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A GENERALIZED THIN FILM EQUATION****

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Abstract

The authors study a generalized thin film equation. Under some assumptions on the initial value, the existence of weak solutions is established by the time-discrete method. The uniqueness and asymptotic behavior of solutions are also discussed.

Keywords Thin film equation, Existence, Uniqueness, Asymptotic behavior
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§ 1 . Introduction

In this paper, we consider the variant version of the thin film equation, namely

$$\frac{\partial u}{\partial t} + \operatorname{div}(|\nabla \Delta u|^{p-2}\nabla \Delta u) = 0, \qquad x \in \Omega,\ t > 0,\ p > 2, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary.

The equation (1.1) is a typical higher order equation, which has a sharp physical background and a rich theoretical connotation. It is relevant to capillary driven flows of thin films of power-law fluids, where u denotes the height from the surface of the oil to the surface of the solid. It was J. R. King [1] who first derived the equation. J. R. King [1] studied the Cauchy problem of the equation in one-dimension, exploiting local analyses about the edge of the support and special closed form solutions such as travelling waves, separable solutions, instantaneous source solutions.

We restrict ourselves to the two dimensional case, which has a particular physical derivation as mentioned in [2], modelling the spreading of an oil film over an solid surface. On the basis of physical consideration, as usual the equation (1.1) is supplemented with the natural boundary value conditions

$$u = \Delta u = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

The boundary value conditions (1.2) are reasonable for the thin film equation or the Cahn-Hillaird equation, (see [3–5]) and initial value condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

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This equation is something quite like the p-Laplacian equation, but many methods used in the p-Laplacian equation such as the methods based on maximun principle are no longer valid for this equation. Because of the degeneracy, the problem (1.1)–(1.3) does not admit classical solutions in general. So, we introduce weak solutions in the sense as following

Definition 1.1. *A function u is said to be a weak solution of the problem* (1.1)–(1.3), *if the following conditions are satisfied:*

(1) $u \in L^\infty(0,T;W^{3,p}(\Omega)) \cap C(0,T;L^2(\Omega))$, u, $\Delta u \in W_0^{1,p}(\Omega)$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;W^{-1,p'}(\Omega))$, *where p' is the conjugate exponent of p;*

(2) *For any $\varphi \in C_0^\infty(Q_T)$, the following integral equality holds:*

$$\iint_{Q_T} u\frac{\partial\varphi}{\partial t}dxdt + \iint_{Q_T} |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\varphi dxdt = 0;$$

(3) $u(x,0) = u_0(x)$ *in* $L^2(\Omega)$.

This paper is arranged as following: We first discuss the existence of weak solutions in Section 2. Our method for investigating the existence of weak solutions is based on the time discrete method to construct an approximate solutions. By means of the uniform estimates on solutions of the time difference equations, we prove the existence of weak solutions of the problem (1.1)–(1.3). Using energy techniques, Poincaré inequality and Friedrichs inequality, we also prove the uniqueness and asymptotic behavior subsequently.

§ 2. Existence of Weak Solutions

In this section, we are going to prove the existence of weak solutions.

Theorem 2.1. *Let $u_0 \in W^{3,p}(\Omega), u_0, \Delta u_0 \in W_0^{1,p}(\Omega)$, $p > 2$. Then the problem* (1.1)–(1.3) *admits at least one weak solution.*

To prove the existence, we first consider the following time-discrete problem

$$\frac{1}{h}(u_{k+1} - u_k) + \mathrm{div}(|\nabla\Delta u_{k+1}|^{p-2}\nabla\Delta u_{k+1}) = 0, \tag{2.1}$$

$$u_{k+1}|_{\partial\Omega} = \Delta u_{k+1}|_{\partial\Omega} = 0, \qquad k = 0, 1, \cdots, N-1, \tag{2.2}$$

where $h = \frac{T}{N}$, u_0 is the initial value.

Lemma 2.1. *For any fixed k, if $u_k \in H_0^1(\Omega)$, then the problem* (2.1)–(2.2) *admits weak solutions $u_{k+1} \in W^{3,p}(\Omega)$, $u_{k+1}, \Delta u_{k+1} \in W_0^{1,p}(\Omega)$ such that for any $\varphi \in C_0^\infty(\Omega)$, there holds*

$$\frac{1}{h}\int_\Omega (u_{k+1} - u_k)\varphi dx - \int_\Omega |\nabla\Delta u_{k+1}|^{p-2}\nabla\Delta u_{k+1}\nabla\varphi dx = 0. \tag{2.3}$$

Proof. First we define the space

$$\mathcal{U} = \{u \mid u \in W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega); \Delta u \in W_0^{1,p}(\Omega)\}.$$

It is not difficult to conclude that the space $\mathcal{U}$ is a Banach space. Let us consider the following functionals on the space $\mathcal{U}$,

$$\begin{aligned} F[u] &= \frac{1}{p}\int_\Omega |\nabla\Delta u|^p dx, \\ G[u] &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx, \\ H[u] &= F[u] + \frac{1}{h}G[u] - \int_\Omega f\Delta u dx, \end{aligned}$$

where $f \in H_0^1(\Omega)$ is a known function. By using the Young inequality, there exists $C_1 > 0$, such that

$$\begin{aligned} H[u] &= \frac{1}{p}\int_\Omega |\nabla\Delta u|^p dx + \frac{1}{2h}\int_\Omega |\nabla u|^2 dx - \int_\Omega f\Delta u dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla\Delta u|^p dx - C_1\int_\Omega |\nabla f|^2 dx. \end{aligned}$$

We need to check that $H[u]$ satisfies the coercive condition. For this purpose, we notice that by $u|_{\partial\Omega} = 0$, we have

$$\int_\Omega |D^3 u|^p dx \leq C\Big(\int_\Omega |\nabla\Delta u|^p dx + \int_\Omega |\Delta u|^p dx\Big),$$

and by the L^p theory for elliptic equation (see [8]),

$$\|u\|_{W^{2,p}} \leq C\|\Delta u\|_{L^p}.$$

Again by $\Delta u \in W_0^{1,p}(\Omega)$ and the Poincaré inequality, we get

$$\|\Delta u\|_{L^p} \leq C\|\nabla\Delta u\|_{L^p}.$$

Therefore $\|u\|_{W^{3,p}} \leq C\|\nabla\Delta u\|_{L^p}$, and hence $H[u] \to +\infty$, as $\|u\|_{W^{3,p}} \to +\infty$. On the other hand, $H[u]$ is clearly weakly lower semicontinuous on $\mathcal{U}$. So, it follows from the theory in [6] that there exists $u_* \in W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\Delta u_* \in W_0^{1,p}(\Omega)$, such that

$$H[u_*] = \inf H[u],$$

and u_* is the weak solution of the Euler equation corresponding to $H[u]$, namely

$$\frac{1}{h}u + \operatorname{div}(|\nabla\Delta u|^{p-2}\nabla\Delta u) = f.$$

Taking $f = \frac{1}{h}u_k$, we get the conclusion of the lemma. The proof is complete.

Now, we construct an approximate solution u^h of the problem (1.1)–(1.3) by defining

$$\begin{aligned} u^h(x,t) &= u_k(x), \qquad kh < t \leq (k+1)h,\ k = 0, 1, \cdots, N-1, \\ u^h(x,0) &= u_0(x). \end{aligned}$$

The desired solution of the problem (1.1)–(1.3) will be obtained as the limit of some subsequence of $\{u^h\}$. To this purpose, we need some uniform estimates on u^h.

Lemma 2.2. *Let u_k be the weak solution of the problem* (2.1)–(2.2). *Then the following estimates*

$$h\sum_{k=1}^{N}\int_\Omega |\nabla\Delta u_k|^p dx \le C, \tag{2.4}$$

$$\sup_{0<t<T}\int_\Omega |\nabla\Delta u^h(x,t)|^p dx \le C \tag{2.5}$$

hold, where C is a constant independent of h,k.

Proof. (i) We take $\varphi = \Delta u_{k+1}$ in the integral equality (2.3) (we can easily prove that for $\varphi\in W_0^{1,p}(\Omega)$, (2.3) also holds) and obtain

$$\frac{1}{h}\int_\Omega (u_{k+1}-u_k)\Delta u_{k+1}dx - \int_\Omega |\nabla\Delta u_{k+1}|^{p-2}\nabla\Delta u_{k+1}\nabla\Delta u_{k+1}dx = 0.$$

Integrating by parts, we have

$$\frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx + \int_\Omega |\nabla\Delta u_{k+1}|^p dx = \frac{1}{h}\int_\Omega \nabla u_k\nabla u_{k+1}dx,$$

and by Young inequality, we have

$$\frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx + \int_\Omega |\nabla\Delta u_{k+1}|^p dx \le \frac{1}{2h}\int_\Omega |\nabla u_k|^2dx + \frac{1}{2h}\int_\Omega |\nabla u_{k+1}|^2dx,$$

and hence

$$\frac{1}{2}\int_\Omega |\nabla u_{k+1}|^2dx + h\int_\Omega |\nabla\Delta u_{k+1}|^p dx \le \frac{1}{2}\int_\Omega |\nabla u_k|^2dx. \tag{2.6}$$

Summing up these inequalities for k from 0 to $N-1$, we have

$$h\sum_{k=1}^{N}\int_\Omega |\nabla\Delta u_k|^p dx \le \frac{1}{2}\int_\Omega |\nabla u_0|^2dx.$$

Hence (2.4) holds.

(ii) Choosing $\varphi = \Delta u_{k+1} - \Delta u_k$ in the integral equality (2.3) and integrating by parts, we have

$$\frac{1}{h}\int_\Omega |\nabla u_{k+1}-\nabla u_k|^2dx + \int_\Omega |\nabla\Delta u_{k+1}|^{p-2}\nabla\Delta u_{k+1}\nabla\Delta(u_{k+1}-u_k)dx = 0.$$

Since the first term is nonnegative, it follows that

$$\begin{aligned}\int_\Omega |\nabla\Delta u_{k+1}|^p dx &\le \int_\Omega |\nabla\Delta u_{k+1}|^{p-2}\nabla\Delta u_{k+1}\nabla\Delta u_k dx\\ &\le \frac{p-1}{p}\int_\Omega |\nabla\Delta u_{k+1}|^p dx + \frac{1}{p}\int_\Omega |\nabla\Delta u_k|^p dx,\end{aligned}$$

which implies that

$$\int_\Omega |\nabla\Delta u_{k+1}|^p dx \le \int_\Omega |\nabla\Delta u_k|^p dx.$$

For any $m, 1 \le m \le N-1$, summing the above inequality for k from 0 to $m-1$, we have

$$\int_\Omega |\nabla \Delta u_m|^p dx \le \int_\Omega |\nabla \Delta u_0|^p dx,$$

hence (2.5) holds.

Lemma 2.3. *Let u_{k+1} be the weak solution of the problem* (2.1)–(2.2). *Then the following estimate*

$$-Ch \le \int_\Omega |\nabla u_{k+1}|^2 dx - \int_\Omega |\nabla u_k|^2 dx \le 0 \tag{2.7}$$

holds, where C is a constant independent of h.

Proof. To prove the first inequality, we let $\varphi = \Delta u_k$ in (2.3). Integrating by parts and using the boundary value condition, we obtain

$$\frac{1}{h}\int_\Omega |\nabla u_k|^2 dx = \frac{1}{h}\int_\Omega \nabla u_{k+1} \nabla u_k dx + \int_\Omega |\nabla \Delta u_{k+1}|^{p-2} \nabla \Delta u_{k+1} \nabla \Delta u_k dx.$$

Applying the Hölder inequality and the estimate (2.5), we have

$$\begin{aligned}\frac{1}{h}\int_\Omega |\nabla u_k|^2 dx &\le \frac{1}{h}\int_\Omega \nabla u_{k+1}\nabla u_k dx + \frac{p-1}{p}\int_\Omega |\nabla \Delta u_{k+1}|^p + \frac{1}{p}\int_\Omega |\nabla \Delta u_k|^p dx \\ &\le \frac{1}{2h}\int_\Omega |\nabla u_{k+1}|^2 dx + \frac{1}{2h}\int_\Omega |\nabla u_k|^2 dx + C,\end{aligned}$$

that is

$$-Ch \le \int_\Omega |\nabla u_{k+1}|^2 dx - \int_\Omega |\nabla u_k|^2 dx.$$

By (2.6) again, we have

$$\int_\Omega |\nabla u_{k+1}|^2 dx - \int_\Omega |\nabla u_k|^2 dx \le 0.$$

The proof is complete.

Proof of Theorem 2.1. First, we define the operator A^t,

$$A^t(\nabla \Delta u^h) = |\nabla \Delta u_k|^{p-2} \nabla \Delta u_k, \qquad \Delta^h u^h = u_{k+1} - u_k,$$

where $kh < t \le (k+1)h,\ k = 0, 1, \cdots, N-1$. By the discrete equation (2.1) and (2.4) in Lemma 2.2, we see that

$$\frac{1}{h}\Delta^h u^h \quad \text{is bounded in} \quad L^\infty(0,T;(W^{1,p}(\Omega))'). \tag{2.8}$$

By (2.3), (2.5), (2.8) and using the compactness results (see [7]), we see that there exists a subsequence of $\{u^h\}$ (which we denote as the original sequence), such that

$$\begin{aligned} u^h &\overset{\star}{\rightharpoonup} u && \text{in } \ L^\infty(0,T;W^{3,p}(\Omega)),\\ u^h &\to u && \text{in } \ C(0,T;L^2(\Omega)),\\ \frac{1}{h}(u_{k+1}-u_k) &\overset{\star}{\rightharpoonup} \frac{\partial u}{\partial t} && \text{in } \ L^\infty(0,T;(W^{1,p}(\Omega))'),\\ A^t(\nabla \Delta u^h) &\overset{\star}{\rightharpoonup} w && \text{in } \ L^\infty(0,T;L^{p'}(\Omega)),\end{aligned}$$

where p' is the conjugate exponent of p. Then from (2.3), we see that, for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \Big(\frac{1}{h}\Delta^h u^h \varphi - A^t(\nabla \Delta u^h)\nabla\varphi\Big)dxdt = 0.$$

Letting $h \to 0$ yields

$$\frac{\partial u}{\partial t} + \operatorname{div}(w) = 0, \tag{2.9}$$

in the sense of distributions.

It remains to prove that $w = |\nabla\Delta u|^{p-2}\nabla\Delta u$ a.e. in Q_T. Set

$$f_h(t) = \frac{t-kh}{2h}\Big(\int_\Omega |\nabla u_{k+1}|^2 dx - \int_\Omega |\nabla u_k|^2 dx\Big) + \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx,$$

where $kh < t \le (k+1)h,\ k = 0, 1, \cdots, N-1$. By (2.7), we have

$$\frac{1}{2}\int_\Omega |\nabla u_k|^2 dx - Ch \le f_h(t) \le \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx,$$
$$-C \le f_h'(t) \le 0.$$

According to the Ascoli-Arzela theorem, there exists a function $f(t) \in C([0,T])$, such that

$$\lim_{h\to 0} f_h(t) = f(t) \qquad \text{uniformly for} \quad t \in [0,T].$$

Using (2.7), we have

$$\lim_{h\to 0}\frac{1}{2}\int_\Omega |\nabla u^h|^2 dx = f(t) \qquad \text{uniformly for} \quad t\in[0,T]. \tag{2.10}$$

It follows from (2.6) that

$$\frac{1}{2}\int_\Omega |\nabla u_N|^2 dx + \iint_{Q_T} |\nabla\Delta u^h|^p dxdt \le \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx.$$

Letting $h \to 0$ in the above inequality and using (2.10), we have

$$\begin{aligned}
&\lim_{h\to 0}\iint_{Q_T} |\nabla\Delta u^h|^p dxdt\\
&\le f(0) - f(T)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{T-\varepsilon}(f(t) - f(t+\varepsilon))dt\\
&= \lim_{\varepsilon\to 0}\lim_{h\to 0}\frac{1}{2\varepsilon}\int_0^{T-\varepsilon}\int_\Omega (|\nabla u^h(x,t)|^2 - |\nabla u^h(x,t+\varepsilon)|^2)dxdt.
\end{aligned}$$

Consider the functional

$$G[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx.$$

Clearly $G[u]$ is convex and

$$\frac{\delta G[u]}{\delta u} = -\Delta u.$$

Thus, we have

$$\frac{1}{2}\int_\Omega |\nabla u^h(x,t)|^2dx - \frac{1}{2}\int_\Omega |\nabla u^h(x,t+\varepsilon)|^2dx$$

$$\leq -\int_\Omega (u^h(x,t)-u^h(x,t+\varepsilon))\Delta u^h(x,t)dx.$$

Thus

$$\lim_{h\to 0}\frac{1}{2\varepsilon}\int_0^{T-\varepsilon}\int_\Omega (|\nabla u^h(x,t)|^2 - |\nabla u^h(x,t+\varepsilon)|^2)dxdt$$

$$\leq -\frac{1}{\varepsilon}\int_0^{T-\varepsilon}\int_\Omega (u(x,t)-u(x,t+\varepsilon))\Delta u dxdt,$$

hence

$$\lim_{h\to 0}\iint_{Q_T} |\nabla\Delta u^h|^p dxdt \leq \int_0^T \Big\langle \frac{\partial u}{\partial t}, \Delta u\Big\rangle dt,$$

where $\langle\cdot,\cdot\rangle$ denotes inner product. From (2.9), we have

$$\lim_{h\to 0}\iint_{Q_T} |\nabla\Delta u^h|^p dxdt \leq \int_0^T\int_\Omega w\nabla\Delta u dxdt. \tag{2.11}$$

Again by

$$\frac{\delta F[u]}{\delta u} = -\Delta(\operatorname{div}(|\nabla\Delta u|^{p-2}\nabla\Delta u))$$

and the convexity of $F[u]$, for any $g\in L^\infty(0,T;W^{3,p}(\Omega))$, we have

$$\frac{1}{p}\iint_{Q_T}|\nabla\Delta g|^p dxdt - \frac{1}{p}\iint_{Q_T}|\nabla\Delta u^h|^p dxdt$$

$$\geq \iint_{Q_T}(|\nabla\Delta u^h|^{p-2}\nabla\Delta u^h)\nabla\Delta(g-u^h)dxdt.$$

By (2.11) and the fact that $F(u)$ is weakly lower semicontinuous, letting $h\to 0$ in the above equality, we have

$$\frac{1}{p}\iint_{Q_T}|\nabla\Delta g|^p dxdt - \frac{1}{p}\iint_{Q_T}|\nabla\Delta u|^p dxdt \geq -\iint_{Q_T} w\nabla\Delta(u-g)dxdt.$$

Replacing g by $\varepsilon g + u$, we see that

$$\frac{1}{\varepsilon}(F[u+\varepsilon g]-F[u]) \geq \iint_{Q_T} w\nabla\Delta g dxdt.$$

Letting $\varepsilon\to 0$, which implies that

$$\iint_{Q_T}\frac{\delta F[u]}{\delta u} g dxdt = \iint |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\Delta g dxdt \geq \iint_{Q_T} w\nabla\Delta g dxdt.$$

Due to the arbitrariness of g, we get the opposite inequality of the above inequality. Therefore

$$w = |\nabla\Delta u|^{p-2}\nabla\Delta u.$$

The strong convergence of u^h in $C(0,T;L^2(\Omega))$ and the fact that $u^h(x,0) = u_0(x)$ imply that u satisfies the initial value condition. The proof is complete.

§ 3 . Uniqueness of Solutions

In this section, we will prove the uniqueness of solutions.

Theorem 3.1. *The problem* (1.1)–(1.3) *admits at most one weak solution.*

To prove Theorem 3.1, we need the following lemma.

Lemma 3.1. *For $\varphi \in L^\infty(t_1, t_2; W_0^{1,p}(\Omega))$ with $\varphi_t \in L^2(\Omega \times (t_1, t_2))$, the weak solution u of the problem* (1.1)–(1.3) *on Q_T satisfies*

$$\int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_{t_1}^{t_2}\int_\Omega \Big(u\frac{\partial\varphi}{\partial t} + |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\varphi\Big)dxdt$$

$$= \int_\Omega u(x,t_2)\varphi(x,t_2)dx.$$

In particular, for $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega (u(x,t_1) - u(x,t_2))\varphi dx - \int_{t_1}^{t_2}\int_\Omega |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\varphi dxdt = 0. \tag{3.1}$$

Proof. For $\varphi \in L^\infty(t_1, t_2; W_0^{1,p}(\Omega))$ and $\varphi_t \in L^2(\Omega \times (t_1, t_2))$, we choose a sequence of functions $\{\varphi_k\}$, such that $\varphi_k(\cdot, t) \in C_0^\infty(\Omega)$. When $k \to \infty$, we have

$$\|\varphi_{kt} - \varphi_t\| \to 0, \qquad \|\varphi_k - \varphi\|_{L^\infty(t_1,t_2;W_0^{1,p}(\Omega))} \to 0.$$

Choose a function $j(s) \in C_0^\infty(R)$ such that

$$j(s) \ge 0 \quad \text{for} \quad s \in R; \qquad j(s) = 0, \quad \forall\, |s| > 1; \qquad \int_R j(s)ds = 1.$$

For $h > 0$, define

$$j_h(s) = \frac{1}{h}j\Big(\frac{s}{h}\Big), \qquad \eta_h(t) = \int_{t-t_2+2h}^{t-t_1-2h} j_h(s)ds.$$

Clearly $\eta_h(t) \in C_0^\infty(t_1, t_2)$, $\lim\limits_{h\to 0^+} \eta_h(t) = 1$, $\forall\, t \in (t_1, t_2)$.

Taking $\varphi = \varphi_k(x,t)\eta_h(t)$ in the definition of weak solutions, we have

$$\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dxdt - \int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dxdt$$
$$+ \int_{t_1}^{t_2}\int_\Omega u\varphi_{kt}\eta_h dxdt + \int_{t_1}^{t_2}\int_\Omega |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\varphi_k\eta_h dxdt = 0.$$

Observing that

$$\Big|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dxdt - \int_\Omega (u\varphi_k)|_{t=t_1}\Big|$$
$$= \Big|\int_{t_1+h}^{t_1+3h}\int_\Omega u\varphi_k j_h(t-t_1-2h)dxdt - \int_{t_1+h}^{t_1+3h}\int_\Omega (u\varphi_k)|_{t=t_2} j_h(t-t_1-2h)dxdt\Big|$$
$$\le \sup_{t_1+h<t<t_1+3h}\int_\Omega |(u\varphi_k)|_t - (u\varphi_k)|_{t_1}|dx,$$

and $u \in C(0,T;L^2(\Omega))$, we see that the right hand side tends to zero as $h \to 0$.

Similarly

$$\Big|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dxdt - \int_\Omega (u\varphi_k)|_{t=t_2}\Big| \to 0 \qquad \text{as} \quad h \to 0.$$

Letting $h \to 0, k \to \infty$, we get

$$\begin{aligned}&\int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_{t_1}^{t_2}\int_\Omega \Big(u\frac{\partial\varphi}{\partial t} + |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\varphi\Big)dxdt\\ =&\int_\Omega u(x,t_2)\varphi(x,t_2)dx.\end{aligned}$$

Particularly, for $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega (u(x,t_1)-u(x,t_2))\varphi dx - \int_{t_1}^{t_2}\int_\Omega |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla\varphi dxdt = 0.$$

For fixed $\tau \in (0,T)$, and any h with $0<\tau<\tau+h<T$, letting $t_1=\tau$, $t_2=\tau+h$, and multiplying (3.1) by $\frac{1}{h}$, for $\varphi \in W_0^{1,p}(\Omega)$, we obtain

$$\int_\Omega (u_h(x,\tau))_\tau\varphi(x)dx + \int_\Omega (|\nabla\Delta u|^{p-2}\nabla\Delta u)_h(x,\tau)\nabla\varphi dx = 0, \tag{3.2}$$

where

$$u_h(x,t) = \begin{cases}\dfrac{1}{h}\displaystyle\int_t^{t+h} u(\cdot,\tau)d\tau, & t\in(0,T-h),\\ 0, & t>T-h.\end{cases}$$

Proof of Theorem 3.1. Suppose that u_1,u_2 are two solutions of the problem (1.1)–(1.3). Then we have

$$\begin{aligned}&\int_\Omega (u_1(x,\tau)-u_2(x,\tau))_{h\tau}\varphi(x)dx\\ &-\int_\Omega (|\nabla\Delta u_1|^{p-2}\nabla\Delta u_1 - |\nabla\Delta u_2|^{p-2}\nabla\Delta u_2)_h(x,\tau)\nabla\varphi dx = 0.\end{aligned}$$

For fixed τ, taking $\varphi(x) = [\Delta(u_1-u_2)]_h \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}&\int_\Omega \nabla(u_1(x,\tau)-u_2(x,\tau))_{h\tau}\nabla(u_1-u_2)_h dx\\ =&-\int_\Omega [(|\nabla\Delta u_1|^{p-2}\nabla\Delta u_1 - |\nabla\Delta u_2|^{p-2}\nabla\Delta u_2)_h](x,\tau)\nabla\Delta(u_1-u_2)_h dx.\end{aligned}$$

Integrating the above equality with respect to τ over $(0,t)$, we have

$$\int_\Omega |\nabla(u_1-u_2)_h|^2(x,t)dx \le 0.$$

By Poincaré inequality, it follows that

$$\int_\Omega |(u_1-u_2)_h|^2 dx = 0,$$

therefore $u_1 = u_2$.

§ 4. Asymptotic Behavior

This section is devoted to the asymptotic behavior of solutions. To this purpose, we first show the following theorem.

Theorem 4.1. *The weak solution u obtained in Theorem 2.1 satisfies, for any $0 \le \rho \in C^2(\overline{\Omega})$,*

$$\begin{aligned}&\frac{1}{2}\int_\Omega \rho(x)|\nabla u(x,t)|^2dx - \frac{1}{2}\int_\Omega \rho(x)|\nabla u_0(x)|^2dx\\ =& -\iint_{Q_t} |\nabla\Delta u|^{p-2}\nabla\Delta u\nabla \mathrm{div}(\rho(x)\nabla u)dx, \end{aligned}\tag{4.1}$$

where $Q_t = \Omega \times (0,t)$.

Proof. In the proof of Theorem 2.1, we have

$$f(t) = \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2dx \in C([0,T]).$$

Similarly, we can also easily prove that for any $0 \le \rho \in C^2(\overline{\Omega})$,

$$f_\rho(t) = \frac{1}{2}\int_\Omega \rho(x)|\nabla u(x,t)|^2dx \in C([0,T]).$$

Consider the functional

$$\Phi_\rho[v] = \frac{1}{2}\int_\Omega \rho(x)|\nabla v|^2dx.$$

It is easy to see that $\Phi_\rho[v]$ is a convex functional on $H_0^1(\Omega)$.

For any $\tau \in (0,T)$ and $h > 0$, we have

$$\Phi_\rho[u(\tau+h)] - \Phi_\rho[u(\tau)] \ge \langle u(\tau+h)-u(\tau), -\mathrm{div}(\rho(x)\nabla u(x,\tau))\rangle.$$

By

$$\frac{\delta\Phi_\rho[v]}{\delta v} = -\mathrm{div}(\rho(x)\nabla v),$$

for any fixed $t_1, t_2 \in [0,T]$, $t_1 < t_2$, integrating the above inequality with respect to τ over (t_1,t_2) , we have

$$\int_{t_2}^{t_2+h}\Phi_\rho[u(\tau)]d\tau - \int_{t_1}^{t_1+h}\Phi_\rho[u(\tau)]d\tau \ge \int_{t_1}^{t_2}\langle u(\tau+h)-u(\tau), -\mathrm{div}(\rho(x)\nabla u)\rangle d\tau.$$

Multiplying the both side of the above equality by $\frac{1}{h}$, and letting $h \to 0$, we obtain

$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] \ge \int_{t_1}^{t_2}\Big\langle \frac{\partial u}{\partial t}, -\mathrm{div}(\rho(x)\nabla u)\Big\rangle d\tau.$$

Similarly, we have

$$\Phi_\rho[u(\tau)] - \Phi_\rho[u(\tau-h)] \le \langle (u(\tau)-u(\tau-h)), -\mathrm{div}(\rho(x)\nabla u)\rangle.$$

Thus
$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] \le \int_{t_1}^{t_2} \Big\langle \frac{\partial u}{\partial t}, -\mathrm{div}(\rho(x)\nabla u)\Big\rangle d\tau,$$
and hence
$$\Phi_\rho[u(t_2)] - \Phi_\rho[u(t_1)] = \int_{t_1}^{t_2} \Big\langle \frac{\partial u}{\partial t}, -\mathrm{div}(\rho(x)\nabla u)\Big\rangle d\tau.$$
Taking $t_1 = 0, t_2 = t$, we get from the definition of solutions that
$$\begin{aligned}\Phi_\rho[u(t)] - \Phi_\rho[u(0)] &= \int_0^t \langle -\mathrm{div}(|\nabla\Delta u|^{p-2}\nabla\Delta u), -\mathrm{div}(\rho(x)\nabla u(\tau))\rangle d\tau \\ &= -\int_0^t \langle |\nabla\Delta u|^{p-2}\nabla\Delta u, \nabla[\mathrm{div}(\rho(x)\nabla u(\tau))]\rangle d\tau.\end{aligned}$$

Theorem 4.2. *Let u be the weak solution of the problem* (1.1)–(1.3)*, $p > 2$. Then*
$$\int_\Omega |u(x,t)|^2 dx \le \frac{C_3}{(C_1 t + C_2)^\alpha}, \qquad \alpha = \frac{2}{p-2},\ C_i > 0,\ i = 1,2,3.$$

Proof. Taking $\rho(x) = 1$ in the equality (4.1), we have
$$\frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx = -\int_0^t \int_\Omega |\nabla\Delta u|^p dx dt. \tag{4.2}$$
Let
$$f(t) = \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx.$$
Then, by (4.2), we have
$$f'(t) = -\int_\Omega |\nabla\Delta u|^p dx \le 0.$$

Similarly to the discussion in the proof of Lemma 2.1, by $u \in W^{3,p}(\Omega), u, \Delta u \in W_0^{1,p}(\Omega)$, we see that
$$\int_\Omega |\nabla u(x,t)|^2 dx \le C\int_\Omega |\nabla\Delta u|^2 dx \le C\Big(\int_\Omega |\nabla\Delta u|^p dx\Big)^{2/p},$$
that is $f(t) \le C|f'(t)|^{2/p}$.

Again by $f'(t) \le 0$, we have $f'(t) \le -Cf(t)^{p/2}$, and hence
$$\int_\Omega |\nabla u(x,t)|^2 dx \le \frac{1}{(C_1 t + C_2)^\alpha}, \qquad \alpha = \frac{2}{p-2},\ C_i > 0,\ i = 1,2.$$
By Poincaré inequality, we have
$$\int_\Omega |u|^2 dx \le C\int_\Omega |\nabla u|^2 dx.$$
Therefore
$$\int_\Omega |u(x,t)|^2 dx \le \frac{C_3}{(C_1 t + C_2)^\alpha}, \qquad \alpha = \frac{2}{p-2}.$$
The proof is complete.

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