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SEMI-LINEAR SYSTEMS OF BACKWARD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN $\mathbb{R}^n$**

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Abstract

This paper explores the diffeomorphism of a backward stochastic ordinary differential equation (BSDE) to a system of semi-linear backward stochastic partial differential equations (BSPDEs), under the inverse of a stochastic flow generated by an ordinary stochastic differential equation (SDE). The author develops a new approach to BSPDEs and also provides some new results. The adapted solution of BSPDEs in terms of those of SDEs and BSDEs is constructed. This brings a new insight on BSPDEs, and leads to a probabilistic approach. As a consequence, the existence, uniqueness, and regularity results are obtained for the (classical, Sobolev, and distributional) solution of BSPDEs. The dimension of the space variable x is allowed to be arbitrary n, and BSPDEs are allowed to be nonlinear in both unknown variables, which implies that the BSPDEs may be nonlinear in the gradient. Due to the limitation of space, however, this paper concerns only classical solution of BSPDEs under some more restricted assumptions.

Keywords Semi-linear system of backward stochastic partial differential equation, Backward stochastic differential equation, Stochastic differential equation, Probabilistic representation, Stochastic flow
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§ 1 . Introduction

Backward stochastic differential equations (BSDEs) differ from stochastic differential equations (SDEs) at least in the two aspects: (1) the boundary conditions of BSDEs are given at the terminal time, while the boundary conditions of SDEs are given at the initial time; (2) the solutions of BSDEs consist of a pair of processes, while the solutions of SDEs consist of a single process.

BSDEs were initially formulated by Bismut [6–10] when he studied the stochastic maximum principle for optimal stochastic controls. They appeared in a linear form as the adjoint equations in the stochastic maximum principle. The general Lipschitz nonlinear case was solved by Pardoux and Peng [25]. The Riccati equation associated with a linear quadratic optimal stochastic control problem was also suitably formulated by Bismut [9] as a special nonlinear (but not Lipschitz) BSDE. The solution of the Riccati equation had been left

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open for a long time, and it has been solved recently by Tang [34] for the interesting case of Brownian motion-driven stochastic differential systems. A detailed systematic account of the theory and applications of BSDEs is available in El Karoui, Peng and Quenez [13], Pardoux [24], and Yong and Zhou [35].

Backward stochastic partial differential equations (SPDEs), which are a natural generalization of BSDEs, arise in many applications of probability theory and stochastic processes, for instance in the optimal control of processes with incomplete information, as an adjoint equation of the Duncan-Mortensen-Zakai filtration equation. For example, see [4, 5, 15, 29, 36, 37, 21, 32, 33]. A class of fully nonlinear BSPDEs, the so-called backward stochastic Hamilton-Jacobi-Bellman equations, are also introduced in the study of controlled non-markovian processes by Peng [29].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which is defined a d-dimensional standard Brownian motion $w = \{w(t) : t \in [0,T]\}$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by w and augmented by all the P-null sets in $\mathcal{F}$. In this paper, we consider the following semi-linear system of BSPDEs:

$$\begin{cases} du(t,x) = -\mathcal{L}(t,x)u(t,x) - \mathcal{M}v - f(t,x,u,v+\partial u\sigma) \\ \qquad\qquad + \displaystyle\sum_{r=1}^{d} v^{\cdot r}(t,x)\, dw_t^r, \qquad (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(T,x) = g(x), \qquad\qquad\qquad x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here we have defined

$$\begin{aligned} &\partial_i := \frac{\partial}{\partial x^i}, \qquad \partial_{ij}^2 := \frac{\partial^2}{\partial x^i \partial x^j}, \\ &\mathcal{L}(t,x) := \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=1}^{d} \sigma^{ir}\sigma^{jr}(t,x)\partial_{ij}^2 + \sum_{i=1}^{n} b^i(t,x)\partial_i, \\ &\mathcal{M}^r(t,x) := \sum_{i=1}^{n} \sigma^{ir}(t,x)\frac{\partial}{\partial x^i}, \\ &\partial u := (\partial_j u^i)_{1\leq i\leq m, 1\leq j\leq n}, \qquad \mathcal{L}u := (\mathcal{L}u^1, \cdots, \mathcal{L}u^n)', \\ &\mathcal{M}v := \Big(\sum_{r=1}^{d}\mathcal{M}^r v^{1r}, \cdots, \sum_{r=1}^{d}\mathcal{M}^r v^{mr}\Big)'. \end{aligned} \tag{1.2}$$

Our aim is to find a pair of random fields $(u,v) : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^m \times \mathbb{R}^{m\times d}$ in suitable spaces such that the system (1.1) is satisfied in some sense.

The solution will be constructed by using the solutions of both SDEs and BSDEs in the following way. Consider the SDE (its solution will be denoted by $X_\cdot(t,x)$):

$$\begin{cases} dX_s = b(s,X_s)\, ds + \displaystyle\sum_{r=1}^{d} \sigma^{\cdot r}(s,X_s)\, dw_s^r, \qquad t \leq s \leq T, \\ X_t = x. \end{cases} \tag{1.3}$$

and the following BSDE (its adapted solution will be denoted by $(Y_\cdot(t,x), Z_\cdot(t,x))$):

$$\begin{cases} dY_s = -f(s, X_s(t,x), Y_s, Z_s)\, ds + \sum_{r=1}^{d} Z_s^{;r}\, dw_s^r, \qquad t \le s \le T, \\ Y_T = g(X_T(t,x)). \end{cases} \tag{1.4}$$

Under suitable assumptions on the coefficients b, σ, f, and g of Equations (1.3) and (1.4), Equation (1.3) defines a stochastic flow $X_s(t, \cdot)$, which can be proved to have an inverse $X_s^{-1}(t, \cdot)$, and Equation (1.4) defines a pair of random fields $\{(Y_s(t,x), Z_s(t,x)); (s,x) \in [t,T] \times \mathbb{R}^n\}$. Then, it can be shown by using a suitable form of generalized Itô's formula that the pair of random fields (u, v) defined by

$$\begin{aligned} u(t,x) &:= Y_t(0, X_t^{-1}(0,x)), \\ v(t,x) &:= Z_t(0, X_t^{-1}(0,x)) - \partial u(t,x)\sigma(t,x) \end{aligned} \tag{1.5}$$

is a unique solution of the system (1.1). In this way, we can construct both parts of the solution of the system (1.1) using SDE (1.3) and BSDE (1.4). Furthermore, with the help of the SDE of ∂X and the BSDE of $(\partial Y, \partial Z)$, we could obtain various estimates of (u, v) defined by (1.5).

The above-described probabilistic point of view seems to be new. It permits us to study the system of BSPDEs (1.1) via the properties of SDEs and BSDEs depending on a parameter, which have been well studied (see for example [11, 17–19, 26]). The link plays a crucial role, and constitutes a distinct feature of the paper.

The previous method to BSPDEs is analytic (see [4, 20, 21, 36, 37]): the main feature is to analyze the involved differential operators so as to get some a priori estimates. The existence proof is based on a finite-dimensional approximation. Instead, we use stochastic flows (defined by an Itô differential equation) and a pair of random fields (as the solutions of the associated BSDEs) to construct and then to estimate the solution. In this way we could avoid the analyses on differential operators and the limiting arguments presented in [4, 20, 21, 36, 37]. However, it is worth noting that the method of stochastic flows has already been used to study stochastic partial differential equations, and the reader is referred to [1, 2].

Our method also provides new results. It allows us to treat a more general class of BSPDEs than that in [20–22]. Since Hu, Ma, and Yong [14] appeared, it has been very challenging to solve semi-linear BSPDEs for arbitrary finite state dimension n. We provide an answer to this problem. Note that Hu and Peng [15] also discussed a class of nonlinear BSPDEs but in the language of semigroups.

The rest of this paper is organized as follows. Notations and known results and tools are collected in Section 2. In Section 3, the pair of random fields generated by a BSDE are studied in detail, and some new properties are derived. They guarantee the applicability of the generalized Itô's formula, and are the basis of the subsequent arguments. In Section 4, a new pair of random fields are constructed by using the pair of random fields generated by a BSDE and the inverse flow. Some useful bounded estimates are obtained. In Section 5, we show that there is unique adapted classical solution for BSPDEs. Section 6 is devoted to the comparison property of BSPDEs, which is reduced to that of BSDEs. In general, our

BSPDEs (1.1) is degenerate parabolic in the sense of [22, Definition 1.1, Chapter 5, p.104]. However, it may be super-parabolic by restricting the random sources of the coefficients to some partial components of the Brownian motion w. The details are explained in Section 7. Section 8 contains some remarks.

§ 2 . Preliminaries

Notations

Let $\mathcal{A} := \{\alpha = (\alpha_1, \cdots, \alpha_n) : \alpha_i, i = 1, \cdots, n$ are nonnegative integers$\}$ be the set of multi-indices. For any $\alpha \in \mathcal{A}$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, denote

$$|\alpha| := \sum_{i=1}^n \alpha_i, \qquad \partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

The inner product in a Euclidean space $\mathbb{E}$ is denoted by $\langle \cdot, \cdot \rangle$, and the norm in $\mathbb{E}$ by $|\cdot|_{\mathbb{E}}$ or simply by $|\cdot|$ when there is no confusion.

Let $\mathbb{B}$ be a Banach space, k be a positive integer, and $a \in [0, 1]$. We say that a mapping $f : \mathbb{R}^n \to \mathbb{B}$ is of class $C^{k,\alpha}$ or belongs to $C^{k,\alpha}(\mathbb{R}^n; \mathbb{B})$ if f is k-times continuously differentiable and if all the partial derivatives of order k are a-Hölder continuous on $\mathbb{R}^n$. Write for $f \in C^{k,0}(\mathbb{R}^n; \mathbb{B})$,

$$\|f\|_{C^{k,0}} := \sup_{x \in \mathbb{R}^n} \|f(x)\|_{\mathbb{B}} + \sum_{1 \le |\alpha| \le k} \sup_{x \in \mathbb{R}^n} \|\partial^\alpha f(x)\|_{\mathbb{B}}, \tag{2.1}$$

and for $f \in C^{k,a}(\mathbb{R}^n; \mathbb{B})$,

$$\|f\|_{C^{k,a}} := \|f\|_{C^{k,0}} + \sum_{|\alpha| = k} \sup_{x, \tilde{x} \in \mathbb{R}^n} \frac{\|\partial^\alpha f(x) - \partial^\alpha f(\tilde{x})\|_{\mathbb{B}}}{|x - \tilde{x}|^a}. \tag{2.2}$$

$L_n^\infty(\mathbb{E}) := L^\infty(\mathbb{R}^n; \mathbb{E})$ for a Euclidean space $\mathbb{E}$, $L_{n,m}^\infty := L_n^\infty(\mathbb{R}^m)$, and $L_{n,m\times d}^\infty := L_n^\infty(\mathbb{R}^{m\times d})$.

For any integer $k \ge 1$, $M_{\otimes k}^n(\mathbb{E})$ is the space consisting of all the bounded multi-linear maps from $(\mathbb{R}^n)^{\otimes^k}$ to Euclidean space $\mathbb{E}$. $M_{\otimes k}^n(\mathbb{R}^m)$ and $M_{\otimes k}^n(\mathbb{R}^{m\times d})$ will be abbreviated as $M_{\otimes k}^{n,m}$ and $M_{\otimes k}^{n,m\times d}$, respectively. Define $M_{\otimes 0}^{n,m} := \mathbb{R}^m$ and $M_{\otimes 0}^{n,m\times d} := \mathbb{R}^{m\times d}$. Note that $M_{\otimes 1}^{n,m}$ is identical to $\mathbb{R}^{m\times n}$.

For an integer k, a Banach space $\mathbb{B}$, and $u \in C^k(\mathbb{R}^n; \mathbb{B})$, we use (for any $x \in \mathbb{R}^n$) $\partial^k u(x)$ to denote the multi-linear map of $(\mathbb{R}^n)^{\otimes^k}$ into $\mathbb{B}$ which is derived from the partial derivatives of order k of u at x, and use $\partial^k u(x_0)(x_1, \cdots, x_k)$ to denote the image of the map $\partial^k u : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{B}$ at $(x_0, x_1, \cdots, x_k)$. Throughout the paper, for any integer $k > 1$, $\partial^k u\sigma$ is meant by $\sum_{i=1}^n \partial^{k-1}\partial_i u \sigma^{i\cdot}$ where $\sigma^{i\cdot}$ is the i-th row of σ.

For a given Banach space $\mathbb{B}$ and a real number $p > 1$, denote by $\mathcal{L}_{\mathcal{F}}^{\infty,p}(0, T; \mathbb{B})$ the Banach space of all $\mathbb{B}$-valued adapted continuous processes X which satisfy the following

$$\|X\|_{\mathcal{L}_{\mathcal{F}}^{\infty,p}(0,T;\mathbb{B})} := \Big(E \sup_{0 \le t \le T} \|X(t)\|_{\mathbb{B}}^p\Big)^{1/p} < \infty,$$

and denote by $\mathcal{L}^{k,p}_{\mathcal{F}}(0,T;\mathbb{B})$ ($k\in[1,\infty)$) the set of $\mathbb{B}$-valued adapted processes Z satisfying the following

$$\|X\|_{\mathcal{L}^{k,p}_{\mathcal{F}}(0,T;\mathbb{B})} := \Big(E\Big(\int_0^T \|Z(t)\|^k_{\mathbb{B}}\,dt\Big)^{p/k}\Big)^{1/p} < \infty.$$

$\mathcal{L}^{p}_{\mathcal{F}}(0,T;\mathbb{B}) := \mathcal{L}^{p,p}_{\mathcal{F}}(0,T;\mathbb{B})$ with $p\in(1,\infty]$. $L^p(\mathcal{F}_T;\mathbb{B})$ is the Banach space consisting of $\mathbb{B}$-valued $\mathcal{F}_T$-measurable L^p-integrable variables.

Consider the following SDE:

$$\begin{cases} dX_s = b(s,X_s)\,ds + \sigma(s,X_s)\,dw_s, & t\le s\le T,\\ X_t = x. \end{cases} \tag{2.3}$$

We introduce the following assumption.

$(C1)_k$ The coefficients σ and b satisfy the following

$$\sigma\in\mathcal{L}^{\infty}_{\mathcal{F}}(0,T;C^k(\mathbb{R}^n;\mathbb{R}^{n\times d})) \quad\text{and}\quad b\in\mathcal{L}^{\infty}_{\mathcal{F}}(0,T;C^k(\mathbb{R}^n;\mathbb{R}^n)). \tag{2.4}$$

Note that $(C1)_k$ implies that the partial derivatives in x of σ and b up to order k are uniformly bounded in (ω,t,x) by a common positive constant. Under $(C1)_1$, SDE (2.3) has a unique strong solution, which will be denoted by $\{X_s(t,x), t\le s\le T\}$. Set $X_s(x):=X_s(0,x)$ for $s\in[0,T]$.

Lemma 2.1. (see [16, Theorem 2.8.6]) *Suppose that $(C1)_k$ is satisfied for some positive integer k. Then, the process $X\in C^k(\mathbb{R}^n;\mathcal{L}^{\infty,p}(0,T;\mathbb{R}^n))$ for any $p\ge 2$. Moreover, the gradient ∂X of $X_t(x)$ satisfies the following SDE:*

$$\begin{cases} \partial X_t(y) = b_x(t,X_t(y))\partial X_t(y)\,dt + \sigma_x(t,X_t(y))\partial X_t(y)\ dw_t,\\ \partial X_0(y) = I_{n\times n}. \end{cases} \tag{2.5}$$

Lemma 2.2. (see [17]) *Suppose that $(C1)_k$ is satisfied for some positive integer k. For all ω from some subset of probability 1 and for each $t\in[0,T]$ the map $X_t(0,\cdot): x\in\mathbb{R}^n\to X_t(0,x)\in\mathbb{R}^n$ is a diffeomorphism of class C^{k-1} of $\mathbb{R}^n$ into itself, and if $k\ge 3$, the inverse map $X_t^{-1}(0,x)$ solves the following SPDE:*

$$\begin{cases} du(t,x) = (\mathcal{M}^r\mathcal{M}^r-\mathcal{L})(t,x)u(t,x)\,dt - \mathcal{M}^r(t,x)u(t,x)\,dw^r_t, & 0\le t\le T,\\ u(0,x)=x. \end{cases} \tag{2.6}$$

Lemma 2.3. (see [18, Exercise 4.6.8]) *Let $(C1)_1$ be satisfied. Then $[\partial X_t(y)]^{-1}$ exists and satisfies the following SDE:*

$$\begin{cases} [\partial X_0(y)]^{-1} = I_{n\times n},\\ d[\partial X_t(y)]^{-1} = -[\partial X_t(y)]^{-1}\{b_x(t,X_t(y)) - [\sigma_x(t,X_t(y))]^2\}\,dt\\ \qquad\qquad\qquad -[\partial X_t(y)]^{-1}\sigma_x(t,X_t(y))\,dw_t, \qquad 0\le t\le T. \end{cases} \tag{2.7}$$

Let us now recall some basic results (existence and uniqueness, a priori estimates, comparison theorem, and the Malliavin differentiability) of BSDEs, which will be used in our subsequent arguments. They are more or less well known now.

Consider the following BSDE:

$$\begin{cases} dy_t = -f(t, y_t, z_t)\, dt + z_t\, dw_t, \\ y_T = \xi. \end{cases} \tag{2.8}$$

Here f is the generator and ξ is the terminal condition. Its solution consists of a pair of adapted processes (y, z). The BSDE whose generator and terminal condition are f and ξ respectively will be referred to as BSDE (f, ξ).

Lemma 2.4. (see [13, Theorem 5.1]) *Let the predictable random fields $f(t, y, z)$ be Lipschitz in (y, z), uniformly with respect to (t, y, z). Moreover assume that the generator f and the terminal condition ξ satisfy the following for some $p > 1$,*

$$f(\cdot, 0, 0) \in \mathcal{L}^p_{\mathcal{F}}(0, T; \mathbb{R}^m), \qquad \xi \in L^p(\mathcal{F}_T; \mathbb{R}^m). \tag{2.9}$$

Then, BSDE (2.8) *has a unique adapted solution $(y, z) \in \mathcal{L}^{\infty,p}_{\mathcal{F}}(0, T; \mathbb{R}^m) \times \mathcal{L}^{2,p}_{\mathcal{F}}(0, T; \mathbb{R}^{m\times d})$.*

Lemma 2.5. (see [13, Proposition 5.1]) *Let the predictable random fields $f(t, y, z)$ be Lipschitz in (y, z), uniformly with respect to (t, y, z). Moreover assume that the generator f and the terminal condition ξ satisfy* (2.9) *for some $p > 1$. Then an adapted solution (y, z) of BSDE* (2.8) *satisfies the following*

$$\|y.\|^p_{\mathcal{L}^{\infty,p}_{\mathcal{F}}(t,T;\mathbb{R}^m)} + E\int_t^T |y_s|^{p-2}|z_s|^2\, ds \le C_p\{\|f(\cdot, 0, 0)\|^p_{\mathcal{L}^p_{\mathcal{F}}(t,T;\mathbb{R}^m)} + \|\xi\|^p_{L^p(\mathcal{F}_T;\mathbb{R}^m)}\} \tag{2.10}$$

and

$$\|z.\|^p_{\mathcal{L}^{2,p}_{\mathcal{F}}(t,T;\mathbb{R}^{m\times d})} \le C_p\{\|f(\cdot, 0, 0)\|^p_{\mathcal{L}^p_{\mathcal{F}}(t,T;\mathbb{R}^m)} + \|\xi\|^p_{L^p(\mathcal{F}_T;\mathbb{R}^m)}\} \tag{2.11}$$

for some positive constant C_p, which depends on p.

We can check that all the expectation operators involved in the estimates (2.10) and (2.11) of Lemma 2.5 may be replaced with the conditional expectation with respect to $\mathcal{F}_t$. That is, we have

Lemma 2.6. *Let the assumptions of Lemma* 2.5 *be satisfied. Then an adapted solution (y, z) of BSDE* (2.8) *satisfies the following*

$$E^{\mathcal{F}_t} \sup_{t\le s\le T} |y_s|^p + E^{\mathcal{F}_t}\int_t^T |y_s|^{p-2}|z_s|^2\, ds \le C\Big\{E^{\mathcal{F}_t}|\xi|^p + E^{\mathcal{F}_t}\Big(\int_t^T |f(s, 0, 0)|\, ds\Big)^p\Big\} \tag{2.12}$$

and

$$E^{\mathcal{F}_t}\Big(\int_t^T |z_s|^2\, ds\Big)^{p/2} \le C_p\Big\{E^{\mathcal{F}_t}|\xi|^p + E^{\mathcal{F}_t}\Big(\int_t^T |f(s, 0, 0)|\, ds\Big)^p\Big\} \tag{2.13}$$

for some positive constant C_p. In particular, we have

$$|y_t|^p \le C_p\Big\{E^{\mathcal{F}_t}|\xi|^p + E^{\mathcal{F}_t}\Big(\int_t^T |f(s, 0, 0)|\, ds\Big)^p\Big\}. \tag{2.14}$$

Lemma 2.7. (see [13, Theorem 2.2]) *Let $m = 1$. Consider two BSDEs (f_1, ξ_1) and (f_2, ξ_2) whose generators f_1 and f_2 are predictable random fields and are Lipschitz in (y, z), uniformly with respect to all the arguments. Moreover, assume that $f_i(\cdot, 0, 0) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R})$ and $\xi_i \in L^2(\mathcal{F}_T; \mathbb{R})$ for $i = 1, 2$. Let (y_1, z_1) and (y_2, z_2) be the adapted solutions of BSDEs (f_1, ξ_1) and (f_2, ξ_2), respectively. If $f_1(\omega, t, y, z) \geq f_2(\omega, t, y, z)$ for a.s.a.e., $(\omega, t) \in \Omega \times [0, T]$ and any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, and if $\xi_1 \geq \xi_2$ almost surely, then $y_1(t) \geq y_2(t)$ almost surely for any $t \in [0, T]$.*

§ 3. Random Fields Generated by BSDEs

Consider the following BSDE:

$$\begin{cases} dY_t = -f(t, X_t(x), Y_t, Z_t)\, dt + Z_t\, dw_t, \qquad 0 \leq t < T; \\ Y_T = g(X_t(x)). \end{cases} \tag{3.1}$$

Obviously, the adapted solution $\{(Y_t, Z_t), 0 \leq t \leq T\}$ depends on the parameter x, and thus will be denoted by $\{(Y_t(x), Z_t(x)) : (t, x) \in [0, T] \times \mathbb{R}^n\}$ or $\{(Y(x, t), Z(x, t)) : (t, x) \in [0, T] \times \mathbb{R}^n\}$ when necessary or convenient.

We introduce the following assumption.

$(C2)_k$ The coefficients f and g satisfy the following

$$f \in \mathcal{L}^\infty_{\mathcal{F}}(0, T; C^k(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}; \mathbb{R}^m)) \quad \text{and} \quad g \in L^\infty(\mathcal{F}_T; C^k(\mathbb{R}^n; \mathbb{R}^m)). \tag{3.2}$$

Lemma 3.1. (see [26]) *Suppose that $(C1)_k$ and $(C2)_k$ are satisfied for $k = 1$ or $k = 2$. Let $(Y(x), Z(x))$ be the unique adapted solution of BSDE (3.1). Then $Y \in C^k(\mathbb{R}^n; \mathcal{L}^{\infty,p}_{\mathcal{F}}(0, T; \mathbb{R}^m))$ and $Z \in C^k(\mathbb{R}^n; \mathcal{L}^{2,p}_{\mathcal{F}}(0, T; \mathbb{R}^{m \times d}))$ for any $p \geq 2$. For any $p \geq 2$ and any nonnegative integer $\beta \leq k - 1$, we have*

$$\begin{aligned} &\|\partial^\beta Y_\cdot(x_1) - \partial^\beta Y_\cdot(x_2)\|^p_{\mathcal{L}^{\infty,p}_{\mathcal{F}}(t,T;M^{n,m}_{\otimes\beta})} \\ &\quad + E\int_t^T |\partial^\beta Y_s(x_1) - \partial^\beta Y_s(x_2)|^{p-2}_{M^{n,m}_{\otimes\beta}} |\partial^\beta Z_s(x_1) - \partial^\beta Z_s(x_2)|^2_{M^{n,m\times d}_{\otimes\beta}}\, ds \\ &\leq C_p|x_1 - x_2|^p, \qquad \forall\, x_1, x_2 \in \mathbb{R}^n \end{aligned} \tag{3.3}$$

and

$$\|\partial^\beta Z_\cdot(x_1) - \partial^\beta Z_\cdot(x_2)\|^p_{\mathcal{L}^{2,p}_{\mathcal{F}}(t,T;M^{n,m\times d}_{\otimes\beta})} \leq C_p|x_1 - x_2|^p, \qquad \forall\, x_1, x_2 \in \mathbb{R}^n. \tag{3.4}$$

Furthermore, the gradient $(\partial Y, \partial Z)$ satisfies the following BSDE:

$$\begin{cases} \partial Y_t(y) = \{-f_x(\Xi^y_t)\partial X_t(y) - f_y(\Xi^y_t)\partial Y_t(y) - f_z(\Xi^y_t)\partial Z_t(y)\}\, dt + \partial Z_t(y)\, dw_t, \\ \partial Y_T(y) = \partial g(X_T(y))\partial X_T(y), \end{cases} \tag{3.5}$$

where we have used the notation $\Xi^y_t := (t, X_t(y), Y_t(y), Z_t(y))$ and $\partial Z_t(y)\, dw_t := \sum_{r=1}^d \partial Z^r_t(y)$ dw^r_t. We have the following estimates:

$$\|\partial Y_\cdot(x)\|^p_{\mathcal{L}^{\infty,p}_{\mathcal{F}}(t,T;\mathbb{R}^{m\times n})} + E\int_t^T |\partial Y_s(x)|^{p-2}_{\mathbb{R}^{m\times n}} |\partial Z_s(x)|^2_{M^{n,m\times d}_{\otimes 1}}\, ds \leq C_p, \qquad \forall\, x \in \mathbb{R}^n \tag{3.6}$$

and

$$\|\partial Z_\cdot(x)\|^p_{\mathcal{L}^{2,p}_{\mathcal{F}}(t,T;M^{n,m\times d}_{\otimes 1})} \le C_p, \qquad \forall\, x \in \mathbb{R}^n. \tag{3.7}$$

We introduce the following assumption.

$(C3)$ The function $f(t,x,y,z)$ is linear in z with f_z being bounded and being independent of (x,y,z).

We have

Theorem 3.1. *Suppose that $(C1)_k$ and $(C2)_k$ are satisfied for some positive integer k and that $(C3)$ is satisfied. Then if $(Y(x),Z(x))$ is the unique adapted solution of BSDE (3.1), we have $Y \in C^k(\mathbb{R}^n;\mathcal{L}^{\infty,p}_{\mathcal{F}}(0,T;\mathbb{R}^m))$ and $Z \in C^k(\mathbb{R}^n;\mathcal{L}^{2,p}_{\mathcal{F}}(0,T;\mathbb{R}^{m\times d}))$ for any $p \ge 2$. For any $p \ge 2$ and any nonnegative integer $\beta \le k-1$, we have*

$$\begin{aligned}
&\|\partial^\beta Y_\cdot(x_1) - \partial^\beta Y_\cdot(x_2)\|^p_{\mathcal{L}^{\infty,p}_{\mathcal{F}}(t,T;M^{n,m}_{\otimes\beta})}\\
&\quad + E\int_t^T |\partial^\beta Y_s(x_1) - \partial^\beta Y_s(x_2)|^{p-2}_{M^{n,m}_{\otimes\beta}} |\partial^\beta Z_s(x_1) - \partial^\beta Z_s(x_2)|^2_{M^{n,m\times d}_{\otimes\beta}}\, ds\\
&\le C_p|x_1-x_2|^p, \qquad \forall\, x_1,x_2 \in \mathbb{R}^n
\end{aligned} \tag{3.8}$$

and

$$\|\partial^\beta Z_\cdot(x_1) - \partial^\beta Z_\cdot(x_2)\|^p_{\mathcal{L}^{2,p}_{\mathcal{F}}(t,T;M^{n,m\times d}_{\otimes\beta})} \le C_p|x_1-x_2|^p, \qquad \forall\, x_1,x_2 \in \mathbb{R}^n. \tag{3.9}$$

Proof. We use the principle of induction. As $k=1$, Theorem 3.1 is true in view of Lemma 3.1. Suppose that Theorem 3.1 is true for integer $k \ge 1$.

We have

$$\begin{cases}
\partial^k Y_t(x) = -\{f_y(t,X_t(x),Y_t(x))\partial^k Y_t(x) + f_z(t)\partial^k Z_t(x)\}\,dt\\
\qquad\qquad -\{f_x(t,X_t(x),Y_t(x))\partial^k X_t(x) + \mathcal{P}_{k-1}(t,x)\}\,dt + \partial^k Z_t(x)\,dw_t,\\
\partial^k Y_T(x) = \partial^k[g(X_T(x))].
\end{cases} \tag{3.10}$$

Here $\mathcal{P}_k(t,x)$ belongs to $M^{n,m}_{\otimes k}$, and it takes values in $\mathbb{R}^m$ whose components are polynomials of the partial derivatives up to order k of the components of $X_t(x)$ and $Y_t(x)$. From the induction assumption, we easily see that $\mathcal{P}_{k-1} \in C^1(\mathbb{R}^n;\mathcal{L}^{\infty,p}_{\mathcal{F}}(t,T;M^{n,m}_{\otimes k}))$ and moreover,

$$\|\mathcal{P}_{k-1}(\cdot,x_1) - \mathcal{P}_{k-1}(\cdot,x_2)\|^p_{\mathcal{L}^{\infty,p}_{\mathcal{F}}(t,T;M^{n,m}_{\otimes k})} \le C_p|x_1-x_2|^p.$$

In view of Lemma 3.1 and the fact that $g \in C^{k+1}(\mathbb{R}^n;\mathbb{R}^m)$, we also see that $\partial^k[g(X_T)] \in C^1(\mathbb{R}^n;L^p(0,T;\mathbb{R}^m))$. Therefore the desired result follows.

Now we establish the general $C^{m,\alpha}$-estimates for the solutions.

Theorem 3.2. *Let $(C1)_k$ and $(C2)_k$ be satisfied with $k > l + n/2$ for some positive integer $l \ge 2$, and $(C3)$ be satisfied. For any compact subset K, we have*

$$E\sup_{0\le t\le T}\|Y_t(\cdot)\|^2_{C^{k-1,\alpha};K} < \infty \quad \textit{and} \quad E\int_0^T \|Z_t(\cdot)\|^2_{C^{l,\alpha};K}\,dt < \infty. \tag{3.11}$$

Therefore, $Y_t(x)$ is a continuous C^{k-1}-process and a continuous C^l-semimartingale. The characteristic of $Y_t(x)$ is $(f(t,X_t(x),Y_t(x),Z_t(x)),Z_t(x)Z_t(y)')$ satisfying

$$\begin{aligned} &E\int_0^T \sup_{x\in K} |\partial^\beta f(t,X_t(x),Y_t(x),Z_t(x))|_{\mathbb{R}^m}\,dt<\infty,\\ &E\int_0^T \sup_{x,y\in K} |\partial_x^\beta\partial_y^\beta Z_t(x)Z_t(y)'|_{\mathbb{R}^{m\times m}}\,dt<\infty \end{aligned} \tag{3.12}$$

for any $\beta\in\mathcal{A}$ such that $|\beta|\le l$, and any compact subset $K\subset\mathbb{R}^n$.

Proof. It is immediate consequences of [18, Theorem 1.4.1], [31, Lemma 1] and Theorem 3.1.

Definition 3.1. *Define*

$$\begin{aligned} u(t,x)&:=Y_t(0,X_t^{-1}(0,x)),\\ v^{,r}(t,x)&:=Z_t^{,r}(0,X_t^{-1}(0,x))-\partial u(t,x)\sigma^{,r}(t,x),\qquad r=1,2,\cdots,d, \end{aligned} \tag{3.13}$$

where $\partial u:=(\partial_1 u,\cdots,\partial_n u)$.

§ 4. Bounded Estimates for (u,v) and Their Derivatives

In this section, we shall establish various bounded estimates for the pair of random fields (u,v) with the help of SDE for ∂X and BSDE for $(\partial Y,\partial Z)$. The key point is that $\partial u(\cdot,X.)$ and $\partial^2 u(\cdot,X.)$ are governed by two ordinary BSDEs, which can be obtained with SDE (2.3) and BSDE (3.1). Let C_p be a universal positive constant depending on p. Note that it also may depend on the bounds of the coefficients b,σ,f, and g and their partial derivatives of suitable orders depending on the situations.

4.1. Bounded estimate of (u,v)

We have

Lemma 4.1. *If $\phi\in\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m})$, then $\phi(\cdot,X)$ and $\phi(\cdot,X^{-1})$ lie in $\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m})$.*

Proof. For any $\psi\in\mathcal{L}^1_{\mathcal{F}}(0,T;C(\mathbb{R}^n;\mathbb{R}^m))$, we have

$$\begin{aligned} \int_0^T\int_{\mathbb{R}^n}\langle\phi(t,X_t^{-1}(x)),\psi(t,x)\rangle\,dxdt&=\int_0^T\int_{\mathbb{R}^n}\langle\phi(t,y),\psi(t,X_t(y))\rangle\det(\partial X(y))\,dydt\\ &\le\|\phi\|_{\mathcal{L}^\infty_{\mathcal{F}}}\int_0^T\int_{\mathbb{R}^n}|\psi(t,X_t(y))|\det(\partial X(y))\,dydt\\ &=\|\phi\|_{\mathcal{L}^\infty_{\mathcal{F}}}\int_0^T\int_{\mathbb{R}^n}|\psi(t,x)|\,dxdt=\|\phi\|_{\mathcal{L}^\infty_{\mathcal{F}}}\|\psi\|_{\mathcal{L}^1}. \end{aligned}$$

Therefore, $\phi(\cdot,X^{-1})\in\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m})$. It is proved in an identical way that

$$\phi(\cdot,X)\in\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m}).$$

Theorem 4.1. *Let* $(C1)_1$ *be satisfied. Let the predictable field* $f(t,x,y,z)$ *be Lipschitz in* (y,z)*, uniformly with respect to* (t,x,y,z)*. Assume that*

$$g \in L^\infty(\mathcal{F}_T; L^\infty_{n,m}) \quad \text{and} \quad f(\,\cdot\,,\,\cdot\,,0,0) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m}). \tag{4.1}$$

Then for any $p > 1$*, we have*

$$\|Y_t(x)\|^p_{L^\infty(\mathcal{F}_t;\mathbb{R}^m)} \le C_p\{\|g\|^p_{L^\infty(\mathcal{F}_T;L^\infty_{n,m})} + \|f(\,\cdot\,,\,\cdot\,,0,0)\|^p_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m})}\} \tag{4.2}$$

and

$$E^{\mathcal{F}_t}\Big(\int_t^T |Z_s(x)|^2\,ds\Big)^{p/2} \le C_p\{\|g\|^p_{L^\infty(\mathcal{F}_T;L^\infty_{n,m})} + \|f(\,\cdot\,,\,\cdot\,,0,0)\|^p_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m})}\}. \tag{4.3}$$

Proof. In view of Lemma 4.1, the assumption (4.1) implies that

$$g(X_T) \in L^\infty(\mathcal{F}_T; L^\infty_{n,m}) \quad \text{and} \quad f(\,\cdot\,,X.,0,0) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m}). \tag{4.4}$$

Then from Lemma 2.6, we deduce Theorem 4.1.

In view of Definition 3.1, we have $u(t,X_t) = Y_t$. Therefore, we have

Theorem 4.2. *Let* $(C1)_1$ *be satisfied. Let the predictable field* $f(t,x,y,z)$ *be Lipschitz in* (y,z)*, uniformly with respect to* (t,x,y,z)*. Assume that*

$$g \in L^\infty(\mathcal{F}_T; L^\infty_{n,m}) \quad \text{and} \quad f(\,\cdot\,,\,\cdot\,,0,0) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m}).$$

Then we have

$$\|u\|^p_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m})} \le C_p\{\|g\|^p_{L^\infty(\mathcal{F}_T;L^\infty_{n,m})} + \|f(\,\cdot\,,\,\cdot\,,0,0)\|^p_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m})}\}. \tag{4.5}$$

4.2. Gradient estimates

Write

$$\begin{aligned} \tilde\partial Y_t(y) &:= \partial Y_t(y)[\partial X_t(y)]^{-1}, \\ \tilde\partial Z_t^{;r}(y) &:= \partial Z_t^{;r}(y)[\partial X_t(y)]^{-1}, \qquad r = 1,\cdots,d. \end{aligned} \tag{4.6}$$

In view of Lemma 2.3 and BSDE (3.5), we can use Itô's formula to verify the following lemma.

Lemma 4.2. *Let* $(C1)_1$ *be satisfied. Let the predictable random fields* $f \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;$ $C^1(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d};\mathbb{R}^m))$*, and the partial derivative* $f_x(t,x,y,z)$ *be Lipschitz in* (y,z)*, uniformly with respect to* (t,x,y,z)*. Let* $g \in L^\infty(\mathcal{F}_T;C^1(\mathbb{R}^n;\mathbb{R}^m))$*. Then the pair* $(\tilde\partial Y_t(y), \tilde\partial Z_t^{;r}(y), r = 1,\cdots,d)$ *satisfies the following BSDE:*

$$\begin{cases} d\tilde\partial Y_t(y) = \{-f_x(\Xi^y_t) - f_y(\Xi^y_t)\tilde\partial Y_t(y) - f_z(\Xi^y_t)\tilde\partial Z_t(y) - \tilde\partial Y_t(y) b_x \\ \qquad\qquad - [\tilde\partial Z_t^{;r}(y) - \tilde\partial Y_t(y)\sigma_x^{;r}]\sigma_x^{;r}\}\,dt + [\tilde\partial Z_t^{;r}(y) - \tilde\partial Y_t(y)\sigma_x^{;r}]\,dw_t^r, \\ \tilde\partial Y_T(y) = \partial g(X_T(y)). \end{cases} \tag{4.7}$$

Note that the arguments in b_x and $\sigma_x^{\cdot r}$ are $(\omega, t, X_t(y))$, which have been omitted here for simplicity.

In view of Lemma 2.6, we immediately have

Theorem 4.3. *Let $(C1)_k$ and $(C2)_k$ be satisfied for $k > 2 + \frac{n}{2}$, and $(C3)$ be satisfied. Then, we have*

$$\tilde{\partial} Y \in \mathcal{L}_{\mathcal{F}}^{\infty}(0, T; L_{n,m\times n}^{\infty}), \tag{4.8}$$

and for any $p > 1$, the map

$$(\omega, t, x) \mapsto E^{\mathcal{F}_t}\Big(\int_t^T |\tilde{\partial} Z_s(x)|^2\, ds\Big)^{p/2} \quad \text{lies in } \mathcal{L}_{\mathcal{F}}^{\infty}(0, T; L_{n,1}^{\infty}). \tag{4.9}$$

Lemma 4.3. *Let $(C1)_1$ and $(C2)_1$ be satisfied for $k > 2 + \frac{n}{2}$, and $(C3)$ be satisfied. We have*

$$\begin{aligned} \partial u(t, X_t(y)) &= \tilde{\partial} Y_t(y), \\ \partial v^{\cdot r}(t, X_t(y)) &= \tilde{\partial} Z_t^{\cdot r}(y) - \partial\partial u(t, X_t(y))\sigma^{\cdot r}(t, X_t(y)) - \partial u(t, X_t(y))\sigma_x^{\cdot r}(t, X_t(y)). \end{aligned} \tag{4.10}$$

Proof. Since

$$u(t, X_t(y)) = Y_t(y),$$

we have by differentiating both sides

$$\partial u(t, X_t(y))\partial X_t(y) = \partial Y_t(y). \tag{4.11}$$

This gives the first identity of the lemma.

Since

$$v^{\cdot r}(t, X_t(y)) = Z_t^{\cdot r}(y) - \partial u(t, X_t(y))\sigma^{\cdot r}(t, X_t(y)),$$

we have by differentiating both sides

$$\begin{aligned} \partial v^{\cdot r}(t, X_t(y))\partial X_t(y) = {}& \partial Z_t^{\cdot r}(y) - \partial\partial u(t, X_t(y))\partial X_t(y)\sigma^{\cdot r}(t, X_t(y)) \\ & - \partial u(t, X_t(y))\sigma_x^{\cdot r}(t, X_t(y))\partial X_t(y). \end{aligned} \tag{4.12}$$

The last identity, multiplied with $[\partial X_t(y)]^{-1}$, implies the last identity of this lemma.

Theorem 4.4. *Let $(C1)_k$ and $(C2)_k$ be satisfied for $k > 2 + \frac{n}{2}$, and $(C3)$ be satisfied. Then, $\partial u \in \mathcal{L}_{\mathcal{F}}^{\infty}(0, T; L_{n,m\times n}^{\infty})$.*

Proof. Theorem 4.3 together with the first equality in Lemma 4.3 implies that $\partial u(\,\cdot\,, X_\cdot) \in \mathcal{L}_{\mathcal{F}}^{\infty}(0, T; L_{n,m\times n}^{\infty})$. Then the desired result follows from Lemma 4.1.

4.3. Hessian estimates

Define the following notations:

$$\begin{aligned} \tilde{\partial}_{ij}^2 Y_t(y) &:= \partial_k[\tilde{\partial}_i Y_t(y)][\partial X_t(y)]_{kj}^{-1}, \\ \tilde{\partial}_{ij}^2 Z_t^{\cdot r}(y) &:= \partial_k[\tilde{\partial}_i Z_t^{\cdot r}(y)][\partial X_t(y)]_{kj}^{-1}, \\ \Theta_{ij}(t, y) &:= \tilde{\partial}_{ij}^2 Y_t(y), \end{aligned}$$

$$\theta_{ij}^{;r}(t,y) := \tilde{\partial}_{ij}^2 Z_t^{;r}(y) - \sigma_{x^i}^{k'r}(t,X_t(y))\Theta_{k'j}(t,y)$$
$$- \Theta_{ik'}(t,y)\sigma_{x^j}^{k'r}(t,X_t(y)) - \tilde{\partial}Y_t(y)\sigma_{x^ix^j}^{;r}(t,X_t(y)). \quad (4.13)$$

Here $[\partial X_t(y)]_{ij}^{-1}$ is the (i,j)-component of the matrix $[\partial X_t(y)]^{-1}$ for $i,j = 1,\cdots,n$.

Lemma 4.4. *Let $(C1)_k$ and $(C2)_k$ be satisfied for $k > 2 + \frac{n}{2}$, and $(C3)$ be satisfied. Then we have*

$$\begin{cases} d\Theta_{ij}(t,y) = \{-f_y(\Xi_t^y)\Theta_{ij}(t,y)\}\, dt \\ \qquad - f_{z^{,r}}(t)[\theta_{ij}^{;r}(t,y) + \sigma_{x^i}^{k'r}\Theta_{k'j}(t,y) + \Theta_{ik'}(t,y)\sigma_{x^j}^{k'r} + \tilde{\partial}Y_t(y)\sigma_{x^ix^j}^{;r}]\, dt \\ \qquad - \{b_{x^i}^{k'}\Theta_{k'j}(t,y) + b_{x^j}^{k'}\Theta_{ik'}(t,y) + \sigma_{x^i}^{k'r}\Theta_{k'k''}(t,y)\sigma_{x^j}^{k''r}\}\, dt \\ \qquad - \{\tilde{\partial}Y_t(y)b_{x^ix^j} + [\tilde{\partial}Z_t^{;r}(y) - \tilde{\partial}Y_t(y)\sigma_x^{;r}]\sigma_{x^ix^j}^{;r} + \tilde{\Delta}_{ij}f(\Xi_t^y;\tilde{\partial}\Xi_t^y)\}\, dt \\ \qquad - [\sigma_{x^i}^{k'r}\theta_{k'j}^{;r}(t,y) + \theta_{ik'}^{;r}(t,y)\sigma_{x^j}^{k'r}]\, dt + \theta_{ij}^{;r}(t,y)\, dw_t^r, \\ \Theta_{ij}(T,y) = \partial_{ij}^2 g(X_T(y)). \end{cases} \quad (4.14)$$

Here

$$\widetilde{\Delta}_{ij}f(\Xi_t^y;\tilde{\partial}\Xi_t^y) := f_{x^ix^j}(\Xi_t^y) + f_{y^ky^l}(\Xi_t^y)\tilde{\partial}_iY_t^k(y)\tilde{\partial}_jY_t^l(y) + 2f_{x^iy^l}(\Xi_t^y)\tilde{\partial}_jY_t^l(y), \quad (4.15)$$

and the arguments of the coefficients b and σ and their partial derivatives are $(\omega,t,X_t(y))$, which are omitted for simplicity.

In what follows, for any $(t,y) \in [0,T]\times\mathbb{R}^n$, we define the two multi-linear maps $\Theta(t,y) \in M_{\otimes 2}^{n,m}$ and $\theta(t,y) \in M_{\otimes 2}^{n,m\times d}$ by

$$\Theta(t,y)(x_1,x_2) := \sum_{i,j=1}^n \Theta_{ij}(t,y)x_1^ix_2^j, \qquad \forall (x_1,x_2) \in (\mathbb{R}^n)^{\otimes^2},$$
$$\theta(t,y)(x_1,x_2) := \sum_{i,j=1}^n \theta_{ij}(t,y)x_1^ix_2^j, \qquad \forall (x_1,x_2) \in (\mathbb{R}^n)^{\otimes^2},$$

respectively. In a similar way, we define the multi-linear maps $\tilde{\partial}^2Y_s(y), \widetilde{\Delta}(\Xi_s;\tilde{\partial}\Xi_s) \in M_{\otimes 2}^{n,m}$, and $\tilde{\partial}^2Z_s(y) \in M_{\otimes 2}^{n,m\times d}$.

Theorem 4.5. *Let $(C1)_k$ and $(C2)_k$ be satisfied for $k > 2 + \frac{n}{2}$, and $(C3)$ be satisfied. Then, $\Theta \in \mathcal{L}_{\mathcal{F}}^\infty(0,T;L_n^\infty(M_{\otimes 2}^{n,m}))$ and the map $(\omega,t,x) \mapsto E^{\mathcal{F}_t}\Big(\int_t^T |\theta(s,x)|_{M_{\otimes 2}^{n,m\times d}}^2\, ds\Big)^{p/2}$ lies in the space $\mathcal{L}_{\mathcal{F}}^\infty(0,T;L_{n,1}^\infty)$ for any $p>1$.*

Proof. Applying Lemma 2.6 to BSDE (4.14), we have

$$\|\Theta\|_{\mathcal{L}_{\mathcal{F}}^\infty(0,T;L_n^\infty(M_{\otimes 2}^{n,m}))}^2$$
$$\leq C_p\Big\{\|\tilde{\partial}Y\|_{\mathcal{L}_{\mathcal{F}}^\infty(0,T;L_{n,m\times n}^\infty)}^2 + \Big\|E^{\mathcal{F}_\cdot}\int_\cdot^T |\tilde{\partial}Z_\cdot|_{M_{\otimes 1}^{n,m\times d}}^2\, ds$$
$$+ E^{\mathcal{F}_\cdot}\Big(\int_\cdot^T |\widetilde{\Delta}f(\Xi_s;\tilde{\partial}\Xi_s)|_{M_{\otimes 2}^{n,m}}\, ds\Big)^2\Big\|_{\mathcal{L}_{\mathcal{F}}^\infty(0,T;L_{n,1}^\infty)}\Big\}$$

$$\le C_p\Big\{1+\|\tilde\partial Y\|^4_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m\times n})}+\Big\|E^{\mathcal{F}_\cdot}\int_\cdot^T|\tilde\partial Z_s|^2_{M^{n,m\times d}_{\otimes 1}}\,ds\Big\|_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,1})}\Big\} \tag{4.16}$$

and

$$\begin{aligned}&E^{\mathcal{F}_t}\Big(\int_t^T|\theta(s,x)|^2_{M^{n,m\times d}_{\otimes 2}}\,ds\Big)^{p/2}\\ &\le C_p\Big\{1+\|\tilde\partial Y\|^4_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,m\times n})}+\Big\|E^{\mathcal{F}_\cdot}\int_\cdot^T|\tilde\partial Z_s|^2_{M^{n,m\times d}_{\otimes 1}}\,ds\Big\|_{\mathcal{L}^\infty_{\mathcal{F}}(0,T;L^\infty_{n,1})}\Big\}.\end{aligned} \tag{4.17}$$

In view of Theorem 4.3, we have the desired results.

By differentiating with respect to x and then multiplying with $[\partial X_t(y)]^{-1}$ both sides of each equality in Lemma 4.3, we establish the following lemma.

Lemma 4.5. *Let $(C1)_k$ and $(C2)_k$ be satisfied for $k>2+\frac{n}{2}$, and $(C3)$ be satisfied. We have*

$$\partial^2u(t,X_t(y))=\Theta(t,y), \tag{4.18}$$

$$\partial^2v^{\cdot r}(t,X_t(y))+\partial^2\partial u(t,X_t(y))\sigma^{\cdot r}(t,X_t(y))=\theta^{\cdot r}(t,y). \tag{4.19}$$

From Theorem 4.5, Lemma 4.5, and Lemma 4.1, we deduce the following

Theorem 4.6. *Let $(C1)_k$ and $(C2)_k$ be satisfied for $k>2+\frac{n}{2}$, and $(C3)$ be satisfied. Then, ∂^2u is uniformly bounded.*

4.4. Estimates on $(\partial^k u,\partial^k v)$ with $k\ge 3$

We shall derive, under the condition $(C3)$, the BSDEs for $\{(\partial^k u(t,X_t),\partial^k v(t,X_t)),0\le t\le T\}$ for arbitrary positive integer k.

Since $\partial(\partial^k u(t,X_t))=\partial^{k+1}u(t,X_t)\partial X_t$, we have

$$\partial^{k+1}u(t,X_t)=\partial(\partial^k u(t,X_t))(\partial X_t)^{-1},$$

which provides an iterative way of calculating the differentials of $\partial^{k+1}u(t,X_t)$ by using the differentials of $\partial^k u(t,X_t)$. By induction, also in view of Lemma 2.6, we can prove the following

Theorem 4.7. *Let $(C1)_k$ and $(C2)_k$ be satisfied for an integer $k\ge 1$, and $(C3)$ be satisfied. Then the pair of random fields $\{(\partial^k u(t,X_t),\partial^k v(t,X_t)),0\le t\le T\}$ satisfies the following BSDE:*

$$\begin{cases} d\partial^k u(t,X_t)=-F_{k,k}(t,\partial^k u(t,X_t),\partial^k v(t,X_t)+\partial^{k+1}u(t,X_t)\sigma(t,X_t))\,dt\\ \qquad\qquad -\displaystyle\sum_{l=1}^{k-1}F_{k,l}(t,\partial^l u(t,X_t),\partial^l v(t,X_t)+\partial^{l+1}u(t,X_t)\sigma(t,X_t))\,dt\\ \qquad\qquad -\displaystyle\sum_{j=0}^{k-1}\sum_{i=1}^{k-j}\sum_{k-j\le k_1+\cdots+k_i\le k-1}^{k_1,\cdots,k_i\ge 1}\partial_x^j\partial_y^i f(t,X_t,u(t,X_t))(\partial^{k_1}u(t,X_t),\\ \qquad\qquad \cdots,\partial^{k_i}u(t,X_t))+\{\partial^k v(t,X_t)+\partial^{k+1}u(t,X_t)\sigma(t,X_t)\}\,dw_t,\\ \partial^k u(T,X_T)=\partial^k g(X_T).\end{cases} \tag{4.20}$$

Here the map $F_{k,k} : \Omega \times [0,T] \times M^{n,m}_{\otimes k} \times M^{n,m\times d}_{\otimes k} \to M^{n,m}_{\otimes k}$ *is defined by*

$$\begin{aligned} F_{k,k}(t,*_1,*_2) := {}& \partial_x^k f(t,X_t,u(t,X_t)) + \partial_y f(t,X_t,u(t,X_t))(*_1) + \partial_z f(t)(*_2) \\ &+ *_1[\partial b(t,X_t) + (k-1)(\partial\sigma(t,X_t))^2] + k *_2 \partial\sigma(t,X_t), \end{aligned} \tag{4.21}$$

which is linear in the last two arguments. For $l = 1,\cdots,k-1$, *the map* $F_{k,l} : \Omega \times [0,T] \times M^{n,m}_{\otimes l} \times M^{n,m\times d}_{\otimes l} \to M^{n,m}_{\otimes k}$ *is also linear in the last two arguments with the coefficients being the multi-linear forms of the term* $\partial_z f(t)$ *and the partial derivatives of* b *and* σ *up to order* $k-l$.

Further, the following quantities

$$u(t,x), \quad E^{\mathcal{F}_t}\Big(\int_t^T |v(s,x)|^2_{\mathbb{R}^{m\times d}}\,ds\Big)^p, \qquad p \ge 1$$

and

$$|\partial^i u(t,x)|_{M^{n,m}_{\otimes i}}, \quad E^{\mathcal{F}_t}\Big(\int_t^T |\partial^i v(s,x)|^2_{M^{n,m\times d}_{\otimes i}}\,ds\Big)^p, \qquad i = 1,\cdots,k,\ p \ge 1$$

are uniformly bounded in (ω,t,x).

§ 5. Classical Solutions

Definition 5.1. *A pair of random vector fields* $\{(u(t,x,\omega), v(t,x,\omega)), (t,x,\omega) \in [0,T] \times \mathbb{R}^n \times \Omega\}$ *is called an adapted classical solution of system* (1.1), *if*

$$\begin{cases} u \in C_{\mathcal{F}}([0,T]; L^2(\Omega; C^2(\overline{B}; \mathbb{R}^m))), \\ v \in \mathcal{L}^2_{\mathcal{F}}(0,T; C^1(\overline{B}; \mathbb{R}^{m\times d})) \end{cases} \tag{5.1}$$

for any centered ball $B \subset \mathbb{R}^n$, *such that the following holds almost surely* (*note that* $v_{u,\sigma} := v + \partial u\sigma$) :

$$\begin{aligned} u(t,x) = g(x) + \int_t^T \{\mathcal{L}(s,x)u(s,x) + \mathcal{M}(s,x)v(s,x) + f(s,x,u(s,x),v_{u,\sigma}(s,x))\}\,ds \\ - \int_t^T v(s,x)\,dw_s, \qquad \forall\,(t,x) \in [0,T] \times \mathbb{R}^n. \end{aligned} \tag{5.2}$$

Lemma 5.1. *Let* $(C1)_k$ *and* $(C2)_k$ *be satisfied with* $k > 2 + \frac{n}{2}$, *and* $(C3)$ *be satisfied. Then* $u(t,x)$ *is a* C^2*-valued processes and satisfies that for any compact subset* $K \subset \mathbb{R}^n, p > 1$ *and any* $\beta \in \mathcal{A}$ *such that* $|\beta| \le 2$, *there is a constant* $C_{K,p}$ *such that*

$$E \sup_{0\le t\le T, x\in K} |\partial^\beta u(t,x)|^p \le C_{K,p}, \tag{5.3}$$

$$E \int_0^T \sup_{x\in K} |\partial^\beta v(t,x)|^p_{\mathbb{R}^{m\times d}}\,dt \le C_{K,p}. \tag{5.4}$$

Proof. In view of Theorem 4.7, we use Kunita [18, Theorem 1.4.1, p.31] to get the first estimate and Sznitman [31, Lemma 1, pp.46–47] to get the second.

Lemma 5.2. *Let $(C1)_k$ and $(C2)_k$ be satisfied with $k > 2 + \frac{n}{2}$, and $(C3)$ be satisfied. Then, (u, v) almost surely satisfies Equation* (5.2).

Proof. Recall that

$$X_t(x) := X_t(0, x), \quad Y_t(x) := Y_t(0, x), \quad Z_t(x) := Z_t(0, x), \qquad 0 \le t \le T. \tag{5.5}$$

Note that

$$\begin{cases} dY_t(x) = -f(t, X_t(x), Y_t(x), Z_t(x))\, dt + Z_t(x)\, dw_t, \qquad 0 \le t \le T, \\ Y_T(x) = g(X_T(x)) \end{cases} \tag{5.6}$$

and from Lemma 2.2,

$$\begin{cases} dX_t^{-1}(x) = (\mathcal{M}^r\mathcal{M}^r - \mathcal{L})(t, x)X_t^{-1}(x)\, dt - \mathcal{M}^r(t, x)X_t^{-1}(x)\, dw_t^r, \qquad 0 \le t \le T, \\ X_0^{-1}(x) = x. \end{cases} \tag{5.7}$$

In view of Theorem 3.2, we can use the generalized Itô's formula (see [11, 18, 19]) to compute $u(t, x)$. We have

$$\begin{cases} du(t, x) = -f(t, X_t(X_t^{-1}(x)), u(t, x), Z_t(X_t^{-1}(x)))\, dt + Z_t(X_t^{-1}(x))\, dw_t \\ \qquad\qquad + \partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}(\mathcal{M}^r\mathcal{M}^r - \mathcal{L})(t, x)X_t^{-i}(x)\, dt \\ \qquad\qquad - \partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-i}(x)\, dw_t^r \\ \qquad\qquad + \dfrac{1}{2}\partial_{ij}^2 Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}[\mathcal{M}^r(t, x)X_t^{-i}(x)][\mathcal{M}^r(t, x)X_t^{-j}(x)]\, dt \\ \qquad\qquad - \partial_i Z_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-i}(x)\, dt, \qquad 0 \le t \le T, \\ u(T, x) = g(X_T(X_T^{-1}(x))). \end{cases} \tag{5.8}$$

Here $X_t^{-i}(x)$ is the i-th component of $X_t^{-1}(x)$. We have

$$\begin{aligned} X_t(X_t^{-1}(x)) &= x, \qquad 0 \le t \le T, \\ \mathcal{M}^r(t, x)u(t, x) &= \partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-i}(x), \\ \mathcal{M}^r(t, x)Z_t(X_t^{-1}(x)) &= \partial_i Z_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-i}(x), \end{aligned} \tag{5.9}$$

$$\begin{aligned} \mathcal{M}^r\mathcal{M}^r(t, x)u(t, x) &= \mathcal{M}^r(t, x)(\partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-i}(x)) \\ &= \mathcal{M}^r(t, x)(\partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)})\mathcal{M}^r(t, x)X_t^{-i}(x) \\ &\quad + \partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}(\mathcal{M}^r\mathcal{M}^r(t, x)X_t^{-i}(x)) \\ &= \partial_{ij}^2 Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}[\mathcal{M}^r(t, x)X_t^{-i}(x)][\mathcal{M}^r(t, x)X_t^{-j}(x)] \\ &\quad + \partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}\mathcal{M}^r\mathcal{M}^r(t, x)X_t^{-i}(x), \end{aligned} \tag{5.10}$$

and

$$
\begin{aligned}
\mathcal{L}(t,x)u(t,x) = \partial_i Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}\mathcal{L}(t,x)X_t^{-i}(x) \\
+\frac{1}{2}\partial^2_{ij}Y_t(\tilde{x})|_{\tilde{x}=X_t^{-1}(x)}[\mathcal{M}^r(t,x)X_t^{-i}(x)][\mathcal{M}^r(t,x)X_t^{-j}(x)].
\end{aligned}
\tag{5.11}
$$

Therefore, (u,v) solves BSPDEs (5.2).

Concluding the above, we have

Theorem 5.1. *Let $(C1)_k$ and $(C2)_k$ be satisfied with $k>2+\frac{n}{2}$, and $(C3)$ be satisfied. Then (u,v) is a classical solution of the system* (5.2). *Moreover, $\partial^\alpha u$ is bounded for any $\alpha\in\mathcal{A}$ such that $|\alpha|\le k$.*

In what follows, we are concerned with the uniqueness of the adapted classical solution. We shall use a probabilistic representation method.

Theorem 5.2. *Let $(\tilde{u},\tilde{v})$ be an adapted classical solution of the system* (1.1). *Then we have almost surely for any $(t,x)\in[0,T]\times R^n$,*

$$
\tilde{u}(t,X_t(x))=Y_t(x) \quad \text{and} \quad \tilde{v}(t,X_t(x))=Z_t(x)-\partial\tilde{u}(t,\tilde{x})|_{\tilde{x}=X_t(x)}\sigma(t,X_t(x)) \tag{5.12}
$$

or equivalently

$$
\tilde{u}(t,x)=Y_t(X_t^{-1}(x)) \quad \text{and} \quad \tilde{v}(t,x)=Z_t(X_t^{-1}(x))-\partial[Y_t(X_t^{-1}(x))]\sigma(t,x). \tag{5.13}
$$

Proof. The main idea is to use the generalized Itô's formula to calculate $u(t,X_t(x))$ and to verify that $\{(\tilde{u}(t,X_t(x)),\tilde{v}(t,X_t(x))+\partial\tilde{u}(t,X_t(x))\sigma(t,X_t(x)),(t,x)\in[0,T]\times R^n\}$ is an adapted solution of BSDE (3.1). Then the desired equalities follow.

§ 6. A Comparison Theorem

In this section, we establish the comparison theorem for BSPDEs using the relationship between BSDEs and BSPDEs. In this section, we always assume that $m=1$.

Consider BSPDEs (1.1) and

$$
\begin{cases}
d\tilde{u}(t,x) = -\mathcal{L}(t,x)\tilde{u}(t,x)-\mathcal{M}\tilde{v}-\tilde{f}(t,x,\tilde{u},\tilde{v}+\partial\tilde{u}\sigma) & \\
\qquad\qquad +\tilde{v}(t,x)\,dw_t, & (t,x)\in[0,T)\times R^n, \\
\tilde{u}(T,x)=\tilde{g}(x), & x\in R^n.
\end{cases}
\tag{6.1}
$$

Theorem 6.1. *Suppose that all the assumptions of Theorem* 5.1 *are satisfied for both (f,g) and $(\tilde{f},\tilde{g})$. Let (u,v) and $(\tilde{u},\tilde{v})$ be the classical adapted solutions of BSPDEs* (1.1) *and* (6.1), *respectively. If $g(x)\ge\tilde{g}(x)$ and $f(t,x,u,v)\ge\tilde{f}(t,x,u,v)$ for all $(t,x,u,v)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$ a.s., then for any $t\in[0,T]$, a.s.a.e., $(\omega,x)\in\Omega\times\mathbb{R}^n$, $u(t,x)\ge\tilde{u}(t,x)$.*

Proof. From Lemma 2.7, we have

$$
Y_t(x)\ge\widetilde{Y}_t(x), \qquad \text{a.s. for any } x\in\mathbb{R}^n. \tag{6.2}
$$

Here $(\widetilde{Y}, \widetilde{Z})$ is the unique adapted solution of BSDE $(\tilde{f}, \tilde{g})$. From Theorem 5.1, we see that

$$u(t,x) = Y_t(X^{-1}(x)), \quad \tilde{u}(t,x) = Y_t(X^{-1}(x)), \qquad \text{a.s. for any } \ x \in \mathbb{R}^n.$$

The proof is then complete.

§ 7 . The Super-parabolic Case

In general, BSPDEs (1.1) is degenerate parabolic in the sense of Ma and Yong [22, Definition 1.1, p.104]. However, it may be super-parabolic in some situation. To see this, we introduce the following assumption.

$(C4)$ Let $0 \le d_0 < d$, and $\{\mathcal{G}_t; 0 \le t \le T\}$ be the augmented natural filtration of $\{w_1, \cdots, w_{d_0}\}$. Assume that all the coefficients σ, b, f, and g are $\mathcal{G}_t$-adapted or $\mathcal{G}_T$-measur -able. There is $\alpha > 0$ such that $\sum\limits_{r=d_0+1}^{d} \langle \sigma^{\cdot r}, \sigma^{\cdot r} \rangle \ge \alpha I_{n\times n}$.

Let $\{D_\theta \xi, \theta \in [0,T]\}$ denote the Malliavin derivative of $\xi \in L(\mathcal{F}_T; \mathbb{B})$ on the Wiener space $C([0,T]; \mathbb{R}^d)$ and $D^r_\theta \xi$ denote the i-th component of D_θ for $1 \le r \le d$.

Given separable Hilbert space $\mathbb{H}$, let $\mathbb{D}^{(2)}_{1,p}(\mathbb{H})$ denote the subspace of $L^p(\mathcal{F}_T; \mathbb{H})$ such that whose elements ξ have the partial Malliavin derivatives $D^i \xi \in \mathcal{L}^{2,p}_{\mathcal{F}}(0,T;\mathbb{H})$ for $i = d_0 + 1, \cdots, d$. Define for $\xi \in \mathbb{D}^{(2)}_{1,p}(\mathbb{H})$, $\|\xi\|_{\mathbb{D}^{(2)}_{1,p}(\mathbb{H})} := \Big(E\|\xi\|^p_{\mathbb{H}} + \Big(\int_0^T \sum\limits_{i=d_0+1}^{d} \|D^i_\theta \xi\|^2_{\mathbb{H}}\, d\theta \Big)^{p/2} \Big)^{1/p}$. Let $\mathbb{L}^a_{1,p}(\mathbb{H})$ denote the set of $\mathbb{H}$-valued progressively measurable processes $\{f(\omega,t), (\omega,t) \in \Omega \times [0,T]\}$ such that

(i) For a.e., $t \in [0,T], f(\omega,t) \in \mathbb{D}^{(2)}_{1,p}(\mathbb{H})$.

(ii) The map $(\omega,t) \to D^i f(\omega,t) \in (L^2(0,T;\mathbb{H}))^d$ admits a progressively measurable version for $i = d_0+1, \cdots, d$.

(iii) $\|f\|^a_{1,p} = E\Big[\Big(\int_0^T \|f(t)\|^2_{\mathbb{H}}\, dt \Big)^{p/2} + \Big(\int_0^T \int_0^T \sum\limits_{i=d_0+1}^{d} \|D^i_\theta f(t)\|^2_{\mathbb{H}}\, d\theta dt \Big)^{p/2} \Big] < \infty$.

We have

Lemma 7.1. *Let $(C1)_1, (C2)_1$, and $(C4)$ be satisfied. Let X be the unique solution of SDE (2.3). Let (Y,Z) be the unique adapted solution of BSDE (2.8). Then $(Y,Z) \in L^2(0,T; \mathbb{D}^{(2)}_{1,2}(\mathbb{R}^m) \times (\mathbb{D}^{(2)}_{1,2}(\mathbb{R}^m))^d)$. Moreover, As $\theta \in (s,T]$, we have $D_\theta Y_s = 0$ and $D_\theta Z_s = 0$. As $\theta \le s \le T$, we have*

$$\begin{aligned} D^i_\theta Y_s = \int_s^T & [f_y(\tau, X_\tau, Y_\tau, Z_\tau) D^i_\theta Y_\tau + f_z(\tau, X_\tau, Y_\tau, Z_\tau) D^i_\theta Z_\tau \\ & + f_x(\tau, X_\tau, Y_\tau, Z_\tau) D^i_\theta X_\tau]\, d\tau + g_x(X_T) D^i_\theta X_T - \int_s^T D^i_\theta Z_\tau\, dw^r_\tau \end{aligned} \tag{7.1}$$

for $i = d_0+1, \cdots, d$. Moreover, for $i = d_0+1, \cdots, d$, $D^i_\theta Y_s = \partial Y_s (\partial X_s)^{-1} \sigma^{\cdot i}(\theta, X_\theta)$ for $0 \le \theta \le s \le T$, and $\{D^i_s Y_s, 0 \le s \le T\}$ is a version of $\{Z^{;i}_s, 0 \le s \le T\}$ which implies that $Z^{;i}_\theta = \partial Y_\theta (\partial X_\theta)^{-1} \sigma^{\cdot i}(\theta, X_\theta)$ for a.e., $\theta \in [0,T]$.

Then from Lemma 7.1, we obtain

Theorem 7.1. *Let $(C1)_1$, $(C2)_1$, $(C3)$ and $(C4)$ be satisfied. Let (Y,Z) be the unique adapted solution of BSDE (3.1). Then for $r = d_0+1, \cdots, d$, we have*

$$Z^{;r}_s(x) = \partial Y_s(x) [\partial X_s(x)]^{-1} \sigma^{\cdot r}(s, X_s(x)), \qquad r = d_0+1, \cdots, d, \tag{7.2}$$

which implies

$$Z_s^{\cdot r}(X_t(x)^{-1}) = \partial[Y_s(X_t(x)^{-1})]\sigma^{\cdot r}(s,x), \quad v^{\cdot r}(s,x) = 0, \qquad r = d_0+1,\cdots,d. \tag{7.3}$$

Then BSPDEs (1.1) *is super-parabolic under* ($C4$).

§ 8. Comments

In the paper we present the connection of BSPDEs with SDEs and BSDEs in a new way from a probabilistic point of view. The method is purely probabilistic and constructive. A striking feature in our context is that the coefficients of SDEs and BSDEs are allowed to be random. In this situation, the traditional dynamic programming method will encounter a serious difficulty in providing a probabilistic angle: it could not produce the martingale term of the BSPDEs, and therefore it should not be expected to play a role as in the case of Markovian coefficients. Our flow approach can attack the difficulty.

In the paper, it is shown that in the case of the partial derivative $f_z(t,x,y,z)$ being independent of (x,y,z) and being bounded, the smoothness with suitable growth conditions on the coefficients implies the smoothness of the solutions (u,v) of BSPDEs (1.1) and the boundedness of the partial derivatives $\partial^\alpha u$. It would be very challenging to prove or disprove the property for the case of f being nonlinear in z. In making this efforts, we meet with the following difficulty: to get some regularity of the solution's derivatives of order higher than two, we have to estimate the terms like

$$\int_0^T \sum_{r=1}^d |\partial Z_t^{\cdot r}(x)|^k \, dt \quad \text{and} \quad \int_0^T \int_{\mathbb{R}^n} \sum_{r=1}^d |\partial Z_t^{\cdot r}(x)|^k \, dxdt \quad \text{with} \quad k \geq 3. \tag{8.1}$$

Here $\int \partial Z_t^{\cdot r}(x)\, dw_t$ is identified as the martingale part of a BSDE. While from the property of BSDEs, we can at most in general estimate such kinds of terms

$$\int_0^T \sum_{r=1}^d |\partial Z_t^{\cdot r}(x)|^2 \, dt \quad \text{and} \quad \int_0^T \int_{\mathbb{R}^n} \sum_{r=1}^d |\partial Z_t^{\cdot r}(x)|^2 \, dxdt,$$

which do not imply in general any estimate on the integrals (8.1). This situation suggests that an example be constructed to disprove the property mentioned in the above.

Due to limitation of space, we have only addressed the classical solutions of BSPDEs. The generalized solutions in the senses of Sobolev and Schwartz distributions will be discussed elsewhere.

Finally, we would like to make the two remarks in relevance to the literature.

(1) When the coefficients are deterministic, the system (1.1) becomes a deterministic one (just letting $v \equiv 0$)

$$\begin{cases} \dfrac{d}{dt}u(t,x) = -\mathcal{L}(t,x)u(t,x) + f(t,x,u,\partial u\sigma), & (t,x) \in [0,T) \times R^n, \\ u(T,x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{8.2}$$

The connection between PDEs, SDEs and BSDEs has been addressed from a probabilistic point of view in the sense of viscosity solutions by Peng [28] and in the senses of classical and viscosity solutions by Pardoux and Peng [26], and from an analytical point of view in the sense of classical solutions by Peng [27, 30] and in the sense of Sobolev solutions by Barles and Lesigne [3]. They connect BSDE (1.4) together with the diffusion (1.3) to the deterministic semi-linear parabolic system (8.2) in the following way:

$$u(t,x) = Y_t(t,x). \tag{8.3}$$

The work [26] is a semi-linear generalization of the classical Feynman-Kac formula. A main feature of their context is that the coefficients of the SDEs and BSDEs are Markovian. With the traditional dynamic programming principle (DPP), they study a system of deterministic partial differential equations (PDEs) with the associated SDEs and BSDEs. Unfortunately, the traditional DPP method fails to work in our context of non-markovian coefficients. In the paper, we develop a flow approach instead of the DPP. Since

$$Y_t(0, X_t^{-1}(0,x)) = Y_t(t,x), \tag{8.4}$$

our result coincides with Pardoux and Peng [26].

(2) If further $\sigma = 0$, then both SDE (1.3) and BSDE (1.4) reduce to two ordinary differential equations (ODEs), and PDEs (8.2) is of first-order. In this case, the connection of ODEs and PDEs has been discussed by Diperna and Lions [12] in some rather general conditions. However, their goal is to study ODEs driven by Sobolev space valued vector fields, using the theory of the associated first-order PDEs (the so-called transport equation).

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