R^2)$. Applying Lemma 2.3, we have

$$\begin{aligned}\iint_{Q_n} [\xi_n(v-k_n)^-]^{2\mu} dxdt \le C\Big\{ &\iint_{Q_n} [(v-k_n)^-]^2 |\nabla \xi_n|^2 dxdt \\ &+ \int_{B_{R_n}(x_0) \cap \{x: (v(x,t_0-R^2/2)-k_n)^- > 0\}} \xi_n^2(x)dx \Big\}^\mu,\end{aligned} \tag{4.26}$$

where $\mu = 1 + \frac{2}{N}$. By Lemma 4.1, we have

$$\int_{B_{R_n}(x_0) \cap \{x: (v(x,t_0-R^2/2)-k_n)^- > 0\}} \xi_n^2(x)dx = 0. \tag{4.27}$$

Combining (4.26) with (4.27), we compute

$$(k_n - k_{n+1})^{2\mu} A_{n+1} \le C2^{2\mu} R^{-2\mu} A_n^\mu,$$

where $A_n = |\{(x,t)\in Q_n : v(x,t) < k_n\}|$. Then for $R\in(0, l_1(q_1)^2)$, we have

$$A_{n+1} \le C2^{2\mu n}R^{-2\mu}A_n^{\mu} \tag{4.28}$$

for $n = 1, 2, \cdots$, and C is a positive constant depending only on γ_3. Applying Lemma 2.1 and using (4.28), we have that, if

$$A_1 < [CR^{-2\mu}]^{-1/(\mu-1)}2^{-2\mu/(\mu-1)^2}, \tag{4.29}$$

then

$$\lim_{n\to+\infty} A_n = 0. \tag{4.30}$$

Choose α_2 such that

$$\alpha_2\Big|B_{R/4}(x_0)\times\Big(t_0-\frac{R^2}{2}, t_0+R^2\Big)\Big| = 2^{-1}\cdot[CR^{-2\mu}]^{-1/(\mu-1)}2^{-2\mu/(\mu-1)^2}. \tag{4.31}$$

Clearly, α_2 is a positive constant depending only on γ_1 and N. Taking $\alpha_1 = \alpha_2$ and $q_2 = q_1(\alpha_2)$ and using Lemma 3.2 with (4.29)–(4.31), we conclude that (4.30) holds for

$$R \in (0, \min\{\alpha_2 C_0^{-1}2^{-(2q_2+1)}, l_1(q_2)\}).$$

Choosing

$$R_0 = \min\Big\{2^{-1}\alpha_2C_0^{-1}2^{-(2q_2+1)},\quad \theta\Big(2^{q_2}\vartheta\Big(\frac{\lambda_0}{4}\Big)\Big)\Big\}$$

and using (4.24), we have

$$v(x,t) \ge m_v + \frac{l_1(q_2)}{2}$$

for $(x,t)\in B_{R/8}(x_0)\times\big(t_0-\frac{R^2}{2}, t_0+R^2\big)$ and $R\in(0,R_0)$. Thus the proof is completed.

Theorem 4.1 *Assume that u is a solution of* (1.1)–(1.2) *in Q_T. If* (4.9) *holds then we have*

$$u(x,t) \ge m + \delta_1\omega$$

for $(x,t)\in B_{R/8}(x_0)\times(t_0-\frac{R^2}{2}, t_0+R^2)$ and $R\in(0,R_0)$, where

$$\delta_1 = \min\{2^{-1},\ (2\Lambda_1)^{-1}\vartheta(2^{-1}\theta(2^{-q_2}\lambda))\}. \tag{4.32}$$

Proof If (4.19) holds then, by Lemma 3.3, we have

$$v(x,t) \ge m_v + \frac{l_1(q_2)}{2}$$

for $(x,t)\in B_{R/8}(x_0)\times\big(t_0-\frac{R^2}{2}, t_0+R^2\big)$. By (4.6), (4.24) with (4.32), we get

$$u(x,t) \ge \varphi\Big(m_v+\frac{l_1(q_2)}{2}\Big) \ge \varphi\Big(m_v+\frac{\theta(2^{-q_2}\lambda)}{2}\Big) \ge m+\vartheta(2^{-1}\theta(2^{-q_2}\lambda)) \ge m+\delta_1\omega \tag{4.33}$$

for $(x,t)\in B_{R/8}(x_0)\times\big(t_0-\frac{R^2}{2}, t_0+R^2\big)$ and $R\in(0,R_0)$. In addition, if (4.19) is not true then

$$v(x,t) \ge m + \frac{l}{2} \tag{4.34}$$

for $(x,t)\in B_{R/2}(x_0)\times\big(t_0-\frac{R^2}{2}, t_0+R^2\big)$. Combining (4.33)–(4.34) with (4.17), we obtain the conclusion of Theorem 4.1. Thus the proof is completed.

5 Some Estimates on Super Bound

It is well known that, if (4.9) is not true,

$$\Big|\Big\{(x,t)\in Q_R^\tau : v(x,t) < m_v + \frac{\omega_v}{2}\Big\}\Big| > \alpha|Q_R^\tau|,$$

which implies that

$$\Big|\Big\{(x,t)\in Q_R^\tau : v(x,t) > M_v - \frac{\omega_v}{2}\Big\}\Big| < (1-\alpha)|Q_R^\tau| \tag{5.1}$$

for $\tau = t_0 - R^2$, where $\alpha\in(0,1)$ is defined by Lemma 4.1.

Lemma 5.1 *Assume that* (5.1) *holds. There exists a time*

$$t^*\in\Big(\tau - R^2, \tau - \frac{\alpha R^2}{2}\Big) \tag{5.2}$$

such that

$$\Big|\Big\{x\in B_R(x_0) : v(x,t^*) > M_v - \frac{\omega_v}{2}\Big\}\Big| \le \frac{1-\alpha}{1-\frac{\alpha}{2}}|B_R(x_0)|. \tag{5.3}$$

Proof If (5.3) is not true then

$$\Big|\Big\{x\in B_R(x_0) : v(x,t) > M_v - \frac{\omega_v}{2}\Big\}\Big| > \frac{1-\alpha}{1-\frac{\alpha}{2}}|B_R(x_0)| \tag{5.4}$$

for $t\in\big(\tau - R^2, \tau - \frac{\alpha R^2}{2}\big)$. It follows from (5.4) that

$$\begin{aligned}\Big|\Big\{(x,t)\in Q_R^\tau : v(x,t) > M_v - \frac{\omega_v}{2}\Big\}\Big| &\ge \int_{\tau-R^2}^{\tau-\alpha R^2/2}\Big|\Big\{x\in B_R(x_0) : v(x,t) > M_v - \frac{\omega_v}{2}\Big\}\Big|dt\\ &> (1-\alpha)|Q_R^\tau|,\end{aligned}$$

which contradicts (5.1). Thus the proof is completed.

Denote

$$L = M - \varphi\Big(M_v - \frac{\omega_v}{2}\Big),\quad Q_1 = B_R(x_0)\times(t_0 - 2R^2, t_0 + R^2). \tag{5.5}$$

Lemma 5.2 *Assume that* (5.1) *holds and*

$$H^+ = \|(u-(M-L))^+\|_{L^\infty(Q_1)} > \frac{L}{2}. \tag{5.6}$$

Then there exists a positive integer p_1 independent of R such that

$$\Big|\Big\{x\in B_R(x_0) : u(x,t)\ge M - \frac{L}{2^{p_1}}\Big\}\Big| < \Big[1-\Big(\frac{\alpha}{2}\Big)^2\Big]|B_R(x_0)|$$

for $t\in(t_0-R^2, t_0+R^2)$ and $R\in(0,R_1)$, where R_1 is a positive constant independent of R.

Proof We define a nonnegative function $\xi\in C_0^\infty(B_R(x_0))$ such that

$$\xi(x) = 1,\quad \forall\, x\in B_{R-\sigma R}(x_0);\quad |\nabla\xi|\le C(\sigma R)^{-1},$$

where $\sigma \in (0,1)$ is arbitrary. From Theorem 3.1, it follows that

$$\begin{aligned}
&\sup_{t^*\le t\le t_0+R^2}\int_{B_R(x_0)}\psi^2\Big(H^+,(u(x,t)-(M-L))^+,\frac{L}{2^n}\Big)\xi^2dx\\
&\le \gamma_1\int_{t^*}^{t_0+R^2}\int_{B_R(x_0)}\psi\Big(H^+,(u-(M-L))^+,\frac{L}{2^n}\Big)|\nabla\xi|^2dxdt\\
&\quad+\int_{B_R(x_0)}\psi^2\Big(H^+,(u(x,t)-(M-L))^+,\frac{L}{2^n}\Big)\xi^2(x)dx. \qquad (5.7)
\end{aligned}$$

First, we have

$$\psi\Big(H^+,(u(x,t)-(M-L))^+,\frac{L}{2^n}\Big)\le n\ln 2. \qquad (5.8)$$

Applying Lemma 5.1 and using (5.5), we have

$$\int_{B_R(x_0)}\psi^2\Big(H^+,(u(x,t)-(M-L))^+,\frac{L}{2^n}\Big)\xi^2(x)dx\le n^2\ln^2 2\Big(\frac{1-\alpha}{1-\frac{\alpha}{2}}\Big)|B_R(x_0)|. \qquad (5.9)$$

Using (5.2), we get $0<t_0+R^2-t\le 3R^2$. It follows from (5.8) that

$$\int_{t}^{t_0+R^2}\int_{B_R(x_0)}\psi\Big(H^+,(u-(M-L))^+,\frac{L}{2^n}\Big)|\nabla\xi|^2dxdt\le C\sigma^{-2}n|B_R(x_0)|. \qquad (5.10)$$

Using (5.10) and combining (5.7), (5.9), we have

$$\begin{aligned}
&\sup_{t^*\le t\le t_0+R^2}\int_{B_R(x_0)}\psi^2\Big(H^+,(u(x,t)-(M-L))^+,\frac{L}{2^n}\Big)\xi^2dx\\
&\le\Big[n^2\ln^2 2\Big(\frac{1-\alpha}{1-\frac{\alpha}{2}}\Big)+C\sigma^{-2}n\Big]|B_R(x_0)|. \qquad (5.11)
\end{aligned}$$

We estimate the left hand side of (5.11) below by integrating over the smaller set

$$B_{R-\sigma R}\cap\Big\{x\in B_R(x_0):u(x,t)>M-\frac{L}{2^n}\Big\}.$$

Then on such a set, by (5.6), we obtain

$$\psi\Big(H^+,(u(x,t)-(M-L))^+,\frac{L}{2^n}\Big)\ge\ln\Big(\frac{\frac{L}{2}}{\frac{L}{2^{n-1}}}\Big)=(n-2)\ln 2. \qquad (5.12)$$

Combining (5.12) with (5.11), we conclude that

$$\Big|\Big\{x\in B_{R-\sigma R}:u(x,t)>M-\frac{L}{2^n}\Big\}\Big|\le\Big[\Big(\frac{n}{n-2}\Big)^2\Big(\frac{1-\alpha}{1-\frac{\alpha}{2}}\Big)+C\sigma^{-2}n^{-1}\Big]|B_R(x_0)| \qquad (5.13)$$

for $t\in(t,t_0+R^2)$.

On the other hand,

$$\begin{aligned}
\Big|\Big\{x\in B_R(x_0):u(x,t)>M-\frac{L}{2^n}\Big\}\Big|&\le\Big|\Big\{x\in B_{R-\sigma R}:u(x,t)>M-\frac{L}{2^n}\Big\}\Big|\\
&\quad+|B_R(x_0)\setminus B_{R-\sigma R}|. \qquad (5.14)
\end{aligned}$$

By (5.13) and (5.14), we obtain

$$\Big|\Big\{x \in B_R(x_0) : u(x,t) > M - \frac{L}{2^n}\Big\}\Big| \le \Big[\Big(\frac{n}{n-2}\Big)^2\Big(\frac{1-\alpha}{1-\frac{\alpha}{2}}\Big) + C\sigma^{-2}n^{-1} + C\sigma\Big]|B_R(x_0)| \quad (5.15)$$

for $t \in (t, t_0 + R^2)$. Choose σ so small that

$$C\sigma \le \frac{\alpha^2}{16}$$

and $p_1 = n$ so large that

$$\Big(\frac{n}{n-2}\Big)^2 \le \Big(1 - \frac{\alpha}{2}\Big)(1+\alpha); \quad C\sigma^{-2}n^{-1} \le \frac{\alpha^2}{8}.$$

Then for such a choice of $p_1 = n$

$$\Big|\Big\{x \in B_R(x_0) : u(x,t) > M - \frac{L}{2^{p_1}}\Big\}\Big| \le \Big[1 - \Big(\frac{\alpha}{2}\Big)^2\Big]|B_R(x_0)|$$

for $t \in (t, t_0 + R^2)$ with $t \in (\tau - R^2, \tau - \frac{\alpha R^2}{2})$. Thus the proof is completed.

It is well known that

$$u(x,t) > M - \frac{L}{2^{p_1}} \Longleftrightarrow v(x,t) > M_v - L_1, \quad (5.16)$$

where

$$L_1 = M_v - \phi\Big(M - \frac{L}{2^{p_1}}\Big). \quad (5.17)$$

By (5.5) with (4.6) and (4.7), we have

$$L = \varphi\Big(\frac{\omega_v}{2} + \Big(M_v - \frac{\omega_v}{2}\Big)\Big) - \varphi\Big(M_v - \frac{\omega_v}{2}\Big) \ge \vartheta\Big(\frac{\lambda_0}{2}\Big). \quad (5.18)$$

Using (4.6) and (5.18), we get

$$L_1 \ge \phi(M) - \phi\Big(M - 2^{-p_1}\vartheta\Big(\frac{\lambda_0}{2}\Big)\Big) \ge \theta\Big[2^{-p_1}\vartheta\Big(\frac{\lambda_0}{2}\Big)\Big]. \quad (5.19)$$

Lemma 5.3 *Assume that* (5.1) *and* (5.6) *hold. For any $\beta \in (0,1)$, there exists a positive integer $p \equiv p(\beta, N, \alpha)$ depending only on N, α and β such that*

$$\Big|\Big\{x \in B_R(x_0) : v(x,t) > M_v - \frac{L_1}{2^p}\Big\}\Big| \le \beta|B_R(x_0)|$$

for $t \in (t_0 - R^2, t_0 + R^2)$ and $R \in (0, 2^{-p}R_1)$, where α is defined by Lemma 5.1.

Proof We choose a cut-off function $\xi \in C_0^\infty(B_{2R}(x_0))$ such that $\xi \equiv 1$ on $B_R(x_0)$ with $0 \le \xi \le 1$, $|\nabla\xi| \le CR^{-1}$. Choosing

$$l = M_v - \frac{L_1}{2^n}, \quad k = M_v - \frac{L_1}{2^{n-1}}$$

and using Lemma 2.4, we compute

$$\Big(\frac{L_1}{2^n}\Big)|A_{l,R}(t)| \le \frac{CR^{N+1}}{|B_R(x_0) \setminus A_{k,R}(t)|}\int_{A_{k,R}(t)\setminus A_{l,R}(t)} |\nabla v(x,t)|dx, \quad (5.20)$$

where $A_{k,R}(t) \equiv \{x \in B_R(x_0) : v(x,t) > k\}$. In addition, using Lemma 5.2 and (5.20), we compute

$$\begin{aligned}\Big(\frac{L_1}{2^n}\Big)|A_{l,R}(t)| &\le CR|A_{k,R}(t)\setminus A_{l,R}(t)|^{1/2}\Big(\int_{A_{k,R}(t)\setminus A_{l,R}(t)}|\nabla v(x,t)|^2dx\Big)^{1/2}\\ &\le CR|A_{k,R}(t)\setminus A_{l,R}(t)|^{1/2}\Big(\int_{B_{2R}(x_0)}|\nabla[\xi(v(x,t)-k)^+]|^2dx\Big)^{1/2}\\ &\le CR|A_{k,R}(t)\setminus A_{l,R}(t)|^{1/2}\cdot\Big\{\int_{B_{2R}(x_0)}[(v(x,t)-k)^+]^2|\nabla\xi|^2dx\Big\}^{1/2}\\ &\le C\Big(\frac{L_1}{2^n}\Big)|A_{k,R}(t)\setminus A_{l,R}(t)|^{1/2}|B_R(x_0)|^{1/2}\\ &\le C\Big(\frac{L_1}{2^n}\Big)|A_{k,R}(t)\setminus A_{l,R}(t)|^{1/2}|B_R(x_0)|^{1/2}\end{aligned}$$

for $n = 1, 2, \cdots, p$. This implies that

$$|A_{l,R}(t)| \le C|A_{k,R}(t)\setminus A_{l,R}(t)|^{1/2}|B_R(x_0)|^{1/2} \tag{5.21}$$

for all $n = 1, 2, \cdots, p$ and all $R \in (0, 2^{-p}R_1)$, where C is a positive constant depending only on N, γ_4 and α.

Denote

$$A_n(t) = \Big\{x \in B_R(x_0) : v(x,t) \ge M_v - \frac{L_1}{2^n}\Big\}$$

for $n = 1, 2, \cdots, p$. Then, by (5.21), we have

$$|A_n(t)|^2 \le C|B_R(x_0)|(|A_{n-1}(t)| - |A_n(t)|) \tag{5.22}$$

for $t \in (t_0 - R^2, t_0 + R^2)$, where C is a positive constant depending only on N, α and γ_4. We add (5.22) for $n = 1, 2, \cdots, p$. The right hand side can be majorized with a convergent series and therefore we obtain

$$(p-1)|A_p(t)|^2 \le C|B_R(x_0)|^2$$

and

$$|A_p(t)| \le C^{1/2}(p-1)^{-1/2}|B_R(x_0)| \tag{5.23}$$

for all $R \in (0, 2^{-p}R_1)$. We take $p \equiv p(\beta)$ so large that $C^{1/2}(p-1)^{-1/2} < \beta$. Thus, by (5.23), the proof is completed.

Lemma 5.4 *Assume that* (5.1) *and* (5.6) *hold. There exists a positive number* β_0 *independent of* R *and* p *such that, if*

$$\Big|\Big\{x \in B_R(x_0) : v(x,t) > M_v - \frac{L_1}{2^p}\Big\}\Big| \le \beta_0|B_R(x_0)|$$

for $t \in (t_0 - R^2, t_0 + R^2)$, *we have*

$$v(x,t) \le M_v - \frac{L_1}{2^{p+1}} \tag{5.24}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$.

Proof Let

$$R_n = \frac{R}{2} + \frac{R}{2^n}, \quad n = 1, 2, \cdots$$

and $\xi_n \in C_0^\infty(B_{R_n}(x_0))$ such that

$$\begin{cases} \xi_n(x) = 1, \quad \forall\, x \in B_{R_{n+1}}(x_0), \\ 0 \le \xi_n(x) \le 1, \quad |\nabla \xi_n(x)| \le C2^n R^{-1}, \quad \forall x \in B_{R_n}(x_0), \end{cases}$$

$$k_n = M_v - \frac{L_1}{2^{p+1}} - \frac{L_1}{2^{p+n}}, \quad n = 1, 2, \cdots .$$

From (3.9)–(3.11) with Lemma 3.3, it follows that

$$\int_{B_{R_n}(x_0)} |\nabla[\xi_n(v(x,t) - k_n)^+]|^2 dx \le 36 \int_{B_{R_n}(x_0)} |\nabla \xi_n|^2 [(v(x,t) - k_n)^+]^2 dx$$

for $t \in (t_0 - R^2, t_0 + R^2)$. Using the Sobolev imbedding inequality, we get

$$\int_{B_{R_n}(x_0)} [\xi_n(v(x,t) - k_n)^+]^{2\nu} dx \le C\Big\{ \int_{B_{R_n}(x_0)} |\nabla \xi_n|^2 [(v(x,t) - k_n)^+]^2 dx \Big\}^\nu \tag{5.25}$$

for $t \in (t_0 - R^2, t_0 + R^2)$, where $\nu = \frac{N}{N-2}$. Using (5.25), we compute

$$(k_{n+1} - k_n)^{2\nu} |A_{n+1}(t)| \le C\Big[\Big(\frac{L_1}{2^{p+1}}\Big)^2\Big]^\nu 2^{2\nu n} R^{-2\nu} |A_n(t)|^\nu,$$

where $A_n(t) = \{x \in B_{R_n}(x_0) : v(x,t) > k_n\}$. This implies that

$$|A_{n+1}(t)| \le C 2^{2\nu n} R^{-2\nu} |A_n(t)|^\nu$$

for $t \in (t_0 - R^2, t_0 + R^2)$ and $R \in (0, 2^{-2p} R_1)$. Applying Lemma 2.1, we conclude that if

$$|A_1(t)| < (CR^{-2\nu})^{-1/(\nu-1)} 2^{-2\nu/(\nu-1)^2} \tag{5.26}$$

then

$$\lim_{n\to+\infty} |A_n(t)| = 0. \tag{5.27}$$

Choose $\beta_0 > 0$ such that

$$\beta_0 |B_R(x_0)| = 2^{-1} \cdot (CR^{-2\nu})^{-1/(\nu-1)} 2^{-2\nu/(\nu-1)^2}. \tag{5.28}$$

Therefore, by (5.26)–(5.28), we have that, if $|A_1(t)| \le \beta_0 |B_R(x_0)|$,

$$v(x,t) \le M_v - \frac{L_1}{2^{p+1}}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$. Thus the proof is completed.

Lemma 5.5 *Assume that* (5.1) *and* (5.6) *hold. Then we have*

$$v(x,t) \le M_v - \frac{L_1}{2^{p_0+1}}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$ *and* $R \in (0, R_2)$, *where* $p_0 = p(\beta_0, N, \alpha)$ *and* $R_2 = 2^{-2p_0} R_1$.

Proof For $p_0 = p(\beta_0, N, \alpha)$ and $R_2 = 2^{-2p_0}R_1$, by Lemma 5.3 and Lemma 5.4, we conclude that

$$v(x,t) \le M_v - \frac{L_1}{2^{p_0+1}}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$ and $R \in (0, R_2)$. Thus the proof is completed.

Theorem 5.1 *Assume that u is a solution of* (1.1)–(1.2) *in Q_T. If* (5.1) *holds then we have*

$$u(x,t) \le M - \delta_2\omega \tag{5.29}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$, where

$$\delta_2 = \min\Big\{(4\Lambda)^{-1}\vartheta\Big(\frac{\lambda_0}{2}\Big),\ (2\Lambda)^{-1}\vartheta(2^{-p_0-1}\theta(2^{-p_1}\vartheta(\lambda_0)))\Big\}. \tag{5.30}$$

Proof We consider only two cases:

Case 1 (5.6) is true.

By Lemma 4.5, we have

$$v(x,t) \le M_v - \frac{L_1}{2^{p_0+1}} \tag{5.31}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$ and $R \in (0, R_2)$. Using (4.6), (5.19) and (5.32), we compute

$$\begin{aligned} u(x,t) &\le M - [\varphi(M_v) - \varphi(M_v - 2^{-p_0-1}L_1)] \\ &\le M - [\varphi(M_v) - \varphi(M_v - 2^{-p_0-1}\theta(2^{-p_1}\vartheta(2^{-1}\lambda_0)))] \\ &\le M - \vartheta(2^{-p_0-1}\theta(2^{-p_1}\vartheta(2^{-1}\lambda_0))) \\ &\le M - [(2\Lambda_1)^{-1}\vartheta(2^{-p_0-1}\theta(2^{-p_1}\vartheta(2^{-1}\lambda_0)))]\omega \end{aligned} \tag{5.32}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$.

Case 2 (5.6) is not true.

We have

$$u(x,t) \le M - \frac{L}{2}$$

for $(x,t) \in B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$. Using (4.6)–(4.8) and (5.5), we have

$$u(x,t) \le M - 2^{-1}\vartheta\Big(\frac{\lambda_0}{2}\Big) \le M - \Big[(4\Lambda_1)^{-1}\vartheta\Big(\frac{\lambda_0}{2}\Big)\Big]\omega \tag{5.33}$$

for $(x,t) \in B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$. Combining (5.32)–(5.33) with (5.30), we obtain the conclusion of Theorem 5.1. Thus the proof is completed.

6 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1.

In fact, if the conclusion of Theorem 1.1 is not true then there exist a point $(x_0, t_0) \in Q_T$ and a positive number λ such that

$$\lim_{R\to 0^+} \omega(R) = \lambda, \tag{6.1}$$

where

$$\omega(R) = M - m, \quad M = \operatorname{ess}\sup_{Q(R)} u, \quad m = \operatorname{ess}\inf_{Q(R)} u,$$

and $Q(R) = B_R(x_0) \times (t_0 - (2R)^2, t_0 + (2R)^2)$.

By (6.1), there exists a positive number $R_3 \in (0,1)$ such that $Q(R_3) \subset \overline{Q(R_3)} \subset Q_T$ and

$$\omega(R) \geq \lambda \tag{6.2}$$

for all $R \in (0, R_3]$. By Theorem 4.1, if (4.9) is true then

$$u(x,t) \geq m + \delta_1 \omega \tag{6.3}$$

for $(x,t) \in B_{R/2}(x_0) \times (t_0 - R^2, t_0 + R^2)$ and $R \in (0, R_0)$, where δ_1 and R_0 are defined in Theorem 3.1. In addition, if (4.9) is not true then (6.1) holds. By Theorem 5.1, we have

$$u(x,t) \leq M - \delta_2 \omega \tag{6.4}$$

for $(x,t) \in B_{R/4}(x_0) \times (t_0 - R^2, t_0 + R^2)$ and $R \in (0, R_2)$, where δ_2 and R_2 are defined in Theorem 5.1. Combining (6.4) with (6.3), we get

$$\omega\Big(\frac{R}{8}\Big) \leq \delta \omega(R)$$

for all $R \in (0, R_4]$, where $\delta = \max\{\frac{1}{2}, 1 - \delta_1, 1 - \delta_2\} \in (0,1)$ and $R_4 = \min\{R_0, R_2, R_3\}$. Choosing $R = \frac{R_4}{8^{n-1}}$, we obtain

$$\omega\Big(\frac{R_4}{8^n}\Big) \leq \delta \omega\Big(\frac{R_4}{8^{n-1}}\Big)$$

for $n = 1, 2, \cdots$. Therefore, we conclude that

$$\omega\Big(\frac{R_4}{8^n}\Big) \leq \delta^n \omega(R_4) \tag{6.5}$$

for $n = 1, 2, \cdots$. By (6.5) and (6.1), we have

$$\omega(R_4) \geq \lambda \delta^{-n}$$

for $n = 1, 2, \cdots$. Letting $n \to +\infty$, we obtain

$$\omega(R_4) \geq \lim_{n \to +\infty} \lambda \delta^{-n} = +\infty,$$

which contradicts $u \in L^\infty(Q_T)$. Thus the proof is completed.

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