Chin. Ann. Math.
25B:3(2004),401–412.

ON JENSEN'S INEQUALITY FOR g-EXPECTATION***

JIANG Long*   CHEN Zengjing**

Abstract

Briand et al. gave a counterexample showing that given g, Jensen's inequality for g-expectation usually does not hold in general. This paper proves that Jensen's inequality for g-expectation holds in general if and only if the generator $g(t,z)$ is super-homogeneous in z. In particular, g is not necessarily convex in z.

Keywords Backward stochastic differential equation, Jensen's inequality, g-expectation, Conditional g-expectation, Comparison theorem
2000 MR Subject Classification 60H10

§ 1. Introduction

It is by now well known that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE in short) of type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dB_s, \qquad 0 \le t \le T, \tag{1.1}$$

providing that the generator g is Lipschitz in both variables y and z, and that ξ and the process $g(\cdot, 0, 0)$ are square integrable. We denote the unique solution of the BSDE (1.1) by $(y^\xi(t), z^\xi(t))_{t\in[0,T]}$.

In [1], $y^\xi(0)$, denoted by $\mathcal{E}_g[\xi]$, is called g-expectation of ξ. The notion of g-expectation can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying g-expectation comes from the theory of expected utility, which is the foundation of modern mathematical economics. Z. Chen and L. Epstein [2] gave an application of g-expectation to recursive utility. Since the notion of g-expectation was introduced, many properties of g-expectation have been studied in [1, 3–5]. Some properties of classical expectation are preserved (monotonicity for instance), and some results on Jensen's inequality for g-expectation were obtained in [3, 5]. But also in [3], the authors gave a counterexample to indicate that even for a linear function φ, which is obviously convex, Jensen's inequality for g-expectation usually does not hold. This yields a natural question:

What kind of generator g can make Jensen's inequality for g-expectation hold in general? Roughly speaking, for convex function $\varphi : \mathbf{R} \to \mathbf{R}$, what conditions should be given

Manuscript received March 4, 2003. Revised November 9, 2003.
*School of Mathematics and System Sciences, Shandong University, Jinan 250100, China.
Department of Mathematics, China University of Mining and Technology, Xuzhou 221008, Jiangsu, China. **E-mail:** jianglong@math.sdu.edu.cn
**School of Mathematics and System Sciences, Shandong University, Jinan 250100, China.
***Project supported by the National Natural Science Foundation of China (No.10131030).

to the generator g such that the following inequality

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t] \geq \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]$$

will hold in general?

The objective of this paper is to investigate this problem and to prove that Jensen's inequality for g-expectation holds in general if and only if $g(t,z)$ is super-homogeneous, and if g is convex, then Jensen's inequality for g-expectation holds in general if and only if $g(t,z)$ is a positive-homogeneous generator; For monotonic convex function φ, we also get two necessary and sufficient conditions.

§ 2 . Preliminaries

2.1. Notations and Assumptions

Let $(\Omega,\mathcal{F},P)$ be a probability space and $(B_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion on this space such that $B_0 = 0$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by this Brownian motion

$$\mathcal{F}_t = \sigma\{B_s, s\in[0,t]\}\vee\mathcal{N}, \qquad t\in[0,T],$$

where $\mathcal{N}$ is the set of all P-null subsets.

Let $T>0$ be a given real number. In this paper, we always work in the space $(\Omega,\mathcal{F}_T,P)$, and only consider processes indexed by $t\in[0,T]$. For any positive integer n and $z\in\mathbf{R}^n$, $|z|$ denotes its Euclidean norm.

We define the following usual spaces of processes:

$$\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbf{R}) := \Big\{\psi \text{ continuous and progressively measurable; } \mathbf{E}\Big[\sup_{0\leq t\leq T}|\psi_t|^2\Big]<\infty\Big\};$$

$$\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^n) := \Big\{\psi \text{ progressively measurable; } \|\psi\|_2^2 = \mathbf{E}\Big[\int_0^T|\psi_t|^2dt\Big]<\infty\Big\}.$$

We recall the notion of g-expectation, defined in [1]. We are given a function

$$g:\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d\longrightarrow\mathbf{R}$$

such that the process $(g(t,y,z))_{t\in[0,T]}$ is progressively measurable for each pair (y,z) in $\mathbf{R}\times\mathbf{R}^d$, and furthermore, g satisfies some of the following assumptions:

(A1) There exists a constant $K\geq 0$, such that P -a.s., we have

$$\forall\, t\in[0,T],\ \forall\, y_1,y_2\in\mathbf{R},\ z_1,z_2\in\mathbf{R}^d,$$
$$|g(t,y_1,z_1)-g(t,y_2,z_2)|\leq K(|y_1-y_2|+|z_1-z_2|).$$

(A2) The process $(g(t,0,0))_{t\in[0,T]}\in\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R})$.
(A3) P-a.s., $\forall\,(t,y)\in[0,T]\times\mathbf{R},\ g(t,y,0)\equiv 0$.
(A4) P-a.s., $\forall\,(y,z)\in\mathbf{R}\times\mathbf{R}^d,\ t\to g(t,y,z)$ is continuous.

Remark 2.1. The assumption (A3) implies the assumption (A2).

Let g satisfy the assumptions (A1) and (A2). Then for each $\xi\in L^2(\Omega,\mathcal{F}_T,P)$, there exists a unique pair $(y^\xi(t),z^\xi(t))_{t\in[0,T]}$ of adapted processes in $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbf{R})\times\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ solving the BSDE (1.1) (see [6]). We often denote $(y^\xi(t),z^\xi(t))_{t\in[0,T]}$ by $(y_t,z_t)_{t\in[0,T]}$ in short.

2.2. Definitions and Propositions

For the convenience of readers, we recall the notion of g-expectation and conditional g-expectation defined in [1]. We also list some basic properties of BSDEs and g-expectation. In the following Definitions 2.1 and 2.2, we always assume that g satisfies (A1) and (A3).

Definition 2.1. *The g-expectation $\mathcal{E}_g[\,\cdot\,] : L^2(\Omega, \mathcal{F}_T, P) \longmapsto \mathbf{R}$ is defined by*

$$\mathcal{E}_g[\xi] = y^\xi(0).$$

Definition 2.2. *The conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by*

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = y^\xi(t).$$

The following Comparison Theorem is one of the great achievements of theory of BSDEs, readers can see the proof in [7] or [8].

Proposition 2.1. (cf. [7, 8]) *Let g, $\bar{g}$ satisfy* (A1) *and* (A2)*, let Y_T, $\overline{Y}_T \in L^2(\Omega, \mathcal{F}_T, P)$. Let $(y(t), z(t))_{t\in[0,T]}$, $(\bar{y}(t), \bar{z}(t))_{t\in[0,T]}$ be the solutions of the following two BSDEs:*

$$y_t = Y_T + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dB_s, \qquad 0 \le t \le T;$$
$$\bar{y}_t = \overline{Y}_T + \int_t^T \bar{g}(s, \bar{y}_s, \bar{z}_s)ds - \int_t^T \bar{z}_s dB_s, \qquad 0 \le t \le T.$$

(1) *If $Y_T \ge \overline{Y}_T$, $g(t, \bar{y}_t, \bar{z}_t) \ge \bar{g}(t, \bar{y}_t, \bar{z}_t)$, a.s., a.e., then we have*

$$y_t \ge \bar{y}_t, \qquad \text{a.e., a.s.}$$

(2) *In addition, if we also assume that $P(Y_T - \overline{Y}_T > 0) > 0$, then*

$$P(y_t - \bar{y}_t > 0) > 0, \quad \text{in particular}, \quad y_0 > \bar{y}_0.$$

Propositions 2.2–2.5 come from [1], where g is assumed to satisfy (A1) and (A3).

Proposition 2.2. (1) (Preserving of constants) *For each constant c, $\mathcal{E}_g[c] = c$;*
(2) (Monotonicity) *If $X_1 \ge X_2$, a.s., then $\mathcal{E}_g[X_1] \ge \mathcal{E}_g[X_2]$;*
(3) (Strict Monotonicity) *If $X_1 \ge X_2$, a.s., and $P(X_1 > X_2) > 0$, then $\mathcal{E}_g[X_1] > \mathcal{E}_g[X_2]$.*

Proposition 2.3. (1) *If X is $\mathcal{F}_t$-measurable, then $\mathcal{E}_g[X|\mathcal{F}_t] = X$;*
(2) *For all $t, s \in [0, T]$, $\mathcal{E}_g[\mathcal{E}_g[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}_g[X|\mathcal{F}_{t\wedge s}]$.*

Proposition 2.4. *$\mathcal{E}_g[X|\mathcal{F}_t]$ is the unique random variable η in $L^2(\Omega, \mathcal{F}_t, P)$, such that*

$$\mathcal{E}_g[X 1_A] = \mathcal{E}_g[\eta 1_A] \qquad \text{for all} \quad A \in \mathcal{F}_t.$$

Proposition 2.5. *Let $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \longmapsto \mathbf{R}$ be a given function satisfying* (A1) *and* (A3)*. If g does not depend on y, then we have*

$$\mathcal{E}_g[X + \eta|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t] + \eta, \qquad \forall\, \eta \in L^2(\Omega, \mathcal{F}_t, P), \quad \forall\, X \in L^2(\Omega, \mathcal{F}_T, P).$$

Proposition 2.6. (cf. [3, 8]) *Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, and let the assumptions* (A1) *and* (A2) *hold. If the process $(y_t, z_t)_{t\in[0,T]}$ is the solution of BSDE* (1.1), *then we have*

$$\mathbf{E}\Big[\sup_{t\le s\le T}(e^{\beta s}|y_s|^2)+\int_t^T e^{\beta s}|z_s|^2 ds|\mathcal{F}_t\Big]$$
$$\le C\mathbf{E}\Big[e^{\beta T}|\xi|^2+\Big(\int_t^T e^{(\beta/2)s}|g(s,0,0)|ds\Big)^2\Big|\mathcal{F}_t\Big],$$

where $\beta = 2(K+K^2)$ and C is a universal constant.

Proposition 2.7. (cf. [3]) *Suppose g does not depend on y and g satisfies* (A1) *and* (A3). *Suppose moreover that for each $t \in [0,T]$, P-a.s., $z \longrightarrow g(t,z)$ is convex. Given $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, let $\varphi : \mathbf{R} \longrightarrow \mathbf{R}$ be a convex function such that $\varphi(\xi) \in L^2(\Omega, \mathcal{F}_T, P)$. If P-a.s., $\partial\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]\cap]0,1[^c \ne \emptyset$, then we have*

$$P\text{-a.s.},\quad \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] \le \mathcal{E}_g[\varphi(\xi|\mathcal{F}_t)].$$

Proposition 2.7 can be regarded as an important result on Jensen's inequality for g-expectation, but if $\partial\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]\cap]0,1[^c = \emptyset$, for example $\varphi(x) = x/2,\ \forall\, x \in \mathbf{R}$, Proposition 2.7 can not solve this kind of problems. It also can not tell us what kind of generator g can make Jensen's inequality hold in general.

§ 3. Jensen's Inequality for Super-homogeneous Generator g

In the following, we always consider the situation where the generator g does not depend on y, that is, $g : \Omega \times [0,T] \times \mathbf{R}^d \to \mathbf{R}$. We denote this kind of generator g by $g(t,z)$. We always assume that $g(t,z)$ satisfies (A1) and (A3).

Definition 3.1. *Let g satisfy* (A1) *and* (A3). *We say that g is a super-homogeneous generator in z if g also satisfies*

$$P\text{-a.s.},\quad \forall\,(t,z) \in [0,T]\times\mathbf{R}^d,\ \lambda \in \mathbf{R}:\quad g(t,\lambda z) \ge \lambda g(t,z).$$

Now we introduce our main results on Jensen's inequality for g-expectation.

Theorem 3.1. *Let g satisfy* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent:*

(i) *g is a super-homogeneous generator;*

(ii) *Jensen's inequality for g-expectation holds in general, i.e., for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ and convex function $\varphi : \mathbf{R} \to \mathbf{R}$, if $\varphi(\xi) \in L^2(\Omega, \mathcal{F}_T, P)$, then for each $t \in [0,T]$, P-a.s.,*

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t] \ge \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

Proof. (i)⇒(ii). Given $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ and convex function φ such that $\varphi(\xi) \in L^2(\Omega, \mathcal{F}_T, P)$, for each $t \in [0,T]$, we set $\eta_t = \varphi'_-[\mathcal{E}_g(\xi|\mathcal{F}_t)]$. Then η_t is $\mathcal{F}_t$-measurable. Since φ is convex, we have

$$\varphi(x) - \varphi(y) \ge \varphi'_-(y)(x-y), \qquad \forall\, x, y \in R.$$

Take $x = \xi,\ y = \mathcal{E}_g(\xi|\mathcal{F}_t)$. Then we have

$$\varphi(\xi) - \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] \ge \eta_t[\xi - \mathcal{E}_g(\xi|\mathcal{F}_t)].$$

For each positive integer n, we define

$$\Omega_{t,n} := \{|\mathcal{E}_g(\xi|\mathcal{F}_t)| + |\eta_t| + |\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]| \le n\}.$$

Because $\mathcal{E}_g[\xi|\mathcal{F}_t]$, η_t, $\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]$ are all $\mathcal{F}_t$-measurable, we see that $\Omega_{t,n} \in \mathcal{F}_t$. We denote the indicator function of $\Omega_{t,n}$ by $\mathbf{1}_{\Omega_{t,n}}$. Set $\eta_{t,n} = \mathbf{1}_{\Omega_{t,n}}\eta_t$. Then we have

$$\mathbf{1}_{\Omega_{t,n}}[\varphi(\xi) - \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]] \ge \eta_{t,n}[\xi - \mathcal{E}_g(\xi|\mathcal{F}_t)]. \tag{3.1}$$

Since $\eta_{t,n}$, $\mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]$ are bounded by n and $\xi, \varphi(\xi) \in L^2(\Omega, \mathcal{F}_T, P)$, we deduce that

$$\begin{aligned}\mathbf{1}_{\Omega_{t,n}}\varphi(\xi),\ \eta_{t,n}\xi &\in L^2(\Omega, \mathcal{F}_T, P),\\ \mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] &\in L^2(\Omega, \mathcal{F}_t, P),\\ (\eta_{t,n}\mathcal{E}_g(\xi|\mathcal{F}_s))_{t\le s\le T} &\in S^2_{\mathcal{F}}(t, T; R).\end{aligned}$$

From the well-known Comparison Theorem we know that conditional g-expectation $\mathcal{E}_g[\,\cdot\,|\mathcal{F}_t]$ is nondecreasing. Thus from the inequality (3.1), and by taking conditional g-expectation, we can get

$$\mathcal{E}_g[\mathbf{1}_{\Omega_{t,n}}[\varphi(\xi) - \varphi(\mathcal{E}_g(\xi|\mathcal{F}_t))]|\mathcal{F}_t] \ge \mathcal{E}_g[\eta_{t,n}[\xi - \mathcal{E}_g(\xi|\mathcal{F}_t)]|\mathcal{F}_t].$$

Since $\mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]$, $\eta_{t,n}\mathcal{E}_g[\xi|\mathcal{F}_t] \in L^2(\Omega, \mathcal{F}_t, P)$, it follows from Proposition 2.5 that

$$\mathcal{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathcal{F}_t] - \mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] \ge \mathcal{E}_g[\eta_{t,n}\xi|\mathcal{F}_t] - \eta_{t,n}\mathcal{E}_g[\xi|\mathcal{F}_t]. \tag{3.2}$$

Let $(y_u, z_u)_{u\in[0,T]}$ be the solution of the following BSDE (3.3)

$$y_u = \xi + \int_u^T g(s, z_s)ds - \int_u^T z_s dB_s, \qquad 0 \le u \le T. \tag{3.3}$$

Then for the given $t \in [0, T]$, we have

$$\eta_{t,n}y_u = \eta_{t,n}\xi + \int_u^T \eta_{t,n}g(s, z_s)ds - \int_u^T \eta_{t,n}z_s dB_s, \qquad t \le u \le T. \tag{3.4}$$

We define function $g_1(s, z)$ in this way: for each $(s, z) \in [t, T] \times \mathbf{R}^d$,

$$g_1(s, z) := \begin{cases}\eta_{t,n}g(s, z/\eta_{t,n}), & \text{if } \ \eta_{t,n} \ne 0;\\ 0, & \text{if } \ \eta_{t,n} = 0.\end{cases}$$

Since $\eta_{t,n}$ is bounded, the following BSDE

$$\bar{y}_u = \eta_{t,n}\xi + \int_u^T g_1(s, \bar{z}_s)ds - \int_u^T \bar{z}_s dB_s, \qquad t \le u \le T \tag{3.5}$$

has a unique solution in $\mathcal{S}^2_{\mathcal{F}}(t, T; \mathbf{R}) \times \mathcal{H}^2_{\mathcal{F}}(t, T; \mathbf{R}^d)$. We denote it by $(\bar{y}_s, \bar{z}_s)_{s\in[t,T]}$. Also from that $\eta_{t,n}$ is bounded we know that $(\eta_{t,n}y_s, \eta_{t,n}z_s)_{s\in[t,T]}$ is in $\mathcal{S}^2_{\mathcal{F}}(t, T; \mathbf{R}) \times \mathcal{H}^2_{\mathcal{F}}(t, T; \mathbf{R}^d)$. From (3.4) and the definition of g_1, we conclude that the solution of BSDE (3.5) is just $(\eta_{t,n}y_s, \eta_{t,n}z_s)_{s\in[t,T]}$.

Consider the solutions of BSDE (3.5) and the following BSDE (3.6):

$$\tilde{y}_u = \eta_{t,n}\xi + \int_u^T g(s, \tilde{z}_s)ds - \int_u^T \tilde{z}_s dB_s, \qquad t \le u \le T. \tag{3.6}$$

Due to the super-homogeneity of $g(t,z)$ in z, we can get that for each $s \in [t,T]$, P-a.s.,

$$g(s, \eta_{t,n} z_s) \geq \eta_{t,n} g(s, z_s).$$

Combining this with the definition of g_1, we have, P-a.s.,

$$\forall\, s \in [t,T], \quad g(s, \bar{z}_s) = g(s, \eta_{t,n} z_s) \geq \eta_{t,n} g(s, z_s) = g_1(s, \eta_{t,n} z_s) = g_1(s, \bar{z}_s).$$

Thus from Comparison Theorem, we have, P-a.s.,

$$\mathcal{E}_g[\eta_{t,n}\xi|\mathcal{F}_t] = \tilde{y}_t \geq \bar{y}_t = \eta_{t,n} y_t = \eta_{t,n}\mathcal{E}_g[\xi|\mathcal{F}_t]. \tag{3.7}$$

Coming back to (3.2), we can get

$$\mathcal{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathcal{F}_t] - \mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] \geq \mathcal{E}_g[\eta_{t,n}\xi|\mathcal{F}_t] - \eta_{t,n}\mathcal{E}_g[\xi|\mathcal{F}_t] \geq 0.$$

Applying Lebesgue's dominated convergence theorem to $(\mathbf{1}_{\Omega_{t,n}}\varphi(\xi))_{n=1}^{\infty}$, we can get easily that

$$L^2 - \lim_{n\to\infty} \mathbf{1}_{\Omega_{t,n}}\varphi(\xi) = \varphi(\xi).$$

Since that $\xi \to \mathcal{E}_g(\xi|\mathcal{F}_t)$ is a continuous map from $L^2(\mathcal{F}_T)$ into $L^2(\mathcal{F}_t)$ (see [1, Lemma 36.9]), it follows that

$$L^2 - \lim_{n\to\infty} \mathcal{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathcal{F}_t] = \mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t].$$

Thus for the given $t \in [0,T]$, there exists a subsequence $(\mathcal{E}_g[\varphi(\xi)\mathbf{1}_{\Omega_{t,n_i}}|\mathcal{F}_t])_{i=1}^{\infty}$ such that, P-a.s.,

$$\lim_{i\to\infty} \mathcal{E}_g[\varphi(\xi)\mathbf{1}_{\Omega_{t,n_i}}|\mathcal{F}_t] = \mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t].$$

On the other hand, by the definition of $\Omega_{t,n}$, we can get, P-a.s.,

$$\lim_{n\to\infty} \mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] = \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

Hence we can assert that (i) implies (ii). Indeed, P-a.s.,

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t] = \lim_{i\to\infty} \mathcal{E}_g[\mathbf{1}_{\Omega_{t,n_i}}\varphi(\xi)|\mathcal{F}_t] \geq \lim_{i\to\infty} \mathbf{1}_{\Omega_{t,n_i}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] = \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

(ii)⇒(i). Firstly we show that for each $z \in \mathbf{R}^d$, $t \in [0,T[$,

$$L^2 - \lim_{n\to\infty} n[\mathcal{E}_g(z \cdot (B_{t+1/n} - B_t)|\mathcal{F}_t)] = g(t,z). \tag{3.8}$$

(3.8) is a special case of [3, Proposition 2.3]. But for the convenience of readers and the completeness of our proof, here we give a straightforward proof. For each given $z \in \mathbf{R}^d$, $t \in [0,T[$, we choose a large enough positive integer n, such that $t + 1/n \leq T$. We denote by $(y_{s,n}, z_{s,n})_{s\in[t,t+1/n]}$ the solution of the following BSDE:

$$y_s = z \cdot (B_{t+1/n} - B_t) + \int_s^{t+1/n} g(u, z_u)du - \int_s^{t+1/n} z_u dB_u, \qquad t \leq s \leq t + 1/n. \tag{3.9}$$

We set

$$\bar{y}_{s,n} = y_{s,n} - z \cdot (B_s - B_t), \quad \bar{z}_{s,n} = z_{s,n} - z.$$

Then we have $y_{t,n} = \bar{y}_{t,n}$ and

$$\bar{y}_{s,n} = \int_s^{t+1/n} g(u, \bar{z}_{u,n} + z)du - \int_s^{t+1/n} \bar{z}_{u,n} dB_u, \qquad t \leq s \leq t + 1/n. \tag{3.10}$$

Since

$$\mathcal{E}_g[z \cdot (B_{t+1/n} - B_t)|\mathcal{F}_t] = y_{t,n} = \bar{y}_{t,n} = \mathbf{E}\Big[\int_t^{t+1/n} g(s, \bar{z}_{s,n} + z)ds|\mathcal{F}_t\Big],$$

by the classical Jensen's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\mathbf{E}[n\mathcal{E}_g[z \cdot (B_{t+\frac{1}{n}} - B_t)|\mathcal{F}_t] - g(t,z)]^2 \\
&= \mathbf{E}\Big[n\mathbf{E}\Big[\int_t^{t+\frac{1}{n}} (g(s, \bar{z}_{s,n} + z) - g(t,z))ds|\mathcal{F}_t\Big]\Big]^2 \\
&\le n^2\mathbf{E}\Big[\int_t^{t+1/n} (g(s, \bar{z}_{s,n} + z) - g(t,z))ds\Big]^2 \\
&\le n\mathbf{E}\int_t^{t+1/n} |g(s, \bar{z}_{s,n} + z) - g(t,z)|^2 ds \\
&\le 2n\mathbf{E}\int_t^{t+1/n} |g(s, \bar{z}_{s,n} + z) - g(s,z)|^2 ds \\
&\quad + 2n\mathbf{E}\int_t^{t+1/n} |g(s,z) - g(t,z)|^2 ds.
\end{aligned} \tag{3.11}$$

By (A1), Proposition 2.6 and (A3), we know that there exists a universal constant C such that

$$\begin{aligned}
&2n\mathbf{E}\int_t^{t+1/n} |g(s, \bar{z}_{s,n} + z) - g(s,z)|^2 ds \\
&\le 2nK^2\mathbf{E}\int_t^{t+1/n} |\bar{z}_{s,n}|^2 ds \\
&\le 2nK^2C\mathbf{E}\Big(\int_t^{t+1/n} |g(s,z)|ds\Big)^2 \\
&\le 2nK^2C\mathbf{E}\Big(\int_t^{t+1/n} K|z|ds\Big)^2 \\
&= 2K^4C|z|^2/n,
\end{aligned}$$

where K is the Lipschitz constant.

By (A4), we know that

$$P\text{-a.s.,} \quad \lim_{n\to\infty} 2n\int_t^{t+1/n} |g(s,z) - g(t,z)|^2 ds = 0.$$

In view of (A3) and (A1), we have

$$2n\int_t^{t+1/n} |g(s,z) - g(t,z)|^2 ds \le 2n\int_t^{t+1/n} (2K|z|)^2 ds = 8K^2|z|^2.$$

It follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} 2n\mathbf{E}\int_t^{t+1/n} |g(s,z) - g(t,z)|^2 ds = 0.$$

Then coming back to (3.11), we can get

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{E}[n\mathcal{E}_g(z\cdot(B_{t+1/n}-B_t)|\mathcal{F}_t)-g(t,z)]^2\\ \le&\lim_{n\to\infty}2K^4C|z|^2/n+\lim_{n\to\infty}2n\mathbf{E}\int_t^{t+1/n}|g(s,z)-g(t,z)|^2ds=0.\end{aligned}$$

Therefore we have

$$L^2-\lim_{n\to\infty}n[\mathcal{E}_g(z\cdot(B_{t+1/n}-B_t)|\mathcal{F}_t)]=g(t,z).$$

Secondly we prove that for each triple $(t,z,\lambda)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}$, we have

$$P\text{-a.s.,}\quad g(t,\lambda z)\ge\lambda g(t,z). \tag{3.12}$$

Given $\lambda\in\mathbf{R}$, we define a corresponding convex function $\varphi_\lambda:\mathbf{R}\to\mathbf{R}$, such that $\varphi_\lambda(x)=\lambda x,\forall\, x\in\mathbf{R}$. Given $t\in[0,T[$, let us pick a large enough positive integer n, such that $t+1/n\le T$. Then for each $z\in\mathbf{R}^d$, it is obvious that $\varphi_\lambda(z\cdot(B_{t+1/n}-B_t))\in L^2(\Omega,\mathcal{F}_T,P)$. By (ii), we know that, P-a.s.,

$$\mathcal{E}_g[\varphi_\lambda(z\cdot(B_{t+1/n}-B_t))|\mathcal{F}_t]\ge\varphi_\lambda[\mathcal{E}_g(z\cdot(B_{t+1/n}-B_t)|\mathcal{F}_t)];$$

that is, P-a.s.,

$$\mathcal{E}_g[\lambda z\cdot(B_{t+1/n}-B_t)|\mathcal{F}_t]\ge\lambda[\mathcal{E}_g(z\cdot(B_{t+1/n}-B_t)|\mathcal{F}_t)]. \tag{3.13}$$

Because of (3.8), we know there exists a subsequence $\{n_k\}_{k=1}^\infty$ such that

$$\begin{aligned}&P\text{-a.s.,}\quad\lim_{k\to\infty}n_k[\mathcal{E}_g(\lambda z\cdot(B_{t+1/n_k}-B_t)|\mathcal{F}_t)]=g(t,\lambda z),\\ &P\text{-a.s.,}\quad\lim_{k\to\infty}\lambda n_k[\mathcal{E}_g(z\cdot(B_{t+1/n_k}-B_t)|\mathcal{F}_t)]=\lambda g(t,z).\end{aligned}$$

Thus for the given $t\in[0,T[$, $z\in\mathbf{R}^d,\lambda\in\mathbf{R}$, by (3.13), we have

$$P\text{-a.s.,}\quad g(t,\lambda z)\ge\lambda g(t,z).$$

By (A4), we know that for each z, the process $t\longrightarrow g(t,z)$ is continuous. Hence we have

$$P\text{-a.s.,}\quad g(T,\lambda z)=\lim_{\varepsilon\to0^+}g(T-\varepsilon,\lambda z)\ge\lim_{\varepsilon\to0^+}\lambda g(T-\varepsilon,z)=\lambda g(T,z).$$

Therefore we can get (3.12) immediately. The proof is complete.

Remark 3.1. When we prove that (i) implies (ii), we do not need (A4).

Example 3.1. Let $g:\mathbf{R}\to\mathbf{R}$ be defined as follows: $g(z)=z^4$, if $|z|\le1$ and $g(z)=4|z|-3$, if $|z|>1$. We can see clearly that though g is convex, g is not super-homogeneous. Thus for this generator g, by Theorem 3.1, we know that Jensen's inequality for g-expectation does not hold in general.

In fact, if we take $T=1$, $\xi=B_T-T$ and $\varphi(x)=\frac{x}{3}$, $\forall x\in\mathbf{R}$, then we can verify that $(B_t-t,1)_{t\in[0,T]}$ is the solution of the following BSDE:

$$y_t=\xi+\int_t^Tg(z_s)ds-\int_t^Tz_sdB_s,\qquad0\le t\le T,$$

and $(\frac{B_t}{3}-\frac{26T+t}{81},\frac{1}{3})_{t\in[0,T]}$ is the solution of the following BSDE:

$$\bar{y}_t=\varphi(\xi)+\int_t^T g(\bar{z}_s)ds-\int_t^T \bar{z}_s dB_s, \qquad 0\le t\le T.$$

We can calculate that

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t]-\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]=\frac{26}{81}(t-T)<0, \qquad \text{when} \ \ t<T.$$

Example 3.1 yields a natural question: What kind of convex generator g can make Jensen's inequality for g-expectation hold in general? The following Theorem 3.2 will answer this question.

Definition 3.2. *We call a generator $g(t,z)$ is positive-homogeneous in z if*

$$P\text{-a.s.}, \quad \forall\,\lambda\ge 0,\ t\in[0,T],\ z\in\mathbf{R}^d, \quad g(t,\lambda z)=\lambda g(t,z).$$

Theorem 3.2. *Suppose g satisfies* (A1), (A3) *and* (A4). *Suppose moreover that for each $t\in\mathbf{R}$, P-a.s., $z\longrightarrow g(t,z)$ is convex in z. Then the following two conditions are equivalent:*

(i) *$g(t,z)$ is positive-homogeneous in z;*

(ii) *Jensen's inequality for g-expectation holds in general.*

Proof. By Theorem 3.1, it suffices to prove that if $g(t,z)$ is convex in z and $g(t,0)\equiv 0$, then $g(t,z)$ is positive-homogeneous in z if and only if $g(t,z)$ is super-homogeneous.

Suppose $g(t,z)$ is positive-homogeneous in z. We only need to consider the case when $\lambda\le 0$. For each $\lambda\le 0$, $(t,z)\in[0,T]\times\mathbf{R}^d$, since g is convex and $g(t,0)\equiv 0$, we have, P-a.s.,

$$0=g(t,0)=g\Big(t,\frac{\lambda z}{2}+\frac{(-\lambda)z}{2}\Big)\le\frac{g(t,\lambda z)}{2}+\frac{g(t,-\lambda z)}{2}=\frac{g(t,\lambda z)}{2}+\frac{-\lambda g(t,z)}{2}.$$

Thus we have

$$P\text{-a.s.}, \quad \forall\,\lambda\le 0,\ (t,z)\in[0,T]\times\mathbf{R}^d, \quad g(t,\lambda z)\ge\lambda g(t,z).$$

Hence $g(t,z)$ is super-homogeneous.

Suppose $g(t,z)$ is super-homogeneous. For each given triple $(t,z,\lambda)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}_+$, if $0\le\lambda\le 1$, then by the convexity of g and (A3) we have

$$P\text{-a.s.}, \quad g(t,\lambda z)\le\lambda g(z).$$

Thus by the super-homogeneity of g, we have, P-a.s.,

$$\forall\,\lambda\in[0,1],\ t\in[0,T], \quad g(t,\lambda z)=\lambda g(t,z). \tag{3.14}$$

For $\lambda>1$, it follows from (3.14) that P-a.s.,

$$\lambda g(t,z)=\lambda g\Big(t,\frac{1}{\lambda}\times(\lambda z)\Big)=\lambda\times\frac{1}{\lambda}\times g(t,\lambda z)=g(t,\lambda z).$$

Thus $g(t,z)$ is positive-homogeneous. This completes the proof.

Corollary 3.1. *Given $\mu\ge 0$, let the generator $g(t,z)=\mu|z|$, $\forall\,(t,z)\in[0,T]\times\mathbf{R}^d$. Then Jensen's inequality for g-expectation holds in general.*

This kind of g-expectation $\mathcal{E}_g[\,\cdot\,]$ plays a key role in [4].

§ 4. Jensen's Inequality for Monotonic Convex Function φ

In this section, we will consider the following problem: If g is independent of y, φ is a monotonic convex function, then what conditions should be given to the generator g, such that Jensen's inequality for g-expectation holds for φ? We will give two necessary and sufficient conditions to solve this problem, one condition is for increasing convex function φ, the other condition is for decreasing convex function φ.

Theorem 4.1. *Let g satisfy* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent:*

(i) *P-a.s., $\forall\,(t,z,\lambda)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}_+$, $g(t,\lambda z)\geq\lambda g(t,z)$;*

(ii) *Jensen's inequality for g-expectation holds for increasing convex function, i.e., for each $\xi\in L^2(\Omega,\mathcal{F}_T,P)$ and increasing convex function $\varphi:\mathbf{R}\to\mathbf{R}$, if $\varphi(\xi)\in L^2(\Omega,\mathcal{F}_T,P)$, then for each $t\in[0,T]$, P-a.s.,*

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t]\geq\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

Proof. (i)$\Rightarrow$(ii). Given $\xi\in L^2(\Omega,\mathcal{F}_T,P)$ and increasing convex function φ such that $\varphi(\xi)\in L^2(\Omega,\mathcal{F}_T,P)$. For each $t\in[0,T]$ and positive integer n, just as in the proof of Theorem 3.1, we set or define

$$\eta_t=\varphi'_-[\mathcal{E}_g(\xi|\mathcal{F}_t)],\quad \Omega_{t,n}:=\{|\mathcal{E}_g[\xi|\mathcal{F}_t]|+|\eta_t|+|\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]|\leq n\},\quad \eta_{t,n}=\mathbf{1}_{\Omega_{t,n}}\eta_t.$$

We already know that

$\Omega_{t,n}\in\mathcal{F}_t$, $\eta_{t,n},\mathbf{1}_{\Omega_{t,n}}$ are $\mathcal{F}_t$-measurable;
$\eta_{t,n}$, $\mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]$ are bounded by n;
$\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)$, $\eta_{t,n}\xi\in L^2(\Omega,\mathcal{F}_T,P)$, $\mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]\in L^2(\Omega,\mathcal{F}_t,P)$;
$(\eta_{t,n}\mathcal{E}_g(\xi|\mathcal{F}_s))_{s\in[t,T]}\in\mathcal{S}^2_{\mathcal{F}}(t,T;R)$.

Moreover, we also know that

$$\mathcal{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathcal{F}_t]-\mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)]\geq\mathcal{E}_g[\eta_{t,n}\xi|\mathcal{F}_t]-\eta_{t,n}\mathcal{E}_g[\xi|\mathcal{F}_t].\tag{4.1}$$

Let $(y_u,z_u)_{u\in[0,T]}$ be the unique square integrable solution of the following BSDE:

$$y_u=\xi+\int_u^T g(s,z_s)ds-\int_u^T z_s dB_s,\qquad 0\leq u\leq T.\tag{4.2}$$

Then for the given $t\in[0,T]$, we have

$$\eta_{t,n}y_u=\eta_{t,n}\xi+\int_u^T\eta_{t,n}g(s,z_s)ds-\int_u^T\eta_{t,n}z_s dB_s,\qquad t\leq u\leq T.\tag{4.3}$$

For the given t, again we define function $g_1(s,z)$ in this way: for each $(s,z)\in[t,T]\times\mathbf{R}^d$,

$$g_1(s,z):=\begin{cases}\eta_{t,n}g(s,z/\eta_{t,n}), & \text{if }\ \eta_{t,n}\neq 0;\\ 0, & \text{if }\ \eta_{t,n}=0.\end{cases}$$

Consider the solutions of the following BSDE (4.4) and BSDE (4.5):

$$\bar{y}_u=\eta_{t,n}\xi+\int_u^T g_1(s,\bar{z}_s)ds-\int_u^T\bar{z}_s dB_s,\qquad t\leq u\leq T,\tag{4.4}$$

$$\tilde{y}_u=\eta_{t,n}\xi+\int_u^T g(s,\tilde{z}_s)ds-\int_u^T\tilde{z}_s dB_s,\qquad t\leq u\leq T.\tag{4.5}$$

Analogous to the proof of Theorem 3.1, from (4.3) we deduce that $(\eta_{t,n}y_s, \eta_{t,n}z_s)_{s\in[t,T]}$ is the unique solution of BSDE (4.4).

For the given $t \in [0,T]$ and φ, since φ is increasing, we have

$$\eta_t = \varphi'_-[\mathcal{E}_g(\xi|\mathcal{F}_t)] \geq 0, \quad \eta_{t,n} = \mathbf{1}_{\Omega_{t,n}}\eta_t \geq 0.$$

In view of (i), for each $s \in [t,T], P$-a.s., we have

$$g(s, \eta_{t,n}z_s) \geq \eta_{t,n}g(s, z_s). \tag{4.6}$$

Therefore, for each $s \in [t,T]$, we can get, P-a.s.,

$$g(s, \bar{z}_s) = g(s, \eta_{t,n}z_s) \geq \eta_{t,n}g(s, z_s) = g_1(s, \eta_{t,n}z_s) = g_1(s, \bar{z}_s).$$

Thus from Comparison Theorem we have

$$P\text{-a.s.,} \quad \mathcal{E}_g[\eta_{t,n}\xi|\mathcal{F}_t] = \tilde{y}_t \geq \bar{y}_t = \eta_{t,n}y_t = \eta_{t,n}\mathcal{E}_g[\xi|\mathcal{F}_t]. \tag{4.7}$$

This with (4.1), it follows that

$$\mathcal{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathcal{F}_t] - \mathbf{1}_{\Omega_{t,n}}\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] \geq \mathcal{E}_g[\eta_{t,n}\xi|\mathcal{F}_t] - \eta_{t,n}\mathcal{E}_g[\xi|\mathcal{F}_t] \geq 0.$$

Applying Lebesgue's dominated theorem to $(\mathbf{1}_{\Omega_{t,n}}\varphi(\xi))_{n=1}^\infty$, we can get easily that

$$L^2 - \lim_{n\to\infty} \mathbf{1}_{\Omega_{t,n}}\varphi(\xi) = \varphi(\xi).$$

Similarly to the proof of Theorem 3.1, we can get

$$L^2 - \lim_{n\to\infty} \mathcal{E}_g[\mathbf{1}_{\Omega_{t,n}}\varphi(\xi)|\mathcal{F}_t] = \mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t].$$

Hence for each $t \in [0,T]$, P-a,s., we have

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t] \geq \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

(ii)⇒(i). Given $\lambda \geq 0$, we define a corresponding increasing convex function $\varphi_\lambda : \mathbf{R} \to \mathbf{R}$, such that $\varphi_\lambda(x) = \lambda x$, $\forall\, x \in \mathbf{R}$. For each $t \in [0,T[\,,\ z \in \mathbf{R}^d$, let us pick a large enough positive integer n, such that $t + 1/n \leq T$. It is obvious that $\varphi_\lambda(z\cdot(B_{t+1/n} - B_t)) \in L^2(\Omega, \mathcal{F}_T, P)$. By (ii), we know that Jensen's inequality holds for the increasing function φ_λ. Thus we have, P-a.s.,

$$\mathcal{E}_g[\varphi_\lambda(z\cdot(B_{t+1/n} - B_t))|\mathcal{F}_t] \geq \varphi_\lambda[\mathcal{E}_g(z\cdot(B_{t+1/n} - B_t)|\mathcal{F}_t)];$$

that is, P-a.s.,

$$\mathcal{E}_g[\lambda z\cdot(B_{t+1/n} - B_t)|\mathcal{F}_t] \geq \lambda[\mathcal{E}_g(z\cdot(B_{t+1/n} - B_t)|\mathcal{F}_t)]. \tag{4.8}$$

By (3.8), we know that there exists a subsequence $\{n_k\}_{k=1}^\infty$ such that

$$P\text{-a.s.,} \quad \lim_{k\to\infty} n_k[\mathcal{E}_g(\lambda z\cdot(B_{t+1/n_k} - B_t)|\mathcal{F}_t)] = g(t, \lambda z),$$
$$P\text{-a.s.,} \quad \lim_{k\to\infty} \lambda n_k[\mathcal{E}_g(z\cdot(B_{t+1/n_k} - B_t)|\mathcal{F}_t] = \lambda g(t, z).$$

Thus for each $t \in [0,T[\,,\ z \in \mathbf{R}^d,\ \lambda \geq 0$, it follows from (4.8) that

$$P\text{-a.s.,} \quad g(t, \lambda z) \geq \lambda g(t, z). \tag{4.9}$$

(A4) and (4.9) imply that

$$P\text{-a.s.,} \quad g(T, \lambda z) = \lim_{\varepsilon\to 0^+} g(T-\varepsilon, \lambda z) \geq \lim_{\varepsilon\to 0^+} \lambda g(T-\varepsilon, z) = \lambda g(T, z).$$

Hence (ii) implies (i). The proof is complete.

Corollary 4.1. *Given $\mu \geq 0$, let the generator $g(t,z) = -\mu|z|$, $\forall\,(t,z) \in [0,T] \times \mathbf{R}^d$. Then Jensen's inequality for g-expectation holds for increasing convex function φ.*

Similarly we can get the following

Theorem 4.2. *Let g satisfy* (A1), (A3) *and* (A4). *Then the following two conditions are equivalent:*

(i) *P-a.s., $\forall\,\lambda \leq 0$, $(t,z) \in [0,T] \times \mathbf{R}^d$, $g(t,\lambda z) \geq \lambda g(t,z)$;*

(ii) *Jensen's inequality for g-expectation holds for decreasing convex function, i.e., for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ and decreasing convex function $\varphi : \mathbf{R} \to \mathbf{R}$, if $\varphi(\xi) \in L^2(\Omega, \mathcal{F}_T, P)$, then for each $t \in [0,T]$, P-a.s.,*

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t] \geq \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

Proof. The proof of Theorem 4.2 is similar to that of Theorem 4.1. We omit it.

By Theorem 4.2, we can obtain the following corollary immediately.

Corollary 4.2. *Let g satisfy* (A1) *and* (A3). *If P-a.s., $\forall\,(t,z) \in [0,T]\times\mathbf{R}^d$, $g(t,z) \geq 0$, then Jensen's inequality for g-expectation holds for decreasing convex function φ.*

Acknowledgement. The authors thank Professor S. Peng and Professor J. Mémin for their comments and help, and also thank the referee for his suggestions.

References

[1] Peng, S., BSDE and related g-expectations, Pitman Research Notes in Mathematics Series, **364**, 1997, 141–159.

[2] Chen, Z. & Epstein, L., Ambiguity, risk and asset returns in continuous time, *Econometrica,* **70**(2002), 1403–1443.

[3] Briand, P., Coquet, F., Hu, Y., Mémin, J. & Peng, S., A converse comparison theorem for BSDEs and related properties of g-expectation, *Electon. Comm. Probab.,* **5**(2000), 101–117.

[4] Coquet, F., Hu, Y., Mémin, J. & Peng, S., Filtration consistent nonlinear expectations and related g-expectation, *Probab. Theory Related Fields,* **123**(2002), 1–27.

[5] Chen, Z. & Peng, S., A general downcrossing inequality for g-martingales, *Statistics and Probability Letters,* **46**(2000), 169–175.

[6] Pardoux, E. & Peng, S., Adapted solution of a backward stochastic differential equation, *Systems Control Letters,* **14**(1990), 55–61.

[7] Peng, S., A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation, *Stochastics,* **38:**2(1992), 119–134.

[8] El Karoui, N., Peng, S. & Quenez, M. C., Backward stochastic differential equations in finance, *Math. Finance,* **7:**1(1997), 1–71.