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Stochastic H_2/H_∞ Control with Random Coefficients

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Abstract This paper is concerned with the mixed H_2/H_∞ control for stochastic systems with random coefficients, which is actually a control combining the H_2 optimization with the H_∞ robust performance as the name of H_2/H_∞ reveals. Based on the classical theory of linear-quadratic (LQ, for short) optimal control, the sufficient and necessary conditions for the existence and uniqueness of the solution to the indefinite backward stochastic Riccati equation (BSRE, for short) associated with H_∞ robustness are derived. Then the sufficient and necessary conditions for the existence of the H_2/H_∞ control are given utilizing a pair of coupled stochastic Riccati equations.

Keywords Stochastic H_∞ control, Stochastic H_2/H_∞ control, Linear quadratic (LQ) optimal control, Indefinite backward stochastic Riccati equation
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1 Introduction

H_∞ control is an important robust control design for eliminating the effect of disturbance efficiently, and has been widely employed to deal with the robust performance control problem with uncertain disturbance. There may be more than one solution to H_∞ control problem with desired robustness. In engineering practice, the ideal control is not only to attenuate external disturbance, but also to minimize a desired control performance when the worst case disturbance is imposed, which naturally leads to the mixed H_2/H_∞ control, see [1, 8, 13, 17] for deterministic systems.

In recent years, the study on H_∞ and mixed H_2/H_∞ controller designs for stochastic Itô systems has attracted great attentions, see [3–6, 19–20] and the references therein. In [6], H_∞ control for general linear stochastic Itô systems was discussed very extensively. Moreover, a very useful lemma called the "stochastic bounded real lemma (SBRL, for short)" was given therein using linear matrix inequalities. Later, Zhang and Chen [19] investigated the nonlinear H_∞ control for stochastic affine systems, where a nonlinear SBRL was obtained. As for the H_2/H_∞ control, Chen and Zhang [3] studied the mixed H_2/H_∞ control with state-dependent noise, and then a further discussion on the case of (x, u, v)-dependent noise was given in Zhang et al [20], both of which extended the deterministic H_2/H_∞ control results of Limebeer et al [13] to the stochastic setting.

It should be pointed out that the above mentioned works are concerned only with the stochastic system with deterministic coefficients. To the author's best knowledge, the case of

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random coefficients, except Zhang et al [21], seems to be unsolved. The objective of this paper is to develop an H_2/H_∞ control theory for stochastic Itô systems with random coefficients. A motivation for random coefficients comes from the observation that in many situations, system coefficients are estimated using historical data (e.g. the past and present system information), and not every element in those matrices can be directly observed. Moreover, there exist parameter perturbations in those obtained data due to measurement errors, identification errors and the variations of environment and operating conditions, etc. Therefore, it is more reasonable to assume the system parameters to be random processes rather than deterministic functionals.

Consider the following linear system governed by Itô stochastic differential equations:

$$\begin{cases} \mathrm{d}x_t = [A_t x_t + B_t^1 v_t + B_t^2 u_t]\mathrm{d}t + [A_t^0 x_t + B_t^0 v_t]\mathrm{d}W_t, \quad t \in [0,T], \\ x_0 = x^0, \\ z_t = \begin{pmatrix} C_t x_t \\ D_t^1 v_t \\ D_t^2 u_t \end{pmatrix}, \end{cases} \tag{1.1}$$

where W_t is a one-dimensional standard Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{F}_t, 0 \leqslant t \leqslant T\}$ is the augmented natural filtration of the standard Brownian motion W. u is viewed as the control input, v as an external disturbance and z as the controlled output, respectively. We also assume that all the coefficients are $\mathcal{F}_t$-adapted bounded matrix-valued processes with suitable dimensions.

We discuss a Nash game approach to the state feedback H_2/H_∞ control problem for the system (1.1). Given a disturbance attenuation level $\gamma > 0$ satisfying $\gamma^2 I - (D_t^1)' D_t^1 \geqslant \epsilon I$ for a sufficiently small constant $\epsilon > 0$, we consider two cost functionals as follows:

$$\begin{cases} J_1(u,v) = E\displaystyle\int_0^T [\gamma^2 |v_t|^2 - |z_t|^2]\mathrm{d}t, \\ J_2(u,v) = E\displaystyle\int_0^T |z_t|^2 \mathrm{d}t. \end{cases} \tag{1.2}$$

According to the analysis in the next section, the H_2/H_∞ control problem can be converted into finding the Nash equilibria point (u^*, v^*) such that

$$J_1(u^*, v^*) \leqslant J_1(u^*, v), \quad J_2(u^*, v^*) \leqslant J_2(u, v^*).$$

To this end, we approach the H_2/H_∞ control problem as a linear quadratic (LQ, for short) optimal control problem and obtain the solution by studying the associated Riccati equations.

It should be noted that minimizing $J_2(u, v^*)$ with respect to u under the constraint (1.1) is a standard LQ problem, which means that the state weighting matrices are positive semi-definite and the control weighting matrix is positive definite in the cost. According to Peng [16] or Tang [18], the existence and uniqueness of the solution to the associated Riccati equation can be obtained, by which the optimal control can be constructed explicitly as a linear state feedback. While minimizing $J_1(u^*, v)$ with respect to v under the constraint (1.1) is an indefinite LQ problem whose cost involves a positive definite control weighting matrix and a negative semi-definite state weighting matrix. Such an indefinite LQ problem leads to an indefinite backward stochastic Riccati equation (BSRE, for short) in the following form:

$$\begin{cases} \mathrm{d}P_t = -[A_t'P_t + P_tA_t + (A_t^0)'P_tA_t^0 + (A_t^0)'L_t + L_tA_t^0 - C_t'C_t \\ \qquad\qquad -(P_tB_t^1 + (A_t^0)'P_tB_t^0 + L_tB_t^0)(\gamma^2 I - (D_t^1)'D_t^1 + (B_t^0)'P_t(B_t^0))^{-1} \\ \qquad\qquad \cdot((B_t^1)'P_t + (B_t^0)'P_tA_t^0 + (B_t^0)'L_t)]\mathrm{d}t + L_t\mathrm{d}W_t, \\ P_T = 0, \\ \gamma^2 I - (D_t^1)'D_t^1 + (B_t^0)'P_t(B_t^0) > 0. \end{cases} \tag{1.3}$$

The first two equalities constitute a nonlinear backward stochastic differential equation (BSDE for short) and the last inequality serves as a constraint. The unknown of the BSRE is a pair of matrix-valued $\mathcal{F}_t$-adapted stochastic processes (P, L).

The general form of the indefinite BSRE is a matrix-valued BSDE with high nonlinearity and possible singularity. Several recent papers were devoted to the stochastic LQ problem with random coefficients and the associated indefinite BSRE. Chen and Yong [2] proved the local existence and uniqueness results with the additional regularities of the coefficients (i.e., the conditions on the Malliavin derivatives of the coefficients). Kohlmann and Tang [9–12] used the typical approximation scheme to construct a solution when the control weighting matrix was possibly singular. A scalar-valued indefinite BSRE, which arose from a mean-variance portfolio selection problem for a market with random coefficients, was resolved in [14] for a complete market and in [15] for an incomplete market. In [7], the existence and uniqueness of solutions to certain special indefinite BSREs were established. However, our BSREs in this paper go beyond their discussions.

Noting that the drift of (1.3) is quadratic in L and has a singularity in P, we are not able to prove the existence of the solution in a direct way. Instead we will first treat the case where only $B^0 = 0$. In this case, the generator of BSRE (1.3) depends on the second unknown variable L in a linear form. We prove the existence and uniqueness of the solution by using Bellman's quasi-linearization and a monotone convergence result of symmetric matrices, which is inspired by Peng [16]. To some extent, this result extends the SBRL of Hinrichsen and Pritchard [6] to the counterpart for stochastic systems with random coefficients. Furthermore, this crucial result enables us to obtain sufficient and necessary conditions for the existence of finite horizon mixed H_2/H_∞ control for stochastic systems with random coefficients, which is an extension in comparison with the result of Chen and Zhang [3].

The rest of this paper is organized as follows. In Section 2, some preliminaries about stochastic H_∞ and H_2/H_∞ control are stated. The relationship between the H_∞ robustness and the solvability of indefinite BSRE is presented in Section 3, which plays an essential role in this paper. Based on the results presented in Section 3, we then derive the necessary and sufficient conditions for the existence of stochastic H_2/H_∞ control in Section 4. Concluding remarks are presented in Section 5.

Finally we end this section by introducing some notations.

M' is the transpose of the vector or matrix M. $|M|$ denotes the square root of the summarized squares of all the components of the vector or matrix M. $\langle M_1, M_2\rangle$ is the inner product of two vectors M_1 and M_2. M^{-1} is the inverse of a nonsingular square matrix M. $\mathbb{R}^m$ stands for the m-dimensional Euclidean space. $\mathbb{S}^n$ is the space of all $n \times n$ symmetric matrices. $E^{\mathcal{F}_t}x$ denotes taking conditional expectation with respect to filtration $\mathcal{F}_t$ for random variable x. $C([0,T]; H)$ is the space of H-valued continuous functions on $[0,T]$, endowed with the maximum norm for a given Hilbert space H. $\mathcal{L}^2_{\mathcal{F}}(\tau,T;H)$ is the space of H-valued $\{\mathcal{F}_t, \tau \le t \le T\}$-adapted square-integrable stochastic processes f on $[\tau,T]$, endowed with the norm $\|f\| = [E\int_\tau^T |f_t|^2 \mathrm{d}t]^{\frac{1}{2}}$ for a given Euclidean space H. $\mathcal{L}^\infty_{\mathcal{F}}(0,T;H)$ is the space of H-valued $\mathcal{F}_t$-adapted, essentially bounded stochastic processes f on $[0,T]$, endowed with the norm $\|f\| = \operatorname*{esssup}_{t,\omega}|f(t)|$ for a given Euclidean space H. $L^2(\Omega,\mathcal{F},P;H)$ is the space of H-valued norm-square-integrable random variables on the probability space $(\Omega,\mathcal{F},P)$. $L^\infty(\Omega,\mathcal{F},P;C([0,T];\mathbb{R}^n))$ is the space of $C([0,T];\mathbb{R}^n)$-valued, essentially maximum-norm-bounded random variables f on the probability space $(\Omega,\mathcal{F},P)$, endowed with the norm $\|f\| = \operatorname*{esssup}_{\omega\in\Omega} \max_{0\leqslant t\leqslant T} |f(t,\omega)|$.

2 Preliminaries

In engineering practice, H_2/H_∞ control is employed to attenuate the effect of external disturbance and obtain the desired optimality performance. Let us consider the flight control system, and take the longitudinal movement model of a small aircraft as an example.

Example 2.1 The flight movement system is always a nonlinear system with uncertainty and sensitivity to external perturbation. By linearization and introducing Itô's stochastic differential equations, the longitudinal movement system can be described as (1.1), where W_t denotes the unmodeled uncertainty such as the impact from the transverse movement, the moment of force from propulsion or other neglected small perturbations. Let $x = (\varrho,\alpha,q,\vartheta)'$, z and $u = (\delta_e,\delta_T)'$ represent deviations from desired fixed values of the state, the output, and the control, respectively. Here $\varrho,\alpha,q,\vartheta,\delta_e,\delta_T$ represent velocity, angle of attack, pitch rate, pitch attitude, deflection angle of elevon and deflection of rudder, respectively. Denote by v the wind gusts and atmospheric turbulence, which adversely affects the signal output z (whose ideal value is represented by 0). The coefficient matrices A, A^0, B^0, B^1, B^2 are always assumed to be constant in the conventional model for simplicity. In fact, not every element in those matrices can be directly observed and there exists parameter perturbation. For instance, the parameters related to the aerodynamics may have a significant change as the plane flies at different altitudes. Therefore, it is rational to assume the matrices to be random processes. For such a system, the H_2/H_∞ control design is necessary to ensure flight safety and stability, and meanwhile to minimize the energy input like aircraft fuel and the loss of aircraft components when the plane suffers the worst case of uncertainty and atmospheric disturbances.

Consider the following stochastic linear system:

$$\begin{cases} \mathrm{d}x_t = [A_t x_t + B^1_t v_t + B^2_t u_t]\mathrm{d}t + A^0_t x_t \mathrm{d}W_t, \quad t \in [0,T], \\ x_0 = x^0, \\ z_t = \begin{pmatrix} C_t x_t \\ D^1_t v_t \\ D^2_t u_t \end{pmatrix}, \end{cases} \tag{2.1}$$

where we assume $(D^2)'D^2 = I$. W_t is a one-dimensional standard Brownian motion defined on a given probability space $(\Omega,\mathcal{F},P)$ and $\{\mathcal{F}_t, 0 \leqslant t \leqslant T\}$ is the augmented natural filtration of W. All the coefficients are assumed to be bounded matrix-valued processes with suitable dimensions. For all $0 < T < \infty$ and $(u,v,x^0) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_u}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_v}) \times \mathbb{R}^n$, there exists a unique solution $x_t = x(t, u, v, x^0) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ to the state equation of system (2.1). In applications, such models are often obtained by linearization. We view v as an external disturbance which adversely affects the to-be-controlled output $z \in \mathbb{R}^{n_z}$ (whose desired value is represented by 0). The disturbing effect is to be ameliorated via control input $u \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_u})$. The effect of the disturbance on the to-be-controlled output z of system (2.1) is then described by the perturbation operator $\mathcal{L}_{cl}$: $v \mapsto z$ which (for the zero initial state) maps disturbance signals v into the corresponding output signals z of the closed loop system. The size of this linear operator is measured by the induced norm. The larger this norm is, the larger the effect of the unknown disturbance v on the to-be-controlled output z in the worst case is. Then the H_∞ control problem is to determine whether or not for each $\gamma > 0$ there exists a control input u^* achieving $\|\mathcal{L}_{u^*}\| < \gamma$. Obviously, there may be more than one solution satisfying the required condition. In engineering practice, as pointed out in Example 2.1, we want the control not only to guarantee robust stability, but also to optimize system performance. That is, we wish to:

(1) Find a feedback control $u^* \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_u})$ such that the norm of the perturbation operator of system (2.1) is less than some given number $\gamma > 0$, i.e., $\|\mathcal{L}_{u^*}\| < \gamma$.

(2) We require the control u^* to minimize the output energy z when the worst case disturbance $v^* \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_v})$ is applied to the system (2.1).

As we will show, this problem may be formulated as an LQ nonzero sum game. The two cost functions we use are defined as (1.2). The first is associated with an H_∞ robustness, while the second reflects an H_2 optimality requirement. The aim is to find equilibrium strategies u^* and v^* defined by

$$\begin{aligned} J_1(u^*, v^*) &\leqslant J_1(u^*, v), \quad \forall v \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_v}), \\ J_2(u^*, v^*) &\leqslant J_2(u, v^*), \quad \forall u \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_u}). \end{aligned}$$

If $J_1(u^*, v^*) \geqslant 0$ with $x^0 = 0$, certainly $|z|^2 \leqslant \gamma^2 |v|^2$ for all $v \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_v})$, which ensures $\|\mathcal{L}_{u^*}\| \leqslant \gamma$. The second Nash inequality shows that u^* minimizes the output energy z when the input disturbance is at its worst v^*. Clearly, if the Nash equilibria (u^*, v^*) exists, then u^* is our desired H_2/H_∞ controller, and v^* is the corresponding worst case disturbance.

In Example 2.1, u^* represents the minimum control effort and sensitivity of the control response to the worst-case atmospheric disturbances. v^* means the worst-case weather under the requirement of system stability ($\|\mathcal{L}_{u^*}\| < \gamma$).

In the following, we will give sufficient and necessary conditions for the existence of the linear state feedback pair (u^*, v^*). To this end, we will make some preliminaries in the next section.

3 The Stochastic Bounded Real Lemma with Random Coefficients

Let $(\Omega, \mathcal{F}, P)$ be a given probability space and $\{W_t, 0 \leqslant t \leqslant T\}$ is a one-dimensional standard Brownian Motion on it. $\{\mathcal{F}_t, 0 \leqslant t \leqslant T\}$ is the augmented natural filtration of the standard Brownian motion W. Note that we assume the Brownian motion to be one-dimensional just for simplicity, and there is no essential difficulty in the analysis below for the multi-dimensional cases. Consider the following stochastic linear system:

$$\begin{cases} \mathrm{d}x_t = (A_t x_t + B_t v_t)\mathrm{d}t + A^0_t x_t \mathrm{d}W_t, \quad t \in [0, T], \\ x_0 = x^0 \end{cases} \tag{3.1}$$

and

$$z_t = \begin{pmatrix} C_t x_t \\ D_t v_t \end{pmatrix}. \tag{3.2}$$

Here we assume that all the coefficients A, A^0, B, C and D are $\{\mathcal{F}_t, 0 \leqslant t \leqslant T\}$-progressively measurable bounded matrix-valued processes, defined on $\Omega \times [0,T]$, of dimensions $n \times n$, $n \times n$, $n \times m$, $q \times n$ and $l \times m$, respectively. We also denote $v \in \mathbb{R}^m$ as the external disturbance and $z \in \mathbb{R}^{q+l}$ the controlled output. We use (3.2) rather than the more natural $z_t = C_t x_t + D_t v_t$ to avoid the appearance of cross terms when computing $z'z$.

Definition 3.1 *The system* (3.1)–(3.2) *with the initial state zero is said to be externally stable or L^2 input-output stable if, there exists a constant $\gamma \geqslant 0$ such that*

$$|z(\cdot)| \leqslant \gamma |v(\cdot)|, \quad \forall v \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m). \tag{3.3}$$

Definition 3.2 *Suppose that the system* (3.1)–(3.2) *is externally stable. The operator $\mathcal{L}$: $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \to \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{q+l})$ defined by*

$$(\mathcal{L}v)(t) = \begin{pmatrix} C_t x(t,v;0,0) \\ D_t v_t \end{pmatrix}, \quad \forall (t,v) \in [0,T] \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$$

is called the perturbation operator of (3.1)–(3.2). *Its norm is defined as the minimal $\gamma \geqslant 0$ such that* (3.3) *is satisfied, i.e.,*

$$\|\mathcal{L}\| = \sup_{\substack{v \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \\ v \neq 0 \\ x^0 = 0}} \frac{|(\mathcal{L}v)|}{|v|} = \sup_{\substack{v \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \\ v \neq 0 \\ x^0 = 0}} \frac{\{E\int_0^T (x_t' C_t' C_t x_t + v_t' D_t' D_t v_t)\mathrm{d}t\}^{\frac{1}{2}}}{\{E\int_0^T v_t' v_t \mathrm{d}t\}^{\frac{1}{2}}},$$

$\|\mathcal{L}\|$ is a measure of the worst effect that the stochastic disturbance v may have on the to-be-controlled output z of the system. Therefore, it is important to find a way of determining the norm $\|\mathcal{L}\|$. The stochastic bounded real lemma with random coefficients provides a method for computing $\|\mathcal{L}\|$.

We proceed by associating a finite time quadratic cost functional with the problem parameterized by the initial data $(\tau, \xi) \in [0,T] \times \mathbb{R}^n$, $v \in \mathcal{L}^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m)$ and $z \in \mathcal{L}^2_{\mathcal{F}}(\tau,T;\mathbb{R}^{q+l})$:

$$\begin{aligned} J(v;\tau,\xi) &= E^{\mathcal{F}_\tau} \int_\tau^T [\gamma^2 |v_t|^2 - |z_t|^2]\mathrm{d}t \\ &= E^{\mathcal{F}_\tau} \int_\tau^T [\langle (\gamma^2 I - D_t' D_t) v_t, v_t\rangle - \langle C_t' C_t x_t, x_t\rangle]\mathrm{d}t, \end{aligned} \tag{3.4}$$

where x denotes the solution of (3.1) with $x_\tau = \xi$, for any $\tau \in [0,T]$. Note that for a given $\gamma > 0$, the cost $J(v;0,0)$ is nonnegative for all $v \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ (at $x^0 = 0$) if and only if $\|\mathcal{L}\| \leqslant \gamma$. In order to examine whether or not $J(v;0,0) \geqslant 0$ for all $v \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, we will analyze the finite horizon optimal control problem $\operatorname*{ess\,inf}_{v \in \mathcal{L}^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m)} J(v;\tau,\xi)$. Formally, the problem of minimizing $J(v;\tau,\xi)$ has the form of an optimal control problem and so in our development in this section we will refer to the disturbance v as a "control" and the perturbation operator $\mathcal{L}$ as an "input-output" operator. The value function is defined as

$$V(\tau,\xi) = \operatorname*{ess\,inf}_{v \in \mathcal{L}^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m)} J(v;\tau,\xi), \quad \forall (\tau,\xi) \in [0,T] \times \mathbb{R}^n. \tag{3.5}$$

It is known that the above stochastic LQ problem (3.1) and (3.4)–(3.5) is associated with the following BSRE:

$$\begin{cases} \mathrm{d}P_t = -[A_t'P_t + P_tA_t + (A_t^0)'P_tA_t^0 + (A_t^0)'L_t + L_tA_t^0 - C_t'C_t \\ \qquad\quad -P_tB_t(\gamma^2 I - D_t'D_t)^{-1}B_t'P_t]\mathrm{d}t + L_t\mathrm{d}W_t, \\ P_T = 0. \end{cases} \tag{3.6}$$

This equation is a nonlinear BSDE, the unknown of which is a pair of matrix-valued stochastic processes (P, L).

Definition 3.3 *A stochastic process* $(P, L) \in [\mathcal{L}_{\mathcal{F}}^{\infty}(0, T; \mathbb{S}^n) \cap L^{\infty}(\Omega, \mathcal{F}_T, P; C([0, T]; \mathbb{S}^n))] \times \mathcal{L}_{\mathcal{F}}^2(0, T; \mathbb{S}^n)$ *is called a solution to the BSRE* (3.6) *if it satisfies the first equation of* (3.6) *in the Itô sense as well as the second* (*the terminal condition*)*. A solution* (P, L) *of* (3.6) *is called bounded if* $P \in \mathcal{L}_{\mathcal{F}}^{\infty}(0, T; \mathbb{S}^n)$*, and is called negative* (*negative semi*) *definite if* $P < (\leqslant)\ 0$*.*

Next, we will prove that $\|\mathcal{L}\| < \gamma$ is equivalent to the solvability of the BSRE (3.6), which is called the stochastic bounded real lemma with random coefficients and plays an essential role in this paper.

3.1 Sufficiency for the solution of the BSRE

In this subsection, we will show that the solvability of (3.6) is sufficient for the solvability of the LQ problem (3.1) and (3.4)–(3.5). Moreover, the optimal linear state feedback control can be obtained via the solution to the BSRE. Meanwhile, the perturbation operator can be rendered less than γ (appearing in the BSRE).

Theorem 3.1 *Let* (P, L) *be a solution to BSRE* (3.6) *with* P *being almost surely and almost everywhere* (*abbreviated hereafter as a.s.a.e.*) *negative definite and uniformly bounded, and* $\gamma^2 I - D_t'D_t$ *be uniformly positive for a.s.a.e.* $(t, \omega) \in [0, T] \times \Omega$*. Then, the* LQ *problem* (3.1) *and* (3.4)–(3.5) *is solvable with the optimal control* $v_t = \Psi_t x_t$ *and* $\|\mathcal{L}\| < \gamma$*, where*

$$\Psi_t = -(\gamma^2 I - D_t'D_t)^{-1}B_t'P_t.$$

Proof Suppose that BSRE (3.6) has a solution, and let x be the solution of (3.1) with $\xi = x_\tau$, for any $\tau \in [0, T]$. Applying Itô's formula to $\langle P_t x_t, x_t\rangle$, in view of (3.4) and using the completion of squares, we have

$$\begin{aligned} J(v; \tau, \xi) &= J(v; \tau, \xi) + E^{\mathcal{F}_\tau}\Big\{\int_\tau^T d(x_t'P_tx_t) - x_T'P_Tx_T + x_\tau'P_\tau x_\tau\Big\} \\ &= \xi'P_\tau\xi + E^{\mathcal{F}_\tau}\Big\{\int_\tau^T \langle(\gamma^2 I - D_t'D_t)(v_t - \Psi_tx_t), v_t - \Psi_tx_t\rangle \mathrm{d}t\Big\} \\ &\geqslant \xi'P_\tau\xi, \end{aligned} \tag{3.7}$$

where

$$\Psi_t = -(\gamma^2 I - D_t'D_t)^{-1}B_t'P_t.$$

It follows immediately that the optimal feedback control would be $v_t = \Psi_t x_t$ and the optimal value is $V(\tau, \xi) = \xi'P_\tau\xi$, provided that the corresponding solution to the system equation exists under such a feedback control. In fact, when $v_t = \Psi_t x_t$, the system (3.1) is reduced to

$$\begin{cases} \mathrm{d}x_t = [A_t - B_t(\gamma^2 I - D_t' D_t)^{-1} B_t' P_t] x_t \mathrm{d}t + A_t^0 x_t \mathrm{d}W_t, \quad t \in [\tau, T], \\ x_\tau = \xi. \end{cases} \tag{3.8}$$

As $\gamma^2 I - D_t' D_t \geqslant \epsilon I$, $(\gamma^2 I - D_t' D_t)^{-1}$ is bounded. Moreover, since P is negative definite and uniformly bounded, the coefficients of (3.8) are uniformly bounded. Therefore, SDE (3.8) indeed has a unique strong solution $x \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$, and thus, $v_t = \Psi_t x_t \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$. From (3.7), we derive that $J(v; 0, x^0) \geqslant (x^0)' P_0 x^0$. In particular, if $x^0 = 0$, then $J(v; 0, 0) \geqslant 0$, which is equivalent to $\|\mathcal{L}\| \leqslant \gamma$. To show $\|\mathcal{L}\| < \gamma$, we define an operator

$$\Gamma: \quad \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \mapsto \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m), \quad \Gamma v_t = \widetilde{v}_t := v_t - \Psi_t x_t$$

with its realization

$$\begin{aligned} \mathrm{d}x_t &= (A_t x_t + B_t v_t)\mathrm{d}t + A_t^0 x_t \mathrm{d}W_t, \\ x_0 &= 0, \\ \widetilde{v}_t &= v_t + (\gamma^2 I - D_t' D_t)^{-1} B_t' P_t x_t. \end{aligned}$$

Then Γ^{-1} exists, which is determined by

$$\begin{aligned} \mathrm{d}x_t &= \{[A_t - B_t(\gamma^2 I - D_t' D_t)^{-1} B_t' P_t] x_t + B_t \widetilde{v}_t\}\mathrm{d}t + A_t^0 x_t \mathrm{d}W_t, \\ x_0 &= 0, \\ v_t &= -(\gamma^2 I - D_t' D_t)^{-1} B_t' P_t x_t + \widetilde{v}_t. \end{aligned}$$

According to the inverse operator theorem in functional analysis, $\|\Gamma^{-1}\|$ is bounded. There exists a positive constant $c = \frac{\epsilon}{\|\Gamma^{-1}\|^2}$ such that

$$\begin{aligned} J(v; 0, 0) &= E \int_0^T (v_t - \Psi_t x_t)' (\gamma^2 I - D_t' D_t)(v_t - \Psi_t x_t) \mathrm{d}t \\ &= E \int_0^T (\Gamma v_t)' (\gamma^2 I - D_t' D_t)(\Gamma v_t) \mathrm{d}t \\ &\geqslant \epsilon |\Gamma v_t|^2 \geqslant c |v_t|^2 > 0, \end{aligned}$$

which is equivalent to $\|\mathcal{L}\| < \gamma$.

The proof of Theorem 3.1 is complete.

The preceding theorem implies that if the BSRE (3.6) (including γ) is solvable, the norm of the input-output operator is less than γ.

3.2 Necessity for the solution of the BSRE

In this subsection we shall show that if $\|\mathcal{L}\| < \gamma$, then the corresponding BSRE (3.6) admits a unique solution by using Bellman's principle of quasi-linearization and a monotone convergence result of symmetric matrices.

The following lemma establishes a lower bound for the cost functional, which depends only on the norm of the initial state.

Lemma 3.1 *Suppose that $\|\mathcal{L}\| < \gamma$. Then there exists $\mu > 0$, such that for any $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$ and any $v \in \mathcal{L}^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m)$, we have $J(v; \tau, \xi) \geqslant -\mu|\xi|^2$.*

Proof From the assumption of $\|\mathcal{L}\| < \gamma$ and the definition of $\|\mathcal{L}\|$, there exists a sufficiently small constant $\delta > 0$ such that

$$\begin{aligned} J(v; 0, 0) &= E\int_0^T [\langle(\gamma^2 I - D_t' D_t)v_t, v_t\rangle - \langle C_t' C_t x(t, v; 0, 0), x(t, v; 0, 0)\rangle]\mathrm{d}t \\ &\geqslant \delta^2 E\int_0^T \langle v_t, v_t\rangle \mathrm{d}t, \quad \forall v \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m). \end{aligned}$$

It follows immediately that

$$E\int_\tau^T [\langle(\gamma^2 I - D_t' D_t)v_t, v_t\rangle - \langle C_t' C_t x(t, v; \tau, 0), x(t, v; \tau, 0)\rangle]\mathrm{d}t \geqslant \delta^2 E\int_\tau^T \langle v_t, v_t\rangle \mathrm{d}t.$$

We deduce from the last inequality that for a.s.a.e. $(t, \omega) \in [\tau, T] \times \Omega$,

$$\langle(\gamma^2 I - D_t' D_t - \delta^2 I)v_t, v_t\rangle - \langle C_t' C_t x(t, v; \tau, 0), x(t, v; \tau, 0)\rangle \geqslant 0.$$

Finally we obtain

$$\begin{aligned} J(v; \tau, 0) &= E^{\mathcal{F}_\tau}\Big\{\int_\tau^T [\langle(\gamma^2 I - D_t' D_t)v_t, v_t\rangle - \langle C_t' C_t x(t, v; \tau, 0), x(t, v; \tau, 0)\rangle]\mathrm{d}t\Big\} \\ &\geqslant \delta^2 E^{\mathcal{F}_\tau}\Big\{\int_\tau^T \langle v_t, v_t\rangle \mathrm{d}t\Big\}. \end{aligned} \tag{3.9}$$

On the other hand, denote by P_2 the solution of the following equation:

$$\begin{cases} \mathrm{d}P_t = -(A_t' P_t + P_t A_t + (A_t^0)' P_t A_t^0 + (A_t^0)' L_t + L_t A_t^0 - C_t' C_t)\mathrm{d}t + L_t \mathrm{d}W_t, \\ P_T = 0. \end{cases}$$

By linearity, the solution $x(t, v; \tau, \xi)$ of the system equation (3.1) satisfies

$$x(t, v; \tau, \xi) = x(t, v; \tau, 0) + x(t, 0; \tau, \xi).$$

By Itô's formula, we have

$$\begin{aligned} J(v; \tau, \xi) &= J(v; \tau, \xi) + E^{\mathcal{F}_\tau}\Big\{\int_\tau^T \mathrm{d}[x_t' P_{2,t} x_t]\Big\} + x_\tau' P_{2,\tau} x_\tau - x_T' P_{2,T} x_T \\ &= \xi' P_{2,\tau}\xi + E^{\mathcal{F}_\tau}\Big\{\int_\tau^T [\langle(\gamma^2 I - D_t' D_t)v_t, v_t\rangle \\ &\quad + \langle v_t, B_t' P_{2,t} x(t, v; \tau, \xi)\rangle + \langle x(t, v; \tau, \xi), P_{2,t} B_t v_t\rangle]\mathrm{d}t\Big\}. \end{aligned} \tag{3.10}$$

According to the estimate for SDE, there exists a constant $\alpha_0 > 0$ such that

$$E^{\mathcal{F}_\tau}\Big\{\sup_{\tau \leqslant t \leqslant T} |x_t|^2\Big\} \leqslant \alpha_0 E^{\mathcal{F}_\tau}\Big\{|\xi|^2 + \int_\tau^T |v_t|^2 \mathrm{d}t\Big\}.$$

Then, there exists a constant $\alpha_1 > 0$ such that

$$E^{\mathcal{F}_\tau}\Big\{\int_\tau^T |x(t, 0; \tau, \xi)|^2 \mathrm{d}t\Big\} \leqslant \alpha_1 |\xi|^2. \tag{3.11}$$

In addition, taking $v=0$ in (3.10), we get

$$\xi' P_{2,\tau}\xi = J(0;\tau,\xi) = -E^{\mathcal{F}_\tau}\Big\{\int_\tau^T \langle C_t'C_t x(t,0;\tau,\xi), x(t,0;\tau,\xi)\rangle \mathrm{d}t\Big\} \geqslant -\beta_0|\xi|^2,$$

where β_0 is a positive constant. Therefore,

$$-\beta_0 I \leqslant P_{2,\tau} \leqslant 0, \quad \forall \tau\in[0,T]. \tag{3.12}$$

From (3.10), it is easy to check that for any $(\tau,\xi)\in[0,T]\times\mathbb{R}^n$ and any $v\in\mathcal{L}^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m)$,

$$\begin{aligned}&J(v;\tau,\xi)-J(v;\tau,0)\\ &= \xi' P_{2,\tau}\xi + E^{\mathcal{F}_\tau}\Big\{\int_\tau^T [\langle v_t, B_t'P_{2,t}x(t,0;\tau,\xi)\rangle - \langle x(t,0;\tau,\xi), P_{2,t}B_t v_t\rangle]\mathrm{d}t\Big\}.\end{aligned}$$

In view of (3.9), we get

$$\begin{aligned}J(v;\tau,\xi) &\geqslant E^{\mathcal{F}_\tau}\Big\{\int_\tau^T |\delta v_t + \delta^{-1}B_t'P_{2,t}x(t,0;\tau,\xi)|^2\mathrm{d}t\Big\}\\ &\quad - E^{\mathcal{F}_\tau}\Big\{\int_\tau^T |\delta^{-1}B_t'P_{2,t}x(t,0;\tau,\xi)|^2\mathrm{d}t\Big\} + \xi' P_{2,\tau}\xi\\ &\geqslant \xi' P_{2,\tau}\xi - E^{\mathcal{F}_\tau}\Big\{\int_\tau^T |\delta^{-1}B_t'P_{2,t}x(t,0;\tau,\xi)|^2\mathrm{d}t\Big\}.\end{aligned} \tag{3.13}$$

By virtue of (3.11) and (3.12), there exists $\beta_1>0$ such that

$$E^{\mathcal{F}_\tau}\Big\{\int_\tau^T |\delta^{-1}B_t'P_{2,t}x(t,0;\tau,\xi)|^2\mathrm{d}t\Big\} \leqslant \beta_1|\xi|^2. \tag{3.14}$$

Hence, through (3.13) and (3.14), we obtain

$$J(v;\tau,\xi) \geqslant -(\beta_0+\beta_1)|\xi|^2 := -\mu|\xi|^2.$$

The proof is complete.

The following lemma concerns the solvability of the matrix-valued linear BSDE, which can be found in [16].

Lemma 3.2 *Let $\widehat{A}$, $\widehat{C}$ be $\mathbb{R}^{n\times n}$-valued, and $\widehat{R}$ be $\mathbb{S}^n$-valued, $\mathcal{F}_t$-adapted processes. Assume that they are all bounded. Let $\widehat{Q}$ be a bounded $\mathcal{F}_T$-measurable random variable with values in $\mathbb{S}^n$. Then there exists a pair (K,M) satisfying the following linear equation:*

$$\begin{cases}-\mathrm{d}K_t = [\widehat{A}_t'K_t + K_t\widehat{A}_t + \widehat{C}_t'K_t\widehat{C}_t' + (M_t\widehat{C}_t + \widehat{C}_t'M_t) + \widehat{R}_t]\mathrm{d}t - M_t\mathrm{d}W_t,\\ K_T=\widehat{Q}.\end{cases}$$

Moreover

$$\sup_{t,\omega}|K_t(\omega)|^2 \leqslant k_0,$$

where the constant k_0 depends only on $\sup\limits_{t,\omega}(|\widehat{A}_t|+|\widehat{C}_t|)$ and $\sup\limits_{\omega}(|\widehat{Q}|^2+\int_0^T \widehat{R}_t^2\mathrm{d}t)(\omega)$.

In addition, there exists a representation: for any $(t,h)\in[0,T]\times\mathbb{R}^n$,

$$\langle K_t h, h\rangle = E^{\mathcal{F}_t}\Big\{\int_t^T \langle \widehat{R}_s y_s, y_s\rangle \mathrm{d}s + \langle \widehat{Q} y_T, y_T\rangle\Big\}$$

with y being the solution of

$$\begin{cases}\mathrm{d}y_s = \widehat{A}_s y_s \mathrm{d}t + \widehat{C}_s y_s \mathrm{d}W_s, \quad s\in[t,T],\\ y_t = h.\end{cases}$$

If $\widehat{R}$ and $\widehat{Q}$ are negative (positive) semi-definite almost surely, then K is also negative (positive) semi-definite almost surely.

We are now in a position to prove necessity.

Theorem 3.2 *If $\|\mathcal{L}\|<\gamma$, then there exists a unique solution to (3.6) such that P is negative semi-definite and uniformly bounded and L is square integrable. Moreover, there exists a deterministic constant β such that*

$$E\int_0^T |L_t|^2 \mathrm{d}t \leqslant \beta.$$

Here, β depends on the uniform lower bound of P and all the coefficients.

Proof From the assumption of $\|\mathcal{L}\|<\gamma$ and the definition of $\|\mathcal{L}\|$, there exists a sufficiently small constant $\epsilon>0$ such that $\gamma^2 I - D'D \geqslant \epsilon I$.

(i) Existence: We define $\widehat{U}(P)$: $\mathbb{S}^n\to\mathbb{R}^{n\times m}$ by

$$\widehat{U}(P) = -(\gamma^2 I - D'D)^{-1}B'P$$

and $\widehat{A}(P)$: $\mathbb{S}^n\to\mathbb{R}^{n\times n}$ by

$$\widehat{A}(P) = A + B\widehat{U}(P) = A - B(\gamma^2 I - D'D)^{-1}B'P.$$

We also define $F(P,L;\widetilde{P})$: $\mathbb{S}^n\times\mathbb{S}^n\times\mathbb{S}^n\to\mathbb{S}^n$ by

$$F(P,L;\widetilde{P}) = \widehat{A}(\widetilde{P})'P + P\widehat{A}(\widetilde{P}) + A_0'PA_0 + A_0'L + LA_0 - C'C + \widehat{U}'(\widetilde{P})(\gamma^2 I - D'D)\widehat{U}(\widetilde{P}).$$

With these notations, we can rewrite (3.6) as

$$\begin{cases}-\mathrm{d}P_t = F(P_t, L_t; P_t)\mathrm{d}t - L_t \mathrm{d}W_t,\\ P_T = 0.\end{cases}$$

It is seen that

$$F(P,L;\widetilde{P}) - F(P,L;P) = (\widetilde{P}-P)B(\gamma^2 I - D'D)^{-1}B'(\widetilde{P}-P)\geqslant 0,$$

i.e.,

$$F(P,L;\widetilde{P})\geqslant F(P,L;P). \tag{3.15}$$

We now iteratively construct a sequence of approximating solutions. First, we define (P_1, L_1) by solving the following linear BSDE:

$$\begin{cases} -\mathrm{d}P_{1,t} = F(P_{1,t}, L_{1,t}; 0)\mathrm{d}t - L_{1,t}\mathrm{d}W_t, \\ P_{1,T} = 0. \end{cases}$$

By Lemma 3.2, the last equation has a unique solution P_1 with the representation: For any $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$,

$$\langle P_{1,\tau}\xi, \xi\rangle = E^{\mathcal{F}_\tau}\Big\{ \int_\tau^T \langle -C_t' C_t x_t, x_t\rangle \mathrm{d}t \Big\},$$

where x_t satisfies

$$\begin{cases} \mathrm{d}x_t = A_t x_t \mathrm{d}t + A_t^0 x_t \mathrm{d}W_t, \quad t \in [\tau, T], \\ x_\tau = \xi. \end{cases}$$

Since all the coefficients are bounded, P_1 is bounded and negative semi-definite. Therefore, $\widehat{U}(P_{1,t}) = -(\gamma^2 I - D_t' D_t)^{-1} B_t' P_{1,t}$ is bounded. Then, we define the pairs (P_2, L_2) to be the solution of BSDE

$$\begin{cases} -\mathrm{d}P_{2,t} = F(P_{2,t}, L_{2,t}; P_{1,t})\mathrm{d}t - L_{2,t}\mathrm{d}W_t, \\ P_{2,T} = 0. \end{cases}$$

Again from Lemma 3.2, for any $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$, there is a unique bounded solution (P_2, L_2). Thus $\widehat{U}(P_{2,t})$ is well-defined and bounded. Inductively, we can define (P_{j+1}, L_{j+1}) by the unique bounded solution of

$$\begin{cases} -\mathrm{d}P_{j+1,t} = F(P_{j+1,t}, L_{j+1,t}; P_{j,t})\mathrm{d}t - L_{j+1,t}\mathrm{d}W_t, \\ P_{j+1,T} = 0, \quad j = 1, 2, \cdots. \end{cases} \tag{3.16}$$

Furthermore, by virtue of a classic estimate for BSDE, we have

$$\begin{aligned} &E \sup_{t\in[0,T]} |P_{j+1,t}|^2 + E\int_0^T |L_{j+1,t}|^2 \mathrm{d}t \\ &\leqslant E\int_0^T [-C_t' C_t + \widehat{U}'(P_{j,t})(\gamma^2 I - D_t' D_t)\widehat{U}(P_{j,t})]^2 \mathrm{d}t. \end{aligned} \tag{3.17}$$

We claim that the sequence $\{P_{j,t}\}$ is nonincreasing. Indeed, we have

$$\begin{cases} -\mathrm{d}(P_{j,t} - P_{j+1,t}) = [F(P_{j,t}, L_{j,t}; P_{j-1,t}) - F(P_{j+1,t}, L_{j+1,t}; P_{j,t})]\mathrm{d}t - [L_{j,t} - L_{j+1,t}]\mathrm{d}W_t \\ \qquad = [F(P_{j,t}, L_{j,t}; P_{j-1,t}) - F(P_{j,t}, L_{j,t}; P_{j,t}) + F(P_{j,t}, L_{j,t}; P_{j,t}) \\ \qquad\quad - F(P_{j+1,t}, L_{j+1,t}; P_{j,t})]\mathrm{d}t - [L_{j,t} - L_{j+1,t}]\mathrm{d}W_t \\ \qquad = [F(P_{j,t}, L_{j,t}; P_{j-1,t}) + F(P_{j,t} - P_{j+1,t}, L_{j,t} - L_{j+1,t}; P_{j,t}) \\ \qquad\quad - F(P_{j,t}, L_{j,t}; P_{j,t})]\mathrm{d}t - [L_{j,t} - L_{j+1,t}]\mathrm{d}W_t, \\ P_{j,T} - P_{j+1,T} = 0 \end{cases}$$

with $R_{j,t} \triangleq F(P_{j,t}, L_{j,t}; P_{j-1,t}) - F(P_{j,t}, L_{j,t}; P_{j,t})$. From (3.15), $R_{j,t}$ is nonnegative. Thus, according to Lemma 3.2, $P_{j,t} - P_{j+1,t}$ is also nonnegative, which implies that $\{P_{j,t}\}$ is a non-increasing sequence.

In addition, according to Lemma 3.1, $J(v;\tau,\xi) \geqslant -\mu|\xi|^2$, that is, for any $v \in \mathcal{L}^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m)$,

$$E^{\mathcal{F}_\tau}\Big\{\int_\tau^T [\langle(\gamma^2 I - D_t'D_t)v_t, v_t\rangle - \langle C_t'C_t x(t,v;\tau,\xi), x(t,v;\tau,\xi)\rangle]\mathrm{d}t\Big\} \geqslant -\mu|\xi|^2. \tag{3.18}$$

Setting $v_t = \widehat{U}(P_{j,t})x_t = -(\gamma^2 I - D_t'D_t)^{-1}B_t'P_{j,t}x_t$ in (3.18), where x_t satisfies

$$\begin{cases} \mathrm{d}x_t = \widehat{A}(P_{j,t})x_t\mathrm{d}t + A_0 x_t \mathrm{d}W_t, \\ x_\tau = \xi, \end{cases} \qquad \forall(\tau,\xi) \in [0,T]\times\mathbb{R}^n,$$

we conclude

$$\begin{aligned}\langle P_{j+1,\tau}\xi,\xi\rangle &= E^{\mathcal{F}_\tau}\Big\{\int_\tau^T \langle[-C_t'C_t + \widehat{U}'(P_{j,t})(\gamma^2 I - D_t'D_t)\widehat{U}(P_{j,t})]x_t, x_t\rangle\mathrm{d}t\Big\} \\ &\geqslant -\mu|\xi|^2.\end{aligned}$$

As seen from the afore-mentioned information, we can get

$$0 \geqslant P_1 \geqslant P_2 \geqslant \cdots \geqslant P_j \geqslant \cdots \geqslant -\mu I.$$

It follows that $\{P_{j,t}\}$ converges almost surely to a negative semi-definite S^n-valued process P_t. According to Lebesgue's convergence theorem, we have

$$E\int_0^T |P_{j,t} - P_t|^q \mathrm{d}t \to 0, \quad \text{as } j\to\infty, \quad \forall q > 0.$$

Thus $\{P_{j,t}\}$, and then also $\widehat{U}(P_{j,t})$ is a Cauchy sequence in the above sense. We have also almost everywhere

$$E|P_{j,t} - P_t|^q \to 0, \quad \text{as } j\to\infty, \quad \forall q > 0.$$

By the definition (3.16), we can apply Itô's formula to $|P_{i,t} - P_{j,t}|^2$,

$$\begin{aligned} & E|P_{i,0} - P_{j,0}|^2 + E\int_0^T |L_{i,t} - L_{j,t}|^2 \mathrm{d}t \\ &= 2E\int_0^T \mathrm{tr}(P_{i,t} - P_{j,t})\{[2A_t'(P_{i,t} - P_{j,t}) + (A_t^0)'(P_{i,t} - P_{j,t})A_t^0 + 2(A_t^0)'(L_{i,t} - L_{j,t}) \\ &\quad - \widehat{U}'(P_{i,t})(\gamma^2 I - D_t'D_t)\widehat{U}(P_{i,t}) + \widehat{U}'(P_{j,t})(\gamma^2 I - D_t'D_t)\widehat{U}(P_{j,t})]\mathrm{d}t + (L_{i,t} - L_{j,t})\mathrm{d}W_t\} \\ &\leqslant \frac{1}{2}E\int_0^T |L_{i,t} - L_{j,t}|^2\mathrm{d}t + c_1 E\int_0^T |P_{i,t} - P_{j,t}|^2 \mathrm{d}t \\ &\quad + c_2 E\int_0^T |\widehat{U}'(P_{i,t})(\gamma^2 I - D_t'D_t)\widehat{U}(P_{i,t}) - \widehat{U}'(P_{j,t})(\gamma^2 I - D_t'D_t)\widehat{U}(P_{j,t})|^2\mathrm{d}t, \end{aligned}$$

where "tr" denotes the trace, c_1 and c_2 are positive constants. Thus $\{L_{j,t}\}$ is a Cauchy sequence in $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{S}^n)$. Passing to the limit in (3.16), we obtain that (P_t, L_t) is a solution of (3.6), with

$$L_t = \lim_{j\to\infty} L_{j,t} \quad \text{in } \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{S}^n).$$

(ii) Uniqueness: Let (P, L) and $(\widetilde{P}, \widetilde{L})$ be two pairs in

$$[\mathcal{L}_{\mathcal{F}}^{\infty}(0,T;\mathbb{S}^n) \cap L^{\infty}(\Omega, \mathcal{F}_T, P; C([0,T];\mathbb{S}^n))] \times \mathcal{L}_{\mathcal{F}}^2(0,T;\mathbb{S}^n)$$

satisfying (3.6), such that P and $\widetilde{P}$ are negative semi-definite and uniformly bounded. Now applying Itô's formula to $|P_\tau - \widetilde{P}_\tau|^2$ and using their respective equations, we have

$$\begin{aligned} \mathrm{d}|P_\tau - \widetilde{P}_\tau|^2 &= 2(P_\tau - \widetilde{P}_\tau)\mathrm{d}(P_\tau - \widetilde{P}_\tau) + |L_\tau - \widetilde{L}_\tau|^2 \mathrm{d}\tau \\ &= |L_\tau - \widetilde{L}_\tau|^2 \mathrm{d}\tau. \end{aligned} \tag{3.19}$$

On the other hand, from the proof of Theorem 3.1, we obtain $V(\tau, \xi) = \xi' P_\tau \xi = \xi' \widetilde{P}_\tau \xi$, for all $[\tau, \xi] \in [0,T] \times \mathbb{R}^n$, which implies $P_\tau = \widetilde{P}_\tau$, a.s. ω, for all $\tau \in [0,T]$. Putting this equality into (3.19), we have $L_\tau = \widetilde{L}_\tau$, a.s. ω, a.e. $\tau \in [0,T]$.

(iii) Boundedness: Passing the limit in (3.17) and by the boundedness of P and all the coefficients, we immediately get $E\int_0^T |L_t|^2 \mathrm{d}t \leqslant \beta$ for some constant β, which depends on the uniformly lower bound of P and the upper bound of all the coefficients.

The proof is complete.

From Theorems 3.1 and 3.2, we see that $\|\mathcal{L}\| < \gamma$ is equivalent to that (3.6) has a negative semi-definite and uniformly bounded solution, which tells us that the minimal γ satisfying (3.6) can be taken as an estimate of $\|\mathcal{L}\|$. Moreover, this result to some extent extends the stochastic bounded real lemma in [6] to the case of random coefficients.

4 Solvability of the Stochastic H_2/H_∞ Control

In this section, we shall give necessary and sufficient conditions for the solvability of the stochastic H_2/H_∞ control problem in terms of a pair of coupled Riccati equations. Consider the stochastic linear system (2.1), and the finite horizon stochastic H_2/H_∞ problem can be stated as follows.

Definition 4.1 *Given a scalar $\gamma > 0$, we want to find, if possible, a state feedback control $u^* \in \mathcal{L}_{\mathcal{F}}^2(0,T;\mathbb{R}^{n_u})$, such that with the constraint* (2.1)*, we have that*

(1)

$$\begin{aligned} \|\mathcal{L}_{u^*}\| &:= \sup_{\substack{v \in \mathcal{L}_{\mathcal{F}}^2(0,T;\mathbb{R}^{n_v}) \\ v \neq 0 \\ x^0 = 0}} \frac{|\mathcal{L}_{u^*}(v)|}{|v|} \\ &= \sup_{\substack{v \in \mathcal{L}_{\mathcal{F}}^2(0,T;\mathbb{R}^{n_v}) \\ v \neq 0 \\ x^0 = 0}} \frac{\{E\int_0^T (x_t' C_t' C_t x_t + u_t^{*\prime} u_t^* + v_t'(D_t^1)' D_t^1 v_t)\mathrm{d}t\}^{\frac{1}{2}}}{\{E\int_0^T v_t' v_t)\mathrm{d}t\}^{\frac{1}{2}}} < \gamma, \end{aligned}$$

where $\mathcal{L}_{u^*}(v) := ((C_t x(t,u^*,v,0))', (D_t^1 v_t)', (D_t^2 u_t^*)')'$;

(2) *When the worst case disturbance $v^* \in \mathcal{L}_{\mathcal{F}}^2(0,T;\mathbb{R}^{n_v})$, if it exists, applied to* (2.1)*, u^* minimizes the output energy*

$$J_2(u, v^*) = E\int_0^T (x_t' C_t' C_t x_t + u_t' u_t + (v_t^*)'(D_t^1)' D_t^1 v_t^*)\mathrm{d}t.$$

Here, the so-called worst case disturbance v^* means that for any $v \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_v})$ and any $x^0 \in \mathbb{R}^n$,

$$v^* = \underset{v}{\operatorname{argmin}} J_1(u^*, v) = \underset{v}{\operatorname{argmin}} E\int_0^T (\gamma^2 v_t' v_t - z_t' z_t)\mathrm{d}t.$$

If the previous (u^*, v^*) exists, then the finite horizon H_2/H_∞ control has a pair of solutions (u^*, v^*).

In the following two theorems, we shall give a necessary and sufficient condition for the existence of the linear state feedback pair (u^*, v^*). It generalizes the result of Chen and Zhang [3] to the case of stochastic systems with random coefficients.

Theorem 4.1 *Let P_1, P_2 be the uniformly bounded solutions of the following coupled Riccati equations:*

$$\begin{cases} \mathrm{d}P_{1,t} = -\Big[A_t'P_{1,t} + P_{1,t}A_t + (A_t^0)'P_{1,t}A_t^0 + (A_t^0)'L_{1,t} + L_{1,t}A_t^0 - C_t'C_t \\ \qquad - \begin{pmatrix} P_{1,t} & P_{2,t} \end{pmatrix} \begin{pmatrix} N_t & B_t^2(B_t^2)' \\ B_t^2(B_t^2)' & B_t^2(B_t^2)' \end{pmatrix} \begin{pmatrix} P_{1,t} \\ P_{2,t} \end{pmatrix} \Big]\mathrm{d}t + L_{1,t}\mathrm{d}W_t, \\ P_{1,T} = 0 \end{cases} \tag{4.1}$$

and

$$\begin{cases} \mathrm{d}P_{2,t} = -\Big[A_t'P_{2,t} + P_{2,t}A_t + (A_t^0)'P_{2,t}A_t^0 + (A_t^0)'L_{2,t} + L_{2,t}A_t^0 + C_t'C_t \\ \qquad - \begin{pmatrix} P_{1,t} & P_{2,t} \end{pmatrix} \begin{pmatrix} M_t & N_t \\ N_t & B_t^2(B_t^2)' \end{pmatrix} \begin{pmatrix} P_{1,t} \\ P_{2,t} \end{pmatrix} \Big]\mathrm{d}t + L_{2,t}\mathrm{d}W_t, \\ P_{2,T} = 0. \end{cases} \tag{4.2}$$

Here

$$\begin{aligned} M &= -B^1(\gamma^2 I - (D^1)'D^1)^{-1}(D^1)'D^1(\gamma^2 I - (D^1)'D^1)^{-1}(B^1)', \\ N &= B^1(\gamma^2 I - (D^1)'D^1)^{-1}(B^1)'. \end{aligned}$$

Suppose further that $\gamma^2 I - (D^1)'D^1$ is uniformly positive for a.s. a.e. $(t,\omega) \in [0,T]\times\Omega$. Then, we have that

(1) *the Nash equilibrium strategies are uniquely specified by*

$$\begin{cases} u_t^* = -(B_t^2)'P_{2,t}x_t, \\ v_t^* = -(\gamma^2 I - (D_t^1)'D_t^1)^{-1}(B_t^1)'P_{1,t}x_t, \end{cases} \tag{4.3}$$

(2) *in the case that $u = u^*$ with $x^0 = 0$, $\|\mathcal{L}_{u^*}\| < \gamma$.*

Proof Suppose that the coupled Riccati equations (4.1) and (4.2) have solutions. Let us consider the cost functional $J_1(u, v)$ first. Applying Itô's formula and the completion of squares, we have

$$
\begin{aligned}
J_1(u,v) &= E\int_0^T [(\gamma^2 v_t' v_t - z_t' z_t)\mathrm{d}t + \mathrm{d}x_t' P_{1,t} x_t + (x^0)' P_{1,0} x^0 \mathrm{d}t - x_T' P_{1,T} x_T \mathrm{d}t] \\
&= E\int_0^T [\langle (\gamma^2 I - (D_t^1)' D_t^1)[v + (\gamma^2 I - (D_t^1)' D_t^1)^{-1} (B_t^1)' P_{1,t} x_t], \\
&\quad v + (\gamma^2 I - (D_t^1)' D_t^1)^{-1} (B_t^1)' P_{1,t} x_t \rangle \\
&\quad - u_t' u_t + x_t' P_{2,t} B_t^2 (B_t^2)' P_{2,t} x_t + x_t' P_{2,t} B_t^2 (B_t^2)' P_{1,t} x_t + x_t' P_{1,t} B_t^2 (B_t^2)' P_{2,t} x_t \\
&\quad + u_t' (B_t^2)' P_{1,t} x_t + x_t' P_{1,t} B_t^2 u_t] \mathrm{d}t + (x^0)' P_{1,0} x^0 \\
&= (x^0)' P_{1,0} x^0 + E\int_0^T [\langle (\gamma^2 I - (D_t^1)' D_t^1)(v_t - v_t^*), (v_t - v_t^*)\rangle - u_t' u_t \\
&\quad + x_t' P_{1,t} B_t^2 (u_t - u_t^*) + (u_t - u_t^*)' (B_t^2)' P_{1,t} x_t + (u_t^*)' u_t] \mathrm{d}t, \qquad (4.4)
\end{aligned}
$$

where u^* and v^* are defined as in (4.3). Setting $u = u^*$, we obtain

$$
J_1(u^*, v) = (x^0)' P_{1,0} x^0 + E\int_0^T \langle (\gamma^2 I - (D_t^1)' D_t^1)(v_t - v_t^*), (v_t - v_t^*)\rangle \mathrm{d}t.
$$

Therefore, it follows immediately that $J_1(u^*, v^*) \leqslant J_1(u^*, v)$ and $J_1(u^*, v^*) = (x^0)' P_{1,0} x^0$. We then conclude that v^* is the worst case disturbance with respect to u^*. Moreover, by the discussion in the proof of Theorem 3.1, we obtain $\|\mathcal{L}_{u^*}\| < \gamma$.

Similarly, we have

$$
\begin{aligned}
J_2(u,v) = (x^0)' P_{2,0} x^0 + E\int_0^T [\langle (u_t - u_t^*), (u_t - u_t^*)\rangle + x_t' P_{2,t} B_t^1 (v_t - v_t^*) \\
+ (v_t - v_t^*)' (B_t^1)' P_{2,t} x_t + v_t' (D_t^1)' D_t^1 v_t - (v_t^*)' (D_t^1)' D_t^1 v_t^*] \mathrm{d}t. \qquad (4.5)
\end{aligned}
$$

Setting $v = v^*$ results in $J_2(u^*, v^*) \leqslant J_1(u, v^*)$ and $J_2(u^*, v^*) = (x^0)' P_{2,0} x^0$. It means that $\min_u J_2(u, v^*) = J_2(u^*, v^*)$. By Definition 4.1, the afore-mentioned information implies that the finite horizon H_2/H_∞ control has a pair of solutions (u^*, v^*) with

$$
\begin{aligned}
u_t^* &= -(B_t^2)' P_{2,t} x_t, \\
v_t^* &= -(\gamma^2 I - (D_t^1)' D_t^1)^{-1} (B_t^1)' P_{1,t} x_t.
\end{aligned}
$$

This completes the proof of Theorem 4.1.

Remark 4.1 We establish the signs of P_1 and P_2 as follows:

(i) Using the completion of squares similar to that which lead to (4.4), we derive that

$$
\begin{aligned}
&E^{\mathcal{F}_\tau}\Big\{\int_\tau^T (\gamma^2 v_t' v_t - z_t' z_t)\mathrm{d}t\Big\} \\
&= x_\tau' P_{1,\tau} x_\tau + E^{\mathcal{F}_\tau}\Big\{\int_\tau^T [\langle (\gamma^2 I - (D_t^1)' D_t^1)(v_t - v_t^*), (v_t - v_t^*)\rangle \\
&\quad - u_t' u_t + x_t' P_{1,t} B_t^2 (u_t - u_t^*) + (u_t - u_t^*)' (B_t^2)' P_{1,t} x_t + (u_t^*)' u_t] \mathrm{d}t\Big\}.
\end{aligned}
$$

Setting $u_t = u_t^*$ and $v_t = 0$ yields

$$
x_\tau' P_{1,\tau} x_\tau = E^{\mathcal{F}_\tau}\Big\{\int_\tau^T [-(\gamma^2 I - (D_t^1)' D_t^1)(v_t^*)'(v_t^*) - z_t' z_t] \mathrm{d}t\Big\} \leqslant 0.
$$

Thus, $P_{1,\tau} \leqslant 0,$ a.s. a.e. $(\omega, \tau) \in \Omega \times [0,T]$.

(ii) By the similar calculations that lead to (4.5), setting $u_t = u_t^*$ and $v_t = v_t^*$, we finally obtain

$$x_\tau' P_{2,\tau} x_\tau = E^{\mathcal{F}_\tau}\Big\{ \int_\tau^T z_t' z_t \mathrm{d}t \Big\} \geqslant 0.$$

Consequently, $P_{2,\tau} \geqslant 0,$ a.s. a.e. $(\omega, \tau) \in \Omega \times [0,T]$.

Remark 4.2 If the coupled BSREs (4.1) and (4.2) have solutions, then by

$$x_\tau'(P_{1,\tau} + P_{2,\tau})x_\tau = E^{\mathcal{F}_\tau}\Big\{ \int_\tau^T \gamma^2 (v_t^*)'(v_t^*)\mathrm{d}t \Big\} \geqslant 0,$$

we see that $P_{1,\tau} + P_{2,\tau} \geqslant 0,$ a.s. a.e. $(\omega, \tau) \in \Omega \times [0,T]$.

Notice that by adding $J_1(u^*, v^*)$ to $J_2(u^*, v^*)$, we prove that the energy in the worst case feedback disturbance is given by $|v^*|^2 = \gamma^{-2}(x^0)'(P_{1,0} + P_{2,0})x^0$.

Theorem 4.2 *Assume that the finite horizon H_2/H_∞ control problem admits a pair of solutions (u^*, v^*) with $v_t^* = K_t^1 x_t$, $u_t^* = K_t^2 x_t$, where K^1 and K^2 are bounded adapted processes. Then the coupled stochastic Riccati equations* (4.1) *and* (4.2) *have solutions P_1, P_2 on $[0,T]$ respectively, with P_1 being negative definite and uniformly bounded and P_2 being positive definite and uniformly bounded.*

Proof Assuming that an admissible Nash equilibrium strategy pair exist, we will show that the coupled Riccati equations (4.1) and (4.2) have solutions $P_{1,t} \leqslant 0$ and $P_{2,t} \geqslant 0$, respectively.

(i) Implementing $u_t^* = K_t^2 x_t$ in (2.1), we obtain

$$\begin{cases} \mathrm{d}x_t = [(A_t + B_t^2 K_t^2)x_t + B_t^1 v_t]\mathrm{d}t + A_t^0 x_t \mathrm{d}W_t, \quad x_0 = x^0, \\ z_t = \begin{pmatrix} \begin{pmatrix} C_t \\ D_t^2 K_t^2 \end{pmatrix} x_t \\ D_t^1 v_t \end{pmatrix}, \quad (D_t^2)' D_t^2 = I. \end{cases} \tag{4.6}$$

Since the finite horizon H_2/H_∞ control is solvable, by Definition 4.1, we have $\|\mathcal{L}_{u^*}\| < \gamma$. Moreover, by Theorem 3.2, there exist constants $\lambda_1 > 0$ and $\kappa_1 > 0$ such that the following BSRE

$$\begin{cases} \mathrm{d}P_t = -[(A_t + B_t^2 K_t^2)' P_t + P_t(A_t + B_t^2 K_t^2) + (A_t^0)' P_t A_t^0 + (A_t^0)' L_t + L_t A_t^0 \\ \qquad -C_t' C_t - (K_t^2)' K_t^2 - P_t B_t^1 (\gamma^2 I - (D_t^1)' D_t^1)^{-1} (B_t^1)' P_t]\mathrm{d}t + L_t \mathrm{d}W_t, \\ P_T = 0 \end{cases} \tag{4.7}$$

has a unique solution (P_1, L_1) with $-\lambda_1 I \leqslant P_{1,t} \leqslant 0$, $E\int_0^T |L_{1,t}|^2 \mathrm{d}t \leqslant \kappa_1$. It now follows from Theorem 3.1 that the control problem

$$\min_{v \in \mathcal{L}_\mathcal{F}^2(0,T;R^{n_v})} J_1(u^*, v)$$

has a unique solution $v_t^* = -(\gamma^2 I - (D_t^1)' D_t^1)^{-1} (B_t^1)' P_{1,t} x_t$. Therefore,

$$K_t^1 = -(\gamma^2 I - (D_t^1)' D_t^1)^{-1} (B_t^1)' P_{1,t}.$$

(ii) Substituting $v_t = v_t^* = -(\gamma^2 I - (D_t^1)' D_t^1)^{-1}(B_t^1)' P_{1,t} x_t$ into (4.1), we have

$$\begin{cases} \mathrm{d}x(t) = [(A_t - B_t^1(\gamma^2 I - (D_t^1)' D_t^1)^{-1}(B_t^1)' P_t) x_t + B_t^2 u_t]\mathrm{d}t + A_t^0 x_t \mathrm{d}W_t, \\ x_0 = x^0 \end{cases}$$

and

$$z_t = \left(\begin{pmatrix} C_t \\ -D_t^1(\gamma^2 I - (D_t^1)' D_t^1)^{-1}(B_t^1)' P_{1,t} \end{pmatrix} x_t \atop D_t^2 u_t \right), \quad (D_t^2)' D_t^2 = I.$$

Since

$$\min_{u \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^{n_u})} J_2^T(u, v^*)$$

is a standard stochastic LQ problem, according to Peng [16] or Tang [18], there exists a unique optimal control $u_t^* = -(B_t^2)' P_{2,t} x_t$. Here $P_{2,t}$ is the unique nonnegative solution of

$$\begin{cases} \mathrm{d}P_t = -[(A_t - B_t^1(\gamma^2 I - (D_t^1)' D_t^1)^{-1}(B_t^1)' P_{1,t}) P_t + P_t (A_t - B_t^1(\gamma^2 I - (D_t^1)' D_t^1)^{-1}(B_t^1)' P_{1,t}) \\ \qquad + (A_t^0)' P_t A_t^0 + (A_t^0)' L_t + L_t A_t^0 + C_t' C_t - P_t B_t^2 (B_t^2)' P_t \\ \qquad + P_{1,t} B_{1,t} (\gamma^2 I - (D_t^1)' D_t^1)^{-1} (D_t^1)' D_t^1 (\gamma^2 I - (D_t^1)' D_t^1)^{-1} (B_t^1)' P_{1,t}]\mathrm{d}t + L_t \mathrm{d}W_t, \\ P_T = 0, \end{cases}$$

which is the coupled Riccati equation (4.2). Moreover, there exist constants $\lambda_2 > 0$ and $\kappa_2 > 0$ such that $0 \leqslant P_{2,t} \leqslant \lambda_2 I$, $E\int_0^T |L_{2,t}|^2 \mathrm{d}t \leqslant \kappa_2$. By uniqueness, $K_t^2 = -(B_t^2)' P_{2,t}$. Substituting for K_t^2 in (4.7), we obtain the required BSRE (4.1).

The proof is complete.

Remark 4.3 Theorems 4.1 and 4.2 tell us that, to some extent, the existence of a stochastic H_2/H_∞ state feedback controller is equivalent to the solvability of a pair of coupled BSREs (4.1)–(4.2). However, generally speaking, it is hard to solve the coupled BSREs (4.1)–(4.2) analytically. We can only obtain its approximating solutions by decoupling the equations under very special conditions. For example, for the one-dimensional case, by selecting some special random matrices to decouple the equations such that (4.1) and (4.2) being two independent BSREs, we then need to solve these BSREs which has the following form (the variable t is suppressed):

$$\begin{cases} \mathrm{d}P = -[\alpha P + \beta L + Q - PB(\gamma^2 I - D'D)^{-1} B' P]\mathrm{d}t + L\mathrm{d}W_t, \\ P_T = 0. \end{cases} \tag{4.8}$$

We first consider the linear BSDE (with Q being some symmetric random matrix)

$$\begin{cases} \mathrm{d}P_1 = -[\alpha P_1 + \beta L_1 + Q]\mathrm{d}t + L_1 \mathrm{d}W_t, \\ P_{1,T} = 0, \end{cases}$$

whose solution is

$$P_{1,t} = E^{\mathcal{F}_t}\Big[\int_t^T Q_s \Gamma_s^t \mathrm{d}s\Big], \quad \forall s \in [t, T]$$

with Γ_s^t satisfying

$$\mathrm{d}\Gamma_s^t = \Gamma_s^t[\alpha_s \mathrm{d}s + \beta_s \mathrm{d}W_s], \quad \Gamma_t^t = 1.$$

We then follow the lines of the proof of Theorem 3.2 to obtain the approximating solution to (4.8).

5 Concluding Remarks

In this paper, we have studied the mixed H_2/H_∞ control for stochastic Itô systems with random coefficients when only the state appears in the noise. We approach this problem as an LQ optimal control problem and focus on the associated BSREs. A key part of our derivation is the proof of the existence and uniqueness of the solution to the indefinite BSRE associated with the H_∞ robustness. This pivotal result enables us to present sufficient and necessary conditions for the existence of H_2/H_∞ control in terms of a pair of coupled stochastic Riccati equations.

However, when state and control appear in the noise term, the indefinite BSRE takes the form (1.3). The existence of the solution is hard to prove due to the following reasons. First of all, it is a highly nonlinear BSDE, especially in view of the matrix inverse term $(\gamma^2 I - (D^1)' D^1 + (B^0)' P (B^0))^{-1}$. Secondly, the indefiniteness of parameter matrixes gives rise to the singularity of the term $\gamma^2 I - (D^1)' D^1 + (B^0)' P (B^0)$ when one tries to use the typical approximation scheme to construct a solution. Thirdly, due to possible unboundedness of the martingale term L, the Bellman's quasi-linearization principle finds a difficulty in solving this problem. Finally, (1.3) is a matrix equation. Hence certain terms do not commute, which adds substantial difficulties to the analysis. For these reasons, the solvability of (1.3) stands out on its own as an interesting theoretical problem and merits further studies.

The case of state-dependent noise presented in this paper serves as a preliminary discussion on the H_2/H_∞ control for the stochastic system with random coefficients. The more general case of state- and control- dependent noise will be studied elsewhere.

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