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Exact Synchronization for a Coupled System of Wave Equations with Dirichlet Boundary Controls

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(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract In this paper, the exact synchronization for a coupled system of wave equations with Dirichlet boundary controls and some related concepts are introduced. By means of the exact null controllability of a reduced coupled system, under certain conditions of compatibility, the exact synchronization, the exact synchronization by groups, and the exact null controllability and synchronization by groups are all realized by suitable boundary controls.

Keywords Exact null controllability, Exact synchronization, Exact synchronization by groups
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1 Introduction

Synchronization is a widespread natural phenomenon. Thousands of fireflies may twinkle at the same time; audiences in the theater can applaud with a rhythmic beat; pacemaker cells of the heart function simultaneously; and field crickets give out a unanimous cry. All these are phenomena of synchronization.

In principle, synchronization happens when different individuals possess likeness in nature, that is, they conform essentially to the same governing equation, and meanwhile, the individuals should bear a certain coupled relation.

The phenomenon of synchronization was first observed by Huygens [4]. The theoretical research on synchronization phenomena dates back to Fujisaka and Yamada's study of synchronization for coupled equations in 1983 (see [2]). The previous studies focused on systems described by ODEs, such as

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = f(X_i, t) + \sum_{j=1}^{N} A_{ij} X_j, \quad i = 1, \cdots, N, \tag{1.1}$$

where X_i $(i = 1, \cdots, N)$ are n-dimensional vectors, A_{ij} $(i, j = 1, \cdots, N)$ are $n \times n$ matrices, and $f(X, t)$ is an n-dimensional vector function independent of n. The right-hand side of

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(1.1) shows that every X_i $(i = 1, \cdots, N)$ possesses two basic features, that is, satisfying a fundamental governing equation and bearing a coupled relation among one another.

In this paper, we will consider the synchronization of the following hyperbolic system:

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} - \Delta U + AU = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \Gamma_0, \\ U = H & \text{on } \Gamma_1, \\ t = 0: \ U = U_0, \quad \dfrac{\partial U}{\partial t} = U_1, \end{cases} \tag{1.2}$$

where $U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}$ is the state variable, $A \in \mathbb{M}^N(\mathbb{R})$ is the coupling matrix, and $H = (h^{(1)}, \cdots, h^{(N)})^{\mathrm{T}}$ is the boundary control. Different from the ODE situation, the coupling of PDE systems can be fulfilled by coupling of the equations or (and) the boundary conditions. Our goal is to synchronize the state variable U through boundary control H. Roughly speaking, the problem is to find a $T > 0$, and through boundary control on $[0, T]$, we have that from time $t = T$ on, the system states tend to be the same. That is to say, we hope to achieve the synchronization of the system state not only at the moment $t = T$ under the action of boundary controls on $[0, T]$, but also when $t \geq T$ withdrawing all the controls. This is forever (instead of short-lived) synchronization, as is desired in many actual applications. Obviously, if the system has the exact boundary null controllability, it must have the exact synchronization, but this is a trivial situation that should be excluded beforehand.

The exact synchronization is linked with the exact null controllability. In fact, let $W = (w^{(1)}, \cdots, w^{(N-1)})^{\mathrm{T}}$ with $w^{(i)} = u^{(i+1)} - u^{(i)}$ $(i = 1, \cdots, N-1)$. Then under some conditions of compatibility on the coupling matrix A, the new state W satisfies a reduced system of $N-1$ equations as follows:

$$\begin{cases} \dfrac{\partial^2 W}{\partial t^2} - \Delta W + \overline{A} W = 0 & \text{in } \Omega, \\ W = 0 & \text{on } \Gamma_0, \\ W = \overline{H} & \text{on } \Gamma_1, \\ t = 0: \ W = W_0, \quad \dfrac{\partial W}{\partial t} = W_1, \end{cases} \tag{1.3}$$

where $\overline{A}$ is a matrix of order $N-1$. Under such conditions of compatibility, the exact synchronization of system (1.2) of N equations is equivalent to the exact null controllability of the reduced system (1.3) of $N-1$ equations. Our study will be based on two key points. We will first establish the exact null controllability of system (1.3) via the boundary control $\overline{H}$ of $N-1$ components. We next find some conditions of compatibility on the coupling matrix A to guarantee the reduction of system (1.2) to system (1.3).

There are many works on the exact controllability of hyperbolic systems by means of boundary controls. Generally speaking, one needs N boundary controls for the exact controllability of a system of N wave equations. In the case of less controls, we can not realize the exact controllability in general (see [8]). However, for smooth initial data, the exact controllability of two linear wave equations was proved by means of only one boundary control (see [1, 11]). Li and Rao [9] introduced the asymptotic controllability and established the equivalence between the asymptotic controllability of the original system and the weak observability of the dual system. Moreover, in [12], the optimal polynomial decay rate of energy of distributed systems with less boundary damping terms was studied by means of Riesz basis approach.

The exact synchronization is another way to weaken the notion of exact null controllability. In fact, instead of bringing all the states of system to zero, we only need to steer the states of the system to the same, which is unknown a priori. In terms of degree of freedom, we will use $N-1$ boundary controls to realize the exact synchronization for a system of N equations.

Now we briefly outline the contents of the paper. In Section 2, using a recent result on the observability of compactly perturbed systems of Mehrenberger [13], we establish the exact null controllability for (1.3). In Section 3, we consider the exact synchronization for the coupled system (1.2). We first give necessary conditions of compatibility on the coupling matrix A for the exact synchronization. We next prove that under these conditions of compatibility, the system (1.2) of N equations can be exactly synchronized by means of $N-1$ boundary controls. In Section 4, we generalize the notion of synchronization to the exact synchronization by groups. Section 5 is devoted to a mixed problem of synchronization and controllability. In Section 6, we study the behaviors of the final synchronizable state for a system of two wave equations.

2 Exact Controllability for a Coupled System of Wave Equations

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary Γ of class C^2. Let $\Gamma = \Gamma_1 \cup \Gamma_0$ be a partition of Γ, such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$. Furthermore, we assume that there exists an $x_0 \in \mathbb{R}^n$, such that, by setting $m = x - x_0$, we have

$$(m,\nu) > 0, \quad \forall x \in \Gamma_1, \quad (m,\nu) \le 0, \quad \forall x \in \Gamma_0, \tag{2.1}$$

where ν is the unit outward normal vector, and $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in $\mathbb{R}^n$.

Let

$$W = (w^{(1)}, \cdots, w^{(M)})^{\mathrm{T}}, \quad \overline{H} = (\overline{h}^{(1)}, \cdots, \overline{h}^{(M)})^{\mathrm{T}}, \quad \overline{A} \in \mathbb{M}^M(\mathbb{R}).$$

Consider the following mixed problem for a coupled system of wave equations:

$$\frac{\partial^2 W}{\partial t^2} - \Delta W + \overline{A} W = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$W = 0 \quad \text{on } \Gamma_0, \tag{2.3}$$

$$W = \overline{H} \quad \text{on } \Gamma_1, \tag{2.4}$$

$$t = 0: \ W = W_0, \quad \frac{\partial W}{\partial t} = W_1. \tag{2.5}$$

If the coupling matrix $\overline{A}$ is symmetric and positively definite, the exact controllability of (2.2)–(2.5) follows easily from the classical results (see [4, 10]). In this section, we will establish the exact controllability for any coupling matrix $\overline{A}$. We first establish the observability of the corresponding adjoint problem, and then the exact controllability follows from the standard HUM method of Lions.

Now let

$$\Phi = (\phi^{(1)}, \cdots, \phi^{(M)})^{\mathrm{T}}.$$

Consider the corresponding adjoint problem as follows:

$$\frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi + \overline{A}^{\mathrm{T}} \Phi = 0 \quad \text{in } \Omega, \tag{2.6}$$

$$\Phi = 0 \quad \text{on } \Gamma, \tag{2.7}$$

$$t = 0: \ \Phi = \Phi_0, \quad \frac{\partial \Phi}{\partial t} = \Phi_1. \tag{2.8}$$

It is well-known that the above problem is well-posed in the space $\mathcal{V}\times\mathcal{H}$:

$$\mathcal{V}=(H_0^1(\Omega))^M,\quad \mathcal{H}=(L^2(\Omega))^M. \tag{2.9}$$

Moreover, we have the following direct and inverse inequalities.

Theorem 2.1 *Let $T>0$ be suitably large. Then there exist positive constants c and C, such that for any given initial data $(\Phi_0,\Phi_1)\in\mathcal{V}\times\mathcal{H}$, the solution Φ to (2.6)–(2.8) satisfies the following inequalities*:

$$c\int_0^{\mathrm{T}}\int_{\Gamma_1}\Big|\frac{\partial\Phi}{\partial\nu}\Big|^2\mathrm{d}\Gamma\mathrm{d}t\le\|\Phi_0\|_{\mathcal{V}}^2+\|\Phi_1\|_{\mathcal{H}}^2\le C\int_0^{\mathrm{T}}\int_{\Gamma_1}\Big|\frac{\partial\Phi}{\partial\nu}\Big|^2\mathrm{d}\Gamma\mathrm{d}t. \tag{2.10}$$

Before proving Theorem 2.1, we first give a uniqueness result.

Lemma 2.1 *Let B be a square matrix of order M, and $\Phi\in H^2(\Omega)$ be a solution to the following system*:

$$\Delta\Phi=B\Phi\quad in\ \Omega. \tag{2.11}$$

Assume furthermore that

$$\Phi=0,\quad \frac{\partial\Phi}{\partial\nu}=0\quad on\ \Gamma_1. \tag{2.12}$$

Then we have $\Phi\equiv0$.

Proof Let

$$\widetilde{B}=PBP^{-1}=\begin{pmatrix}\widetilde{b}_{11}&0&\cdots&0\\ \widetilde{b}_{21}&\widetilde{b}_{22}&\cdots&0\\ &&\cdots&\\ \widetilde{b}_{M1}&\widetilde{b}_{M2}&\cdots&\widetilde{b}_{MM}\end{pmatrix},\quad \widetilde{\Phi}=P\Phi,$$

where $\widetilde{B}$ is a lower triangular matrix of complex entries. Then (2.11)–(2.12) can be reduced to

$$\begin{cases}\Delta\widetilde{\phi}^{(k)}=\displaystyle\sum_{p=1}^{k}\widetilde{b}_{kp}\widetilde{\phi}^{(p)} & \text{in }\Omega,\\ \widetilde{\phi}^{(k)}=0,\quad \dfrac{\partial\widetilde{\phi}^{(k)}}{\partial\nu}=0 & \text{on }\Gamma_1\end{cases} \tag{2.13}$$

for $k=1,\cdots,M$. In particular for $k=1$, we have

$$\begin{cases}\Delta\widetilde{\phi}^{(1)}=\widetilde{b}_{11}\widetilde{\phi}^{(1)} & \text{in }\Omega,\\ \widetilde{\phi}^{(1)}=0,\quad \dfrac{\partial\widetilde{\phi}^{(1)}}{\partial\nu}=0 & \text{on }\Gamma_1.\end{cases}$$

Thanks to Carleman's uniqueness result (see [3]), we get

$$\widetilde{\phi}^{(1)}\equiv0. \tag{2.14}$$

Inserting (2.14) into the second set of (2.13) leads to

$$\begin{cases}\Delta\widetilde{\phi}^{(2)}=\widetilde{b}_{22}\widetilde{\phi}^{(2)} & \text{in }\Omega,\\ \widetilde{\phi}^{(2)}=0,\quad \dfrac{\partial\widetilde{\phi}^{(2)}}{\partial\nu}=0 & \text{on }\Gamma_1,\end{cases}$$

and we can repeat the same procedure. Thus, by a simple induction, we get successively that

$$\widetilde{\phi}^{(k)} \equiv 0, \quad k = 1, \cdots, M.$$

This yields that

$$\widetilde{\Phi} \equiv 0 \Rightarrow \Phi \equiv 0.$$

The proof is complete.

Proof of Theorem 2.1 We rewrite (2.6)–(2.8) as

$$\begin{pmatrix} \Phi \\ \Phi' \end{pmatrix}' = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\overline{A}^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi' \end{pmatrix} = \mathcal{A} \begin{pmatrix} \Phi \\ \Phi' \end{pmatrix} + \mathcal{B} \begin{pmatrix} \Phi \\ \Phi' \end{pmatrix}, \tag{2.15}$$

where $I = I_M$ is the unit matrix of order M. It is easy to see that $\mathcal{A}$ is a skiew-adjoint operator with compact resolvent in $\mathcal{V} \times \mathcal{H}$, and $\mathcal{B}$ is a compact operator in $\mathcal{V} \times \mathcal{H}$. Therefore, they generate respectively C^0 groups in the energy space $\mathcal{V} \times \mathcal{H}$.

Following a recent perturbation result of Mehrenberger [13], in order to prove (2.10) for a system of this kind, it is sufficient to check the following assertions:

(i) The direct and inverse inequalities

$$c \int_0^{\mathrm{T}} \int_{\Gamma_1} \Big| \frac{\partial \widetilde{\Phi}}{\partial \nu} \Big|^2 \mathrm{d}\Gamma \mathrm{d}t \leq \|\Phi_0\|^2_{\mathcal{V}} + \|\Phi_1\|^2_{\mathcal{H}} \leq C \int_0^{\mathrm{T}} \int_{\Gamma_1} \Big| \frac{\partial \widetilde{\Phi}}{\partial \nu} \Big|^2 \mathrm{d}\Gamma \mathrm{d}t \tag{2.16}$$

hold for the solution $\widetilde{\Phi} = S_{\mathcal{A}}(t)(\Phi_0, \Phi_1)$ to the decoupled problem (2.6)–(2.8) with $\overline{A} = 0$.

(ii) The system of root vectors of $\mathcal{A} + \mathcal{B}$ forms a Riesz basis of subspaces in $\mathcal{V} \times \mathcal{H}$. That is to say, there exists a family of subspaces $\mathcal{V}_i \times \mathcal{H}_i$ $(i \geq 1)$ composed of root vectors of $\mathcal{A} + \mathcal{B}$, such that for all $x \in \mathcal{V} \times \mathcal{H}$, there exists a unique $x_i \in \mathcal{V}_i \times \mathcal{H}_i$ $(i \geq 1)$, such that

$$x = \sum_{i=1}^{+\infty} x_i, \quad c_1 \|x\|^2 \leq \sum_{i=1}^{+\infty} \|x_i\|^2 \leq c_2 \|x\|^2,$$

where c_1, c_2 are positive constants.

(iii) If $(\Phi, \Psi) \in \mathcal{V} \times \mathcal{H}$ and $\lambda \in \mathbb{C}$, such that

$$(\mathcal{A} + \mathcal{B})(\Phi, \Psi) = \lambda(\Phi, \Psi) \quad \text{and} \quad \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \Gamma_1,$$

then $(\Phi, \Psi) = 0$.

For simplification of notation, we will still denote by $\mathcal{V} \times \mathcal{H}$ the complex Hilbert space corresponding to $\mathcal{V} \times \mathcal{H}$.

Since the assertion (i) is well-known (see [9]), we only have to verify (ii) and (iii).

Verification of (ii). Let $\mu_i^2 > 0$ be an eigenvalue corresponding to an eigenvector e_i of $-\Delta$ with homogeneous Dirichlet boundary condition:

$$\begin{cases} -\Delta e_i = \mu_i^2 e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \Gamma. \end{cases}$$

Let

$$\mathcal{H}_i \times \mathcal{V}_i = \{(\alpha e_i, \beta e_i): \ \alpha, \beta \in \mathbb{C}^M\}.$$

Obviously, the subspaces $\mathcal{H}_i \times \mathcal{V}_i$ $(i = 1, 2, \cdots)$ are mutually orthogonal, and

$$\mathcal{H} \times \mathcal{V} = \bigoplus_{i \geq 1} \mathcal{H}_i \times \mathcal{V}_i. \tag{2.17}$$

In particular, for any given $x \in H \times \mathcal{V}$ $(i \geq 1)$, there exists an $x_i \in H_i \times \mathcal{V}_i$ $(i \geq 1)$, such that

$$x = \sum_{i=1}^{+\infty} x_i, \quad \|x\|^2 = \sum_{i=1}^{+\infty} \|x_i\|^2. \tag{2.18}$$

On the other hand, $\mathcal{H}_i \times \mathcal{V}_i$ is an invariant subspace of $\mathcal{A} + \mathcal{B}$ and of finite dimension $2M$. Then, the restriction of $\mathcal{A} + \mathcal{B}$ in the subspace $\mathcal{H}_i \times \mathcal{V}_i$ is a linear bounded operator, and therefore, its root vectors constitute a basis in the finite dimensional complex space $\mathcal{H}_i \times \mathcal{V}_i$. This together with (2.17)–(2.18) implies that the system of root vectors of $\mathcal{A} + \mathcal{B}$ forms a Riesz basis of subspaces in $\mathcal{H} \times \mathcal{V}$.

Verification of (iii). Let $(\Phi, \Psi) \in \mathcal{V} \times \mathcal{H}$ and $\lambda \in \mathbb{C}$, such that

$$(\mathcal{A} + \mathcal{B})(\Phi, \Psi) = \lambda(\Phi, \Psi) \quad \text{and} \quad \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \Gamma_1.$$

Then we have

$$\Psi = \lambda\Phi, \quad \Delta\Phi - \overline{A}^{\mathrm{T}}\Phi = \lambda\Psi,$$

namely,

$$\begin{cases} \Delta\Phi = (\lambda^2 I + \overline{A}^{\mathrm{T}})\Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \Gamma. \end{cases} \tag{2.19}$$

It follows from the classic elliptic theory that $\Phi \in H^2(\Omega)$. Moreover, we have

$$\frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \Gamma. \tag{2.20}$$

Then, applying Lemma 2.1 to (2.19)–(2.20), we get $\Phi = 0$, then $\Psi = 0$. The proof is then complete.

By a standard application of the HUM method, from Theorem 2.1 we get the following result.

Theorem 2.2 *There exists a positive constant $T > 0$, such that for any given initial data*

$$W_0 \in (L^2(\Omega))^M, \quad W_1 \in (H^{-1}(\Omega))^M, \tag{2.21}$$

there exist boundary control functions

$$\overline{H} \in (L^2(0, T; L^2(\Gamma_1)))^M, \tag{2.22}$$

such that (2.2)–(2.5) *admits a unique weak solution*

$$W \in (C^0([0, T]; L^2(\Omega)))^M, \quad \frac{\partial W}{\partial t} \in (C^0([0, T]; H^{-1}(\Omega)))^M, \tag{2.23}$$

satisfying the null final condition

$$t = T: \ W = 0, \quad \frac{\partial W}{\partial t} = 0. \tag{2.24}$$

Remark 2.1 Note that we do not need any assumption on the coupling matrix $\overline{A}$ in Theorem 2.2.

Remark 2.2 The same result on the controllability for a coupled system of 1-dimensional wave equations in the framework of classical solutions can be found in [7, 14].

3 Exact Synchronization for a Coupled System of Wave Equations

Let

$$U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}, \quad A \in \mathbb{M}^N(\mathbb{R}).$$

Consider the following coupled system of wave equations with Dirichlet boundary controls:

$$\frac{\partial^2 U}{\partial t^2} - \Delta U + AU = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$U = 0 \quad \text{on } \Gamma_0, \tag{3.2}$$

$$U = H \quad \text{on } \Gamma_1, \tag{3.3}$$

$$t = 0: \ U = U_0, \quad \frac{\partial U}{\partial t} = U_1. \tag{3.4}$$

According to the result given in the previous section, we have the exact null controllability of the problem (3.1)–(3.4) by means of N boundary controls. If the number of boundary controls is less than N, generally speaking, we can not realize the exact controllability (see [7], for more general discussion).

Definition 3.1 *Problem* (3.1)–(3.4) *is exactly synchronizable at the moment* $T > 0$*, if for any given initial data* $U_0 \in (L^2(\Omega))^N$ *and* $U_1 \in (H^{-1}(\Omega))^N$*, there exist suitable boundary controls given by a part of* $H \in (L^2(0, +\infty; L^2(\Gamma_1)))^N$*, such that the solution* $U = U(t, x)$ *to* (3.1)–(3.4) *satisfies the following final condition*:

$$t \geq T: \ u^{(1)} \equiv u^{(2)} \equiv \cdots \equiv u^{(N)} := u, \tag{3.5}$$

where $u = u(t, x)$ *is called the synchronizable state.*

Remark 3.1 If problem (3.1)–(3.4) is exactly null controllable, then we have certainly the exact synchronization. This trivial situation should be excluded. Therefore, in Definition 3.1, we should restrict ourselves to the case that the number of the boundary controls is less than N, so that (3.1)–(3.4) can be assumed to be not exactly null controllable.

Theorem 3.1 *Assume that* (3.1)–(3.4) *is exactly synchronizable, but not exactly null controllable. Then the coupling matrix* $A = (a_{ij})$ *should satisfy the following conditions of compatibility*:

$$\sum_{p=1}^{N} a_{kp} := \widetilde{a}, \quad k = 1, \cdots, N, \tag{3.6}$$

where $\widetilde{a}$ *is a constant indepentdent of* $k = 1, \cdots, N$.

Proof By synchronization, there exists a $T > 0$ and a scalar function u, such that

$$u^{(k)}(t, x) \equiv u(t, x), \quad t \geq T, \ k = 1, 2, \cdots, N.$$

Then for $t \geq T$, we have

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \Big(\sum_{p=1}^{N} a_{kp}\Big) u = 0 \quad \text{in } \Omega, \ k = 1, 2, \cdots, N.$$

In particular, we have

$$t \geq T: \quad \Big(\sum_{p=1}^{N} a_{kp}\Big) u = \Big(\sum_{p=1}^{N} a_{lp}\Big) u \quad \text{in } \Omega, \ k, l = 1, \cdots, N.$$

By the non-exact null controllability, there exists at least an initial datum (U_0, U_1) for which the corresponding solution U, or equivalently u, does not identically vanish for $t \geq T$, whatever boundary controls H are chosen. This yields the conditions of compatibility (3.6). The proof is completed.

Theorem 3.2 *Assume that the conditions of compatibility* (3.6) *hold. Then the problem* (3.1)–(3.4) *is exactly synchronizable by means of some boundary controls* H *with compact support on* $[0,T]$ *and* $h^{(1)} \equiv 0$.

Proof Let

$$w^{(i)} = u^{(i+1)} - u^{(i)}, \quad i = 1, \cdots, N-1. \tag{3.7}$$

We will transform the problem (3.1)–(3.4) to a reduced problem on the variable $W = (w^{(1)}, \cdots, w^{(N-1)})^{\mathrm{T}}$. By (3.1), we get

$$\frac{\partial^2 w^{(i)}}{\partial t^2} - \Delta w^{(i)} + \sum_{p=1}^{N} (a_{i+1,p} - a_{ip}) u^{(p)} = 0, \quad i = 1, \cdots, N-1. \tag{3.8}$$

Noting (3.7), we have

$$u^{(i)} = \sum_{j=1}^{i-1} w^{(j)} + u^{(1)}, \quad i = 1, \cdots, N.$$

Then a direct computation gives

$$\begin{aligned}
&\sum_{p=1}^{N} (a_{i+1,p} - a_{ip}) u^{(p)} \\
&= \sum_{p=1}^{N} (a_{i+1,p} - a_{ip}) \Big(\sum_{j=1}^{p-1} w^{(j)} + u^{(1)} \Big) \\
&= \sum_{p=1}^{N} (a_{i+1,p} - a_{ip}) \sum_{j=1}^{p-1} w^{(j)} + \sum_{p=1}^{N} (a_{i+1,p} - a_{ip}) u^{(1)}.
\end{aligned}$$

Because of (3.6), the last term vanishes, and then it follows from (3.8) that

$$\frac{\partial^2 w^{(i)}}{\partial t^2} - \Delta w^{(i)} + \sum_{j=1}^{N-1} \overline{a}_{ij} w^{(j)} = 0, \quad i = 1, \cdots, N-1, \tag{3.9}$$

where

$$\overline{a}_{ij} = \sum_{p=j+1}^{N} (a_{i+1,p} - a_{ip}) = \sum_{p=1}^{j} (a_{ip} - a_{i+1,p}), \quad i, j = 1, \cdots, N-1. \tag{3.10}$$

Correspondingly, for the variable W, we set the new initial data as

$$w_0^{(i)} = u_0^{(i+1)} - u_0^{(i)}, \quad w_1^{(i)} = u_1^{(i+1)} - u_1^{(i)}, \quad i = 1, \cdots, N-1, \tag{3.11}$$

and the new boundary controls as

$$\overline{h}^{(i)} = h^{(i+1)} - h^{(i)}, \quad i = 1, \cdots, N-1. \tag{3.12}$$

Noting (3.9)–(3.12), the new variable W satisfies the reduced problem (2.2)–(2.5) (in which $M = N-1$). Then, by Theorem 2.2, there exist boundary controls $\overline{H} \in L^2(0,T;L^2(\Gamma_1))^{N-1}$, such that the corresponding solution $W = W(t,x)$ to the reduced problem (2.2)–(2.5) satisfies the null final condition. Moreover, taking $\overline{H} \equiv 0$ for $t > T$, it is easy to see that

$$t \geq T: \ w^{(i)} \equiv 0, \quad i = 1, \cdots, N-1. \tag{3.13}$$

In order to determine $h^{(i)}$ $(i = 1, \cdots, N)$ from (3.12), setting $h^{(1)} \equiv 0$, we get

$$h^{(i+1)} = \overline{h}^{(i)} + h^{(i)} = \sum_{j=1}^{i} \overline{h}^{(j)}, \quad i = 1, \cdots, N-1, \tag{3.14}$$

which leads to $H \equiv 0$ for $t \geq T$. Once the controls $h^{(i)}$ $(i = 1, \cdots, N)$ are chosen, we solve the original problem (3.1)–(3.4) to get a solution $U = U(t,x)$. Clearly, the exact synchronization condition (3.5) holds for the solution U. Moreover, from the expression (3.14), we see that $h^{(1)} \equiv 0$ and H are with compact support on $[0,T]$, since $\overline{H}$ are with compact support on $[0,T]$. The proof is complete.

Remark 3.2 In Definition 3.1, the synchronization condition (3.5) should be required for all $t \geq T$. In fact, assuming that (3.5) is realized only at some moment $T > 0$, if we set hereafter $H \equiv 0$ for $t > T$, then the corresponding solution does not satisfy automatically the synchronization condition (3.5) for $t > T$. This is different from the exact null controllability, where the solution vanishes with $H \equiv 0$ for $t \geq T$. To illustrate it, let us consider the following system:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial^2 v}{\partial t^2} - \Delta v = u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ v = h & \text{on } \Gamma. \end{cases} \tag{3.15}$$

Since the first equation is separated from the second one, for any given initial data (u_0, u_1), we can first find a solution u. Once u is determined, we look for a boundary control h, such that the solution v to the second equation satisfies the final synchronization conditions

$$t = T: \ v = u, \quad \frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}. \tag{3.16}$$

If we set $h \equiv 0$ for $t > T$, generally speaking, we can not get $v \equiv u$ for $t \geq T$. So, in order to keep the synchronization for $t \geq T$, we have to maintain the boundary control h in action for $t \geq T$. However, for the sake of applications, it is more interesting to get the exact synchronization by some boundary controls with compact support. Theorems 3.1 and 3.2 guarantee that this can be realized if the coupling matrix A satisfies the conditions of compatibility (3.6).

Remark 3.3 In the reduction of the problem (3.1)–(3.4), we have taken $w^{(i)} = u^{(i+1)} - u^{(i)}$ with $w_0^{(i)} = u_0^{(i+1)} - u_0^{(i)}$ and $w_1^{(i)} = u_1^{(i+1)} - u_1^{(i)}$ for $i = 1, \cdots, N-1$, but it is only a possible choice for proving Theorem 3.2. Since the boundary controls $\overline{h}^{(i)}$ $(i = 1, \cdots, N-1)$ for the exact controllability of the reduced problem depend on the initial data $(w_0^{(i)}, \ w_1^{(i)})$ $(i = 1, \cdots, N-1)$, we should find a suitable permutation σ of $\{1, 2, \cdots, N\}$, such that, setting $w^{(i)} = u^{\sigma(i+1)} - u^{\sigma(i)}$ for $i = 1, \cdots, N-1$, the corresponding initial data $(w_0^{(i)}, \ w_1^{(i)})$ $(i = 1, \cdots, N-1)$ have the smallest energy. On the other hand, in the resolution of (3.12), we have chosen $h^{(1)} \equiv 0$ as a possible choice. A good strategy consists in finding some i_0, such that, by setting $h^{(i_0)} \equiv 0$, the final state u has the smallest energy. These problems would be very interesting.

Theorem 3.3 *Assume that the conditions of compatibility* (3.6) *hold. Then the set of the values* (u, u_t) *at the moment* $t = T$ *of the synchronizable state* $u = (t, x)$ *is actually the whole space* $L^2(\Omega) \times H^{-1}(\Omega)$ *as the initial data* U_0 *and* U_1 *vary in the space* $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.

Proof For $t \geq T$, the synchronizable state $u = u(t, x)$ defined by (3.5) satisfies the following wave equation with homogenous Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \widetilde{a} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{3.17}$$

where $\widetilde{a}$ is given by (3.6). Hence, the evolution of the synchronizable state $u = u(t, x)$ with respect to t is completely determined by the values of (u, u_t) at the moment $t = T$:

$$t = T: \ u = \widehat{u}_0, \quad u_t = \widehat{u}_1. \tag{3.18}$$

Now for any given $(\widehat{u}_0, \widehat{u}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, by solving the backward problem (3.17)–(3.18) on the time interval $[0, T]$, we get the corresponding solution $u = u(t, x)$ with its value (u, u_t) at $t = 0$,

$$t = 0: \ u = u_0, \quad u_t = u_1. \tag{3.19}$$

Then, under the conditions of compatibility (3.6), the function

$$U(t, x) = (u, u \cdots, u)^{\mathrm{T}}(t, x) \tag{3.20}$$

is the solution to (3.1)–(3.3) with the null control $H \equiv 0$ and the initial condition

$$t = 0: \ U = U_0 = (u_0, u_0 \cdots, u_0)^{\mathrm{T}}, \quad U_t = U_1 = (u_1, u_1 \cdots, u_1)^{\mathrm{T}}. \tag{3.21}$$

Therefore, from the initial condition (3.21), by solving (3.1)–(3.3) with null boundary controls, we can reach any given synchronizable state $(\widehat{u}_0, \widehat{u}_1)$ at the moment $t = T$. This fact shows that any given state $(\widehat{u}_0, \widehat{u}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ can be expected to be a synchronizable state. Consequently, the set of the values $(\widehat{u}_0, \widehat{u}_1)$ of the synchronizable state $u = (t, x)$ is actually the whole space $L^2(\Omega) \times H^{-1}(\Omega)$ as the initial data U_0 and U_1 vary in the space $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$. The proof is complete.

Definition 3.2 *Problem* (3.1)–(3.4) *is exactly anti-synchronizable at the moment* $T > 0$, *if for any given initial data* $U_0 \in (L^2(\Omega))^N$ *and* $U_1 \in (H^{-1}(\Omega))^N$, *there exist suitable boundary controls given by a part of* $H \in (L^2(0, +\infty; L^2(\Gamma_1)))^N$, *such that the solution* $U = U(t, x)$ *to* (3.1)–(3.4) *satisfies the final condition*

$$t \geq T: \ u^{(1)} \equiv \cdots \equiv u^{(m)} \equiv -u^{(m+1)} \equiv \cdots \equiv -u^{(N)}. \tag{3.22}$$

Theorem 3.4 *Assume that* (3.1)–(3.4) *is exactly anti-synchronizable, but not exactly null controllable. Then the coupling matrix* $A = (a_{ij})$ *should satisfy the following conditions of* *compatibility*:

$$\begin{cases} \sum_{p=1}^{m} a_{kp} - \sum_{p=m+1}^{N} a_{kp} = \widetilde{a}, & k = 1, \cdots, m, \\ \sum_{p=1}^{m} a_{kp} - \sum_{p=m+1}^{N} a_{kp} = -\widetilde{a}, & k = m+1, \cdots, N, \end{cases} \tag{3.23}$$

where $\widetilde{a}$ *is a constant independent of* $k = 1, \cdots, N$.

Inversely, assume that the conditions of compatibility (3.23) *hold, and then* (3.1)–(3.4) *is exactly anti-synchronizable by means of some boundary controls* H *with compact support and* $h^{(1)} \equiv 0$.

Proof Let us define

$$\widehat{u}^{(i)} = \begin{cases} u^{(i)}, & 1 \le i \le m, \\ -u^{(i)}, & m+1 \le i \le N \end{cases}$$

and

$$\widehat{a}_{ij} = \begin{cases} a_{ij}, & 1 \le i, j \le m \ \text{ or } \ m+1 \le i, j \le N, \\ -a_{ij}, & 1 \le i \le m, \ m+1 \le j \le N \ \text{ or } \ m+1 \le i \le N, \ 1 \le j \le m. \end{cases}$$

Then $\widehat{U} = (\widehat{u}^{(1)}, \cdots, \widehat{u}^{(N)})^{\mathrm{T}}$ satisfies (3.1)–(3.4) with the coupling matrix $\widehat{A} = (\widehat{a}_{ij})$ instead of A. By Theorems 3.1 and 3.2, we obtain that

$$\sum_{p=1}^{m} \widehat{a}_{kp} + \sum_{p=m+1}^{N} \widehat{a}_{kp} = \widetilde{a}, \quad k = 1, 2, \cdots, N \tag{3.24}$$

are necessary and sufficient for the exact synchronization of $\widehat{U}$ by means of $N-1$ boundary controls with compact support. Using the definition of the coefficients $\widehat{a}_{ij}$, we see that (3.24) is precisely (3.23). The proof is complete.

4 Exact Synchronization by Groups

In this section, we will study the exact synchronization by groups. Roughly speaking, let us rearrange the components of U, for example, in two groups, and we look for some boundary controls H, such that $(u^{(1)}, \cdots, u^{(m)})$ and $(u^{(m+1)}, \cdots, u^{(N)})$ are independently synchronized.

Definition 4.1 *Problem* (3.1)–(3.4) *is exactly synchronizable by* 2*-groups at the moment* $T > 0$, *if for any given initial data* $U_0 \in (L^2(\Omega))^N$ *and* $U_1 \in (H^{-1}(\Omega))^N$, *there exist suitable boundary controls given by a part of* $H \in (L^2(0, \infty; L^2(\Gamma_1)))^N$, *such that the solution* $U = U(t, x)$ *to* (3.1)–(3.4) *satisfies the final condition*

$$t \ge T: \ \begin{cases} u^{(1)} \equiv \cdots \equiv u^{(m)} := u, \\ u^{(m+1)} \equiv \cdots \equiv u^{(N)} := v, \end{cases} \tag{4.1}$$

and $\widetilde{U} = (u, v)^{\mathrm{T}}$ *is called to be the synchronizable state by* 2*-groups.*

Our object is to realize the exact synchronization by 2-groups by means of $N-2$ boundary controls. Of course, generally speaking, we can divide the components of U into p groups, and consider the exact synchronization by p-groups. Here we focus our attention only on two groups, but the results obtained in this section can be easily extended to the general case. On the other hand, it is clear that any given exactly synchronizable system is exactly synchronizable by 2-groups. In what follows, we study only the case that the problem is independently synchronizable by 2-groups, and thus the linear independence of components of the synchronizable state $(u, v)^{\mathrm{T}}$ excludes the exact synchronization of (3.1)–(3.4).

Theorem 4.1 *Assume that* (3.1)–(3.4) *is exactly synchronizable by* 2*-groups. Furthermore, assume that at least for some initial data* U_0 *and* U_1*, the synchronizable states* u *and* v *are linearly independent. Then the coupling matrix* $A = (a_{ij})$ *should satisfy the following conditions of compatibility*:

$$\sum_{p=1}^{m} a_{kp} = \sum_{p=1}^{m} a_{lp}, \qquad \sum_{p=m+1}^{N} a_{kp} = \sum_{p=m+1}^{N} a_{lp} \tag{4.2}$$

for $k, l = 1, \cdots, m$ *and* $k, l = m+1, \cdots, N$*, respectively.*

Proof Since (3.1)–(3.4) is exactly synchronizable by 2-groups, for any given initial data U_0 and U_1, there exists a boundary control H, such that (4.1) holds. It follows that for $t \geq T$, we have

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \Big(\displaystyle\sum_{p=1}^{m} a_{kp}\Big) u + \Big(\displaystyle\sum_{p=m+1}^{N} a_{kp}\Big) v = 0 \quad \text{in } \Omega, \ k = 1, \cdots, m, \\ \dfrac{\partial^2 v}{\partial t^2} - \Delta v + \Big(\displaystyle\sum_{p=1}^{m} a_{kp}\Big) u + \Big(\displaystyle\sum_{p=m+1}^{N} a_{kp}\Big) v = 0 \quad \text{in } \Omega, \ k = m+1, \cdots, N. \end{cases}$$

Therefore, we have

$$t \geq T: \ \Big(\sum_{p=1}^{m} a_{kp} - \sum_{p=1}^{m} a_{lp}\Big) u + \Big(\sum_{p=m+1}^{N} a_{kp} - \sum_{p=m+1}^{N} a_{lp}\Big) v = 0 \quad \text{in } \Omega \tag{4.3}$$

for $k, l = 1, \cdots, m$ and $k, l = m+1, \cdots, N$, respectively. Since at least for some initial data U_0 and U_1, the synchronizable states u and v are linearly independent, (4.2) follows directly from (4.3).

Theorem 4.2 *Assume that the conditions of compatibility* (4.2) *hold. Then* (3.1)–(3.4) *is exactly synchronizable by* 2*-groups by means of some boundary controls* H *with compact support and* $h^{(1)} \equiv h^{(m+1)} \equiv 0$.

Proof Let

$$\begin{cases} w^{(j)} = u^{(j+1)} - u^{(j)}, & j = 1, \cdots, m-1, \\ w^{(j)} = u^{(j+2)} - u^{(j+1)}, & j = m, \cdots, N-2. \end{cases} \tag{4.4}$$

Then we have

$$\begin{cases} u^{(j)} = \displaystyle\sum_{s=1}^{j-1} w^{(s)} + u^{(1)}, & j = 1, \cdots, m, \\ u^{(j)} = \displaystyle\sum_{s=m}^{j-2} w^{(s)} + u^{(m+1)}, & j = m+1, \cdots, N. \end{cases} \tag{4.5}$$

By (3.1), it is easy to see that for $1 \leq i \leq N-2$, we have

$$\frac{\partial^2 w^{(i)}}{\partial t^2} - \Delta w^{(i)} + \sum_{p=1}^{N} (a_{i+1,p} - a_{ip}) u^{(p)} = 0.$$

By a direct computation, noting (4.2), we have

$$\begin{aligned}
&\sum_{p=1}^{N}(a_{i+1,p}-a_{ip})u^{(p)}\\
=&\sum_{p=1}^{m}(a_{i+1,p}-a_{ip})u^{(p)}+\sum_{p=m+1}^{N}(a_{i+1,p}-a_{ip})u^{(p)}\\
=&\sum_{p=1}^{m}(a_{i+1,p}-a_{ip})\Big(\sum_{s=1}^{p-1}w^{(s)}+u^{(1)}\Big)+\sum_{p=m+1}^{N}(a_{i+1,p}-a_{ip})\Big(\sum_{s=m}^{p-2}w^{(s)}+u^{(m+1)}\Big)\\
=&\sum_{p=1}^{m}(a_{i+1,p}-a_{ip})\sum_{s=1}^{p-1}w^{(s)}+\sum_{p=m+1}^{N}(a_{i+1,p}-a_{ip})\sum_{s=m}^{p-2}w^{(s)}\\
&+\sum_{p=1}^{m}(a_{i+1,p}-a_{ip})u^{(1)}+\sum_{p=m+1}^{N}(a_{i+1,p}-a_{ip})u^{(m+1)}\\
=&\sum_{s=1}^{m-1}\sum_{p=s+1}^{m}(a_{i+1,p}-a_{ip})w^{(s)}+\sum_{s=m}^{N-2}\sum_{p=s+2}^{N}(a_{i+1,p}-a_{ip})w^{(s)}.
\end{aligned}$$

Then

$$\frac{\partial^2 w^{(i)}}{\partial t^2}-\Delta w^{(i)}+\sum_{s=1}^{N-2}\overline{a}_{is}w^{(s)}=0,\quad 1\le i\le N-2, \tag{4.6}$$

where

$$\overline{a}_{is}=\begin{cases}\displaystyle\sum_{p=s+1}^{m}(a_{i+1,p}-a_{ip}), & 1\le i\le N-2,\ 1\le s\le m-1,\\ \displaystyle\sum_{p=s+2}^{N}(a_{i+1,p}-a_{ip}), & 1\le i\le N-2,\ m\le s\le N-2.\end{cases} \tag{4.7}$$

Corresponding to (4.4), for the variable $W=(w^{(1)},\cdots,w^{(N-2)})^{\mathrm{T}}$, we put

$$\overline{h}^{(j)}=\begin{cases}h^{(j+1)}-h^{(j)}, & j=1,\cdots,m-1,\\ h^{(j+2)}-h^{(j+1)}, & j=m,\cdots,N-2,\end{cases} \tag{4.8}$$

and set the new initial data as follows:

$$w_0^{(j)}=\begin{cases}w_0^{(j+1)}-w_0^{(j)}, & j=1,\cdots,m-1,\\ w_0^{(j+2)}-w_0^{(j+1)}, & j=m,\cdots,N-2,\end{cases} \tag{4.9}$$

$$w_1^{(j)}=\begin{cases}w_1^{(j+1)}-w_1^{(j)}, & j=1,\cdots,m-1,\\ w_1^{(j+2)}-w_1^{(j+1)}, & j=m,\cdots,N-2.\end{cases} \tag{4.10}$$

Noting (4.6)–(4.10), we get again a reduced problem (2.2)–(2.5) on the new variable W with $N-2$ components. By Theorem 2.2 (in which $M=N-2$), there exist boundary controls $\overline{H}\in(L^2(0,T;L^2(\Gamma_1)))^{N-2}$, such that the solution $W=W(t,x)$ to the reduced problem (2.2)–(2.5) satisfies the null final condition. Moreover, taking $\overline{H}\equiv 0$ for $t>T$, we have

$$t\ge T:\ w^{(i)}(t,x)\equiv 0,\quad i=1,\cdots,N-2. \tag{4.11}$$

In order to determine $h^{(i)}$ $(i = 1, \cdots, N)$ from (4.8), setting

$$h^{(1)} \equiv h^{(m+1)} \equiv 0, \tag{4.12}$$

we get

$$\begin{cases} h^{(i+1)} = \sum_{j=1}^{i} \overline{h}^{(j)}, & i = 1, \cdots, m-1, \\ h^{(i+1)} = \sum_{j=m}^{i-1} \overline{h}^{(j)}, & i = m+1, \cdots, N-1, \end{cases} \tag{4.13}$$

which leads to $H \equiv 0$ for $t > T$. Once the controls $h^{(i)}$ $(i = 1, \cdots, N)$ are chosen, we solve the original problem (3.1)–(3.4) to get a solution $U = U(t, x)$, which clearly satisfies the final condition (4.1). Thus the proof is complete.

Theorem 4.3 *Assume that the conditions of compatibility* (4.2) *hold. Then the set of the values* (u, v, u_t, v_t) *of the synchronizable state* $(u, v) = (u(t, x), v(t, x))$ *of* (3.1)–(3.4) *is actually the whole space* $(L^2(\Omega))^2 \times (H^{-1}(\Omega))^2$ *as the initial data* U_0 *and* U_1 *vary in the space* $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$*. In particular, there exist initial data* (U_0, U_1) *and boundary controls* H *with compact support and* $h^{(1)} \equiv h^{(m+1)} \equiv 0$*, such that the synchronizable states by* 2*-groups* u *and* v *of the problem* (3.1)–(3.4) *are linearly independent.*

Proof Let $\widetilde{A} = (\widetilde{a}_{ij})$ be the 2×2 matrix with the entries

$$\begin{cases} \widetilde{a}_{11} = \sum_{p=1}^{m} a_{kp}, \quad \widetilde{a}_{12} = \sum_{p=m+1}^{N} a_{kp}, \quad k = 1, \cdots, m, \\ \widetilde{a}_{21} = \sum_{p=1}^{m} a_{kp}, \quad \widetilde{a}_{22} = \sum_{p=m+1}^{N} a_{kp}, \quad k = m+1, \cdots, N. \end{cases} \tag{4.14}$$

For $t \geq T$, the synchronizable state by 2-groups $\widetilde{U} = (u, v)^{\mathrm{T}}$ defined by (4.1) satisfies the following coupled system of wave equations:

$$\frac{\partial^2}{\partial t^2}\widetilde{U} - \Delta\widetilde{U} + \widetilde{A}\widetilde{U} = 0 \quad \text{in } \Omega \tag{4.15}$$

with the homogeneous boundary condition

$$\widetilde{U} = 0 \quad \text{on } \Gamma. \tag{4.16}$$

Thus, the evolution of $\widetilde{U} = \widetilde{U}(t, x)$ with respect to t is completely determined by the value of $(\widetilde{U}, \widetilde{U}_t)$ at the time $t = T$. In a way similar to that of Theorem 3.3, we get that the set of values $(\widetilde{U}, \widetilde{U}_t)$ at the moment $t = T$ of the synchronizable state by 2-groups $\widetilde{U} = (u, v)^{\mathrm{T}}$ is actually the whole space $(L^2(\Omega))^2 \times (H^{-1}(\Omega))^2$. The proof is complete.

Remark 4.1 Under the condition that, at least for some initial data U_0 and U_1, the synchronizable states u and v are linearly independent, we have shown that the conditions of compatibility (4.2) are necessary and sufficient for the synchronization by 2-groups of the problem (3.1)–(3.4). These conditions are still sufficient for the synchronization by 2-groups of the problem (3.1)–(3.4) without the linear independence of u and v. But we do not know if they are also necessary in that case. This seems to be an open problem to our knowledge.

5 Exact Null Controllability and Synchronization by Groups

In general, a system of N wave equations is not exactly controllable by means of less N boundary controls (see [7, 10]). By Theorem 3.2, (3.1)–(3.4) is exactly synchronizable by means of $N-1$ boundary controls. These results are quite logical from the viewpoint of the degree of freedom system and the number of controls. It suggests us to consider the partial controllability for a system of N equations by means of less boundary controls. Since this is still an open problem in the general situation, we would like to weaken our request by asking if it is possible or not, based on the idea of synchronization, to realize the exact null controllability of $N-2$ components of the solution to a system of N equations by means of $N-1$ boundary controls. This is the goal of this section, in which we will discuss this problem in a more general situation.

Definition 5.1 *Problem* (3.1)–(3.4) *is exactly null controllable and synchronizable by* 2-*groups at the moment* $T>0$, *if for any given initial data* $U_0 \in (L^2(\Omega))^N$ *and* $U_1 \in (H^{-1}(\Omega))^N$, *there exist suitable boundary controls given by a part of* $H \in (L^2(0,+\infty; L^2(\Gamma_1)))^N$, *such that the solution* $U = U(t,x)$ *to* (3.1)–(3.4) *satisfies the final condition*

$$t \geq T: \ u^{(1)} \equiv \cdots \equiv u^{(m)} \equiv 0, \quad u^{(m+1)} \equiv \cdots \equiv u^{(N)} := u, \tag{5.1}$$

and $u = u(t,x)$ *is called the partially synchronizable state.*

Theorem 5.1 *Assume that the problem* (3.1)–(3.4) *is exactly null controllable and synchronizable by* 2-*groups, but not exactly null controllable. Then the coupling matrix* $A = (a_{ij})$ *should satisfy the following conditions of compatibility:*

$$\begin{cases} \displaystyle\sum_{p=m+1}^{N} a_{kp} = 0, & k = 1, \cdots, m, \\ \displaystyle\sum_{p=m+1}^{N} a_{kp} = \sum_{p=m+1}^{N} a_{lp}, & k, l = m+1, \cdots, N. \end{cases} \tag{5.2}$$

Proof By the exact null controllability and synchronization by 2-groups, there exist a $T > 0$ and a scalar function u, such that

$$t \geq T: \ \begin{cases} u^{(k)}(t,x) \equiv 0, & k = 1, \cdots, m, \\ u^{(k)}(t,x) \equiv u(t,x), & k = m+1, \cdots, N. \end{cases}$$

Then for $t \geq T$, we have

$$\begin{cases} \displaystyle\Big(\sum_{p=m+1}^{N} a_{kp}\Big) u = 0 & \text{in } \ \Omega, \ k = 1, \cdots, m, \\ \displaystyle\frac{\partial^2 u}{\partial t^2} - \Delta u + \Big(\sum_{p=m+1}^{N} a_{kp}\Big) u = 0 & \text{in } \ \Omega, \ k = m+1, \cdots, N. \end{cases}$$

Since the problem is not exactly null controllable, we may assume that $u \not\equiv 0$ and this yields the conditions of compatibility (5.2).

Theorem 5.2 *Assume that the conditions of compatibility* (5.2) *hold. Then the problem* (3.1)–(3.4) *is exactly null controllable and synchronizable by* 2-*groups by means of some boundary controls* H *with compact support on* $[0,T]$ *and* $h^{(m+1)} \equiv 0$.

Proof Let

$$\begin{cases} w^{(j)} = u^{(j)}, & j = 1, \cdots, m, \\ w^{(j)} = u^{(j+1)} - u^{(j)}, & j = m+1, \cdots, N-1. \end{cases} \tag{5.3}$$

We have

$$u^{(j)} = \sum_{s=m+1}^{j-1} w^{(s)} + u^{(m+1)}, \quad j = m+1, \cdots, N. \tag{5.4}$$

Then the first m equations of (3.1) become

$$\frac{\partial^2 w^{(i)}}{\partial t^2} - \Delta w^{(i)} + \sum_{p=1}^{m} a_{ip} w^{(p)} + \sum_{p=m+1}^{N} a_{ip} u^{(p)} = 0, \quad i = 1, \cdots, m.$$

Using (5.4) and the first condition in (5.2), we have

$$\begin{aligned} \sum_{p=m+1}^{N} a_{ip} u^{(p)} &= \sum_{p=m+1}^{N} \sum_{s=m+1}^{p-1} a_{ip} w^{(s)} + \Big(\sum_{p=m+1}^{N} a_{ip} \Big) u^{(m+1)} \\ &= \sum_{p=m+1}^{N-1} \sum_{s=p+1}^{N} a_{is} w^{(p)}. \end{aligned}$$

Then

$$\frac{\partial^2 w^{(i)}}{\partial t^2} - \Delta w^{(i)} + \sum_{p=1}^{N-1} \overline{a}_{ip} w^{(p)} = 0, \quad 1 \le i \le m, \tag{5.5}$$

where

$$\overline{a}_{ip} = \begin{cases} a_{ip}, & 1 \le i \le m, \ 1 \le p \le m, \\ \displaystyle\sum_{s=p+1}^{N} a_{is}, & 1 \le i \le m, \ m+1 \le p \le N-1. \end{cases} \tag{5.6}$$

Next, by (3.1) and noting the first part of (5.3), for $m+1 \le i \le N-1$, we have

$$\frac{\partial^2 w^{(i)}}{\partial t^2} - \Delta w^{(i)} + \sum_{p=1}^{m} (a_{i+1,p} - a_{ip}) w^{(p)} + \sum_{p=m+1}^{N} (a_{i+1,p} - a_{ip}) u^{(p)} = 0.$$

By a direct computation, noting the second condition in (5.2), we have

$$\begin{aligned} & \sum_{p=m+1}^{N} (a_{i+1,p} - a_{ip}) u^{(p)} \\ = & \sum_{p=m+1}^{N} (a_{i+1,p} - a_{ip}) \Big(\sum_{s=m+1}^{p-1} w^{(s)} + u^{(m+1)} \Big) \\ = & \sum_{p=m+1}^{N} (a_{i+1,p} - a_{ip}) \sum_{s=m+1}^{p-1} w^{(s)} + \sum_{p=m+1}^{N} (a_{i+1,p} - a_{ip}) u^{(m+1)} \\ = & \sum_{s=m+1}^{N-1} \sum_{p=s+1}^{N} (a_{i+1,p} - a_{ip}) w^{(s)}. \end{aligned}$$

Then

$$\frac{\partial^2 w^{(i)}}{\partial t^2} - \Delta w^{(i)} + \sum_{p=1}^{N-1} \overline{a}_{ip} w^{(p)} = 0, \quad m+1 \le i \le N-1, \tag{5.7}$$

where

$$\overline{a}_{ip} = \begin{cases} a_{i+1,p} - a_{ip}, & m+1 \le i \le N-1,\ 1 \le p \le m, \\ \displaystyle\sum_{s=p+1}^{N} (a_{i+1,s} - a_{is}), & m+1 \le i \le N-1,\ m+1 \le p \le N-1. \end{cases} \tag{5.8}$$

Corresponding to (5.3), for the variable $W = (w^{(1)}, \cdots, w^{(N-1)})^{\mathrm{T}}$, we put

$$\overline{h}^{(i)} = \begin{cases} h^{(i)}, & 1 \le i \le m, \\ h^{(i+1)} - h^{(i)}, & m+1 \le i \le N-1, \end{cases} \tag{5.9}$$

and set the new initial data as follows:

$$w_0^{(i)} = \begin{cases} u_0^{(i)}, & 1 \le i \le m, \\ u_0^{(i+1)} - u_0^{(i)}, & m+1 \le i \le N-1, \end{cases} \tag{5.10}$$

$$w_1^{(i)} = \begin{cases} u_1^{(i)}, & 1 \le i \le m, \\ u_1^{(i+1)} - u_1^{(i)}, & m+1 \le i \le N-1. \end{cases} \tag{5.11}$$

Noting (5.5), (5.7) and (5.10)–(5.11), we get again a reduced problem (2.2)–(2.5) on the new variable W with $N-1$ components. By Theorem 2.2 (in which $M = N-1$), there exist controls $\overline{H} \in (L^2(0,T;L^2(\Gamma_1)))^{N-1}$, such that the solution $W = W(t,x)$ to the reduced problem (2.2)–(2.5) satisfies the null final condition. Moreover, taking $\overline{H} \equiv 0$ for $t > T$, we have

$$t \ge T: \ w^{(i)}(t,x) \equiv 0, \quad i = 1, \cdots, N-1. \tag{5.12}$$

In order to determine $h^{(i)}$ $(i = 1, \cdots, N)$ from (5.9), setting $h^{(m+1)} \equiv 0$, we get

$$\begin{cases} h^{(i)} = \overline{h}^{(i)}, & i = 1, \cdots, m, \\ h^{(i+1)} = \displaystyle\sum_{j=m+1}^{i} \overline{h}^{(j)}, & i = m+1, \cdots, N-1, \end{cases} \tag{5.13}$$

which leads to $H \equiv 0$ for $t > T$. Once the controls $h^{(i)}$ $(i = 1, \cdots, N)$ are chosen, we solve the original problem (3.1)–(3.4) to get a solution $U = U(t,x)$, which clearly satisfies the final condition (5.1). Then the proof is complete.

Remark 5.1 Let

$$\sum_{p=m+1}^{N} a_{kp} = \widetilde{a}, \quad k = m+1, \cdots, N, \tag{5.14}$$

where $\widetilde{a}$ is a constant independent of $k = m+1, \cdots, N$ (see (5.2)). For $t \ge T$, the partially synchronizable state $u = u(t,x)$ satisfies the following wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \widetilde{a} u = 0 \quad \text{in } \Omega \tag{5.15}$$

with the homogeneous boundary condition

$$u = 0 \quad \text{on } \Gamma. \tag{5.16}$$

Hence the evolution of $u = u(t, x)$ with respect to t can be completely determined by its initial values (u, u_t) at the moment $t = T$. Moreover, in a way similar to that of Theorem 3.3, the set of values (u, u_t) at the moment $t = T$ of the partially synchronizable state u is actually the whole space $L^2(\Omega) \times H^{-1}(\Omega)$ as the initial values U_0 and U_1 vary in the space $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.

Remark 5.2 Taking $m = N - 2$ in Theorem 5.2, under the corresponding conditions of compatibility, we can use $N - 1$ (instead of $N - 2$!) boundary controls to realize the exact null controllability for $N - 2$ state variables in U.

6 Approximation of the Final State for a System of Vibrating Strings

Once the exact synchronization is realized at the moment T, the final state $u = u(t, x)$ for $t \geq T$ will be governed by a corresponding wave equation with homogeneous Dirichlet boundary condition (see also (3.17))

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \widetilde{a} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{6.1}$$

where the constant $\widetilde{a}$ is given by (3.6). However, for lack of the value of the final state at the moment T, we do not know how the final state u will evolve henceforth. The goal of this section is to give an approximation of the final state $u = u(t, x)$ for $t \geq T$ for a coupled system of vibrating strings with a perturbation of a synchronizable state as the initial data.

Let $0 < a < 1$. Consider the following 1-dimensional problem:

$$\begin{cases} u'' - u_{xx} - av = 0, \quad 0 < x < \pi, \\ v'' - v_{xx} - au = 0, \quad 0 < x < \pi, \\ u(t, 0) = u(t, \pi) = 0, \\ v(t, 0) = 0, \quad v(t, \pi) = h(t), \\ t = 0: \ (u, v) = (u_0, v_0), \quad (u_t, v_t) = (u_1, v_1). \end{cases} \tag{6.2}$$

Using the spectral analysis as in [8], we can prove that (6.2) is asymptotically controllable, but not exactly controllable. By Theorem 3.2, this problem is exactly synchronizable by means of boundary control h. More precisely, by setting

$$y = v - u, \tag{6.3}$$

for $T > 2\pi$, there exists a boundary control $h \in L^2(0, T)$, which realizes the exact null controllability at the moment T for the following reduced problem:

$$\begin{cases} y'' - y_{xx} + ay = 0, \quad 0 < x < \pi, \\ y(t, 0) = 0, \quad y(t, \pi) = h(t), \\ t = 0: \ y = v_0 - u_0, \quad y' = v_1 - u_1. \end{cases} \tag{6.4}$$

Moreover, by the HUM method (see [9]) or the moment method (see [6]), there exists a positive constant $C > 0$, such that

$$\|h\|_{L^2(0,T)} \leq C \|(v_0 - u_0, v_1 - u_1)\|_{L^2(0,\pi) \times H^{-1}(0,\pi)}. \tag{6.5}$$

In what follows, we will give an expression of the final state $u = v$ of the problem (6.2) for $t \geq T$. To this end, setting

$$w = u + v, \quad w_0 = u_0 + v_0, \quad w_1 = u_1 + v_1, \tag{6.6}$$

we consider the following anti-synchronization problem:

$$\begin{cases} w'' - w_{xx} - aw = 0, & 0 < x < \pi, \\ w(t,0) = 0, \quad w(t,\pi) = h(t), \\ t = 0: \ w = w_0, \quad w' = w_1. \end{cases} \tag{6.7}$$

Assume that $w_0 \in L^2(0,\pi)$ and $w_1 \in H^{-1}(0,\pi)$, whose coefficients a_j^0 and b_j^0 $(j \geq 1)$ on the orthonormal basis $\left(\sqrt{\frac{2}{\pi}}\sin(jx)\right)_{j\geq 1}$ in $L^2(0,\pi)$ and $\left(\sqrt{\frac{2}{\pi}}j\sin(jx)\right)_{j\geq 1}$ in $H^{-1}(0,\pi)$ are respectively given by

$$a_j^0 = \sqrt{\frac{2}{\pi}}\int_0^\pi w_0(x)\sin(jx)\mathrm{d}x, \quad jb_j^0 = \sqrt{\frac{2}{\pi}}\int_0^\pi w_1(x)\sin(jx)\mathrm{d}x. \tag{6.8}$$

Correspondingly, for any given $t \geq T$, the coefficients $a_j(t)$ and $b_j(t)$ $(j \geq 1)$ of the final state $u(t,x)$ and $u'(t,x)$ on the orthonormal basis $\left(\sqrt{\frac{2}{\pi}}\sin(jx)\right)_{j\geq 1}$ in $L^2(0,\pi)$ and $\left(\sqrt{\frac{2}{\pi}}j\sin(jx)\right)_{j\geq 1}$ in $H^{-1}(0,\pi)$ are respectively given by

$$a_j(t) = \sqrt{\frac{2}{\pi}}\int_0^\pi u(t,x)\sin(jx)\mathrm{d}x, \quad jb_j(t) = \sqrt{\frac{2}{\pi}}\int_0^\pi u'(t,x)\sin(jx)\mathrm{d}x. \tag{6.9}$$

Now let $\mu_j = \sqrt{j^2 - a}$ $(j \geq 1)$. Multiplying the equation in (6.7) by $\sin(\mu_j s)\sin(jx)$ and integrating with respect to s and x on $[0,t] \times [0,\pi]$, by integration by parts, we get

$$\begin{aligned} &\Big[\int_0^\pi w'(s,x)\sin(\mu_j s)\sin(jx)\mathrm{d}x\Big]_0^t - \Big[\mu_j\int_0^\pi w(s,x)\cos(\mu_j s)\sin(jx)\mathrm{d}x\Big]_0^t \\ &- \Big[\int_0^t w_x(s,x)\sin(\mu_j s)\sin(jx)\mathrm{d}s\Big]_0^\pi + \Big[j\int_0^t w(s,x)\sin(\mu_j s)\cos(jx)\mathrm{d}s\Big]_0^\pi \\ &+ (-\mu_j^2 + j^2 - a)\int_0^\pi\int_0^t w(s,x)\sin(\mu_j s)\sin(jx)\mathrm{d}s\mathrm{d}x = 0. \end{aligned}$$

It follows that

$$\begin{aligned} &\sin(\mu_j t)\int_0^\pi w'(t,x)\sin(jx)\mathrm{d}x - \mu_j\cos(\mu_j t)\int_0^\pi w(t,x)\sin(jx)\mathrm{d}x \\ &+ \mu_j\int_0^\pi w_0(x)\sin(jx)\mathrm{d}x + (-1)^j j\int_0^t h(s)\sin(\mu_j s)\mathrm{d}s = 0. \end{aligned} \tag{6.10}$$

Noting that $w(t,x) = 2u(t,x)$ and $w'(t,x) = 2u'(t,x)$ for $t \geq T$, by (6.9), we have

$$\int_0^\pi w(t,x)\sin(jx)\mathrm{d}x = 2\int_0^\pi u(t,x)\sin(jx)\mathrm{d}x = \sqrt{2\pi}\,a_j(t), \tag{6.11}$$

$$\int_0^\pi w'(t,x)\sin(jx)\mathrm{d}x = 2\int_0^\pi u'(t,x)\sin(jx)\mathrm{d}x = \sqrt{2\pi}\,jb_j(t). \tag{6.12}$$

Inserting (6.11)–(6.12) into (6.10) and noting (6.8), we get

$$a_j(t)\mu_j\cos(\mu_j t) - jb_j(t)\sin(\mu_j t) = \frac{1}{2}\mu_j a_j^0 + (-1)^j j\sqrt{\frac{1}{2\pi}}\int_0^t h(s)\sin(\mu_j s)\mathrm{d}s. \tag{6.13}$$

Similarly, multiplying the equation in (6.7) by $\cos(\mu_j s)\sin(jx)$ and integrating with respect to s and x on $[0,t]\times[0,\pi]$, by integration by parts, we get

$$\begin{aligned}&\Big[\int_0^\pi w'(s,x)\cos(\mu_j s)\sin(jx)\mathrm{d}x\Big]_0^t + \Big[\mu_j\int_0^\pi w(s,x)\sin(\mu_j s)\sin(jx)\mathrm{d}x\Big]_0^t\\ &-\Big[\int_0^t w_x(s,x)\cos(\mu_j s)\sin(jx)\mathrm{d}s\Big]_0^\pi + \Big[j\int_0^t w(s,x)\cos(\mu_j s)\cos(jx)\mathrm{d}s\Big]_0^\pi\\ &+(-\mu_j^2+j^2-a)\int_0^\pi\int_0^t w(s,x)\cos(\mu_j s)\sin(jx)\mathrm{d}s\mathrm{d}x = 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\cos(\mu_j t)\int_0^\pi w'(t,x)\sin(jx)\mathrm{d}x + \mu_j\sin(\mu_j t)\int_0^\pi w(t,x)\sin(jx)\mathrm{d}x\\ &-\int_0^\pi w_1(x)\sin(jx)\mathrm{d}x + (-1)^j j\int_0^t h(s)\cos(\mu_j s)\mathrm{d}s = 0.\end{aligned} \tag{6.14}$$

Then noting (6.11)–(6.12) and (6.8), we get

$$a_j(t)\mu_j\sin(\mu_j t) + jb_j(t)\cos(\mu_j t) = \frac{1}{2}jb_j^0 - (-1)^j j\sqrt{\frac{1}{2\pi}}\int_0^t h(s)\cos(\mu_j s)\mathrm{d}s. \tag{6.15}$$

It follows from (6.13) and (6.15) that

$$a_j(t) = \frac{\mu_j a_j^0\cos(\mu_j t) + jb_j^0\sin(\mu_j t)}{2\mu_j} + (-1)^j\frac{j}{\mu_j}\sqrt{\frac{1}{2\pi}}\int_0^t h(s)\sin[\mu_j(s-t)]\mathrm{d}s, \tag{6.16}$$

$$b_j(t) = \frac{-\mu_j a_j^0\sin(\mu_j t) + jb_j^0\cos(\mu_j t)}{2j} - (-1)^j\sqrt{\frac{1}{2\pi}}\int_0^t h(s)\cos[\mu_j(s-t)]\mathrm{d}s. \tag{6.17}$$

Now assume that (v_0, v_1) is a small perturbation of (u_0, u_1), so that by (6.5), the optimal control h is small in $L^2(0,T)$. Then

$$\begin{cases}\widetilde{a}_j(t) = \dfrac{\mu_j a_j^0\cos(\mu_j t) + jb_j^0\sin(\mu_j t)}{2\mu_j},\\ \widetilde{b}_j(t) = \dfrac{-\mu_j a_j^0\sin(\mu_j t) + jb_j^0\cos(\mu_j t)}{2j}\end{cases} \tag{6.18}$$

provide an approximation of the coefficients $a_j(t)$ and $b_j(t)$, respectively. Indeed, let

$$\widetilde{u}(t,x) = \sum_{j=1}^{+\infty}\widetilde{a}_j(t)\sin(jx), \quad \widetilde{u}'(t,x) = \sum_{j=1}^{+\infty} j\widetilde{b}_j(t)\sin(jx). \tag{6.19}$$

$(\widetilde{u}, \widetilde{u}')$ would be a good approximation of the final state (u, u') for $t \geq T$. In fact, we have the following result.

Theorem 6.1 *Let $T > 2\pi$. Assume that*

$$(u_0,\ u_1) \in L^2(0,\pi)\times H^{-1}(0,\pi), \quad (v_0,\ v_1) \in L^2(0,\pi)\times H^{-1}(0,\pi). \tag{6.20}$$

Then for all $t \geq T$, we have

$$\begin{aligned} &\|(u(t,\cdot) - \widetilde{u}(t,\cdot),\ u'(t,\cdot) - \widetilde{u}'(t,\cdot))\|_{L^2(0,\pi)\times H^{-1}(0,\pi)} \\ \leq\ & C\|(v_0 - u_0, v_1 - u_1)\|_{L^2(0,\pi)\times H^{-1}(0,\pi)}. \end{aligned} \tag{6.21}$$

Proof Define

$$s_j(t) = \int_0^t h(s)\sin[\mu_j(s-t)]\mathrm{d}s, \quad c_j(t) = \int_0^t h(s)\cos[\mu_j(s-t)]\mathrm{d}s. \tag{6.22}$$

Noting that the reals $\{\mu_j\}_{j\geq 1}$ are distinct, and for all $j \geq 1$, we have the following gap condition:

$$\mu_{j+1} - \mu_j = \frac{2j+1}{\sqrt{(j+1)^2 - a} + \sqrt{j^2 - a}} \geq \frac{2j+1}{2\sqrt{(j+1)^2 - a}} \geq \frac{3}{2\sqrt{4-a}} > 0, \tag{6.23}$$

where the last inequality is due to the growth of the function $x \to \frac{2x+1}{2\sqrt{(x+1)^2-a}}$ $(x \geq 0)$. Then for any fixed $t \geq T$, the system $\{\sin[\mu_j(s-t)], \cos[\mu_j(s-t)]\}_{j\in\mathbb{N}}$ is a Riesz sequence in $L^2(0,T)$. Consequently, there exists a positive constant $C > 0$ independent of t, such that the following Bessel's inequality holds for all $t \geq T$ (see [5, 15]):

$$\sum_{j=1}^{+\infty}(|s_j(t)|^2 + |c_j(t)|^2) \leq C\|h\|^2_{L^2(0,T)} \leq C\|(v_0 - u_0, v_1 - u_1)\|^2_{L^2(0,\pi)\times H^{-1}(0,\pi)}, \tag{6.24}$$

where the last inequality is due to (6.5). Then, it follows from (6.16)–(6.17) that

$$\begin{aligned} &\sum_{j=1}^{+\infty}(|a_j(t) - \widetilde{a}_j(t)|^2 + |b_j(t) - \widetilde{b}_j(t)|^2) \\ \leq\ & C\sum_{j=1}^{+\infty}(|s_j(t)|^2 + |c_j(t)|^2) \leq C\|(v_0 - u_0, v_1 - u_1)\|^2_{L^2(0,\pi)\times H^{-1}(0,\pi)}, \end{aligned} \tag{6.25}$$

which yields (6.21). The proof is complete.

Remark 6.1 Let (v_0, v_1) be a perturbation of (u_0, u_1). Then the norm of their difference $\|(v_0 - u_0, v_1 - u_1)\|_{L^2(0,\pi)\times H^{-1}(0,\pi)}$ is a small quantity, so that (6.21) shows that $(\widetilde{u}, \widetilde{u}')$ is indeed a good approximation of the final state (u, u'). Furthermore, noting that $\frac{j}{\mu_j} \sim 1$ for j large enough, we have

$$|\widetilde{a}_j(t)|^2 + |\widetilde{b}_j(t)|^2 \sim \frac{1}{4}(|a_j^0|^2 + |b_j^0|^2). \tag{6.26}$$

Noting that a_j^0 $(j \geq 1)$ are the coefficients of $w_0 = u_0 + v_0$ in $L^2(0,\pi)$ and b_j^0 $(j \geq 1)$ are the coefficients of $w_1 = u_1 + v_1$ in $H^{-1}(0,\pi)$, (6.26) shows that the approximate final state $(\widetilde{u}, \widetilde{u}')$ has the same norm as that of the average of the initial data for high frequencies.

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