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NEW RECURRENCE FORMULAE FOR THE POLYLOGARITHM FUNCTION

HE Yuan, ZHANG Jia-ling

(*School of Science, Kunming University of Science and Technology, Kunming 650500, China*)

Abstract: In this paper, we perform a further investigation for the polylogarithm function at negative integral arguments. By applying the generating function methods and Padé approximation techniques, we establish some new recurrence formulae for this type function and present some illustrative special cases of main results.

Keywords: polylogarithm function; generating function; Padé approximants; recurrence formulae

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1 Introduction

Let s and z be complex numbers, the polylogarithm function $\mathrm{Li}_s(z)$ is defined by means of the Dirichlet series

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \tag{1.1}$$

which is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$ and can be extended to $|z| \geq 1$ by the process of analytic continuation.

The polylogarithm function at zero and negative integral arguments are referred to as the polypseudologarithms (or polypseudologs) of order n by Lee [8]. It is worth noticing that the values of polypseudologrithms at $z = 1$ are related to the values of the Riemann zeta function $\zeta(s)$ at negative integers and are expressed in terms of the Bernoulli numbers B_n, as follows (see, e.g., [4, 8])

$$\mathrm{Li}_{-n}(-1) = (2^{n+1} - 1)\zeta(-n) = (1 - 2^{n+1})\frac{B_{n+1}}{n+1} \quad (n = 1, 2, \cdots). \tag{1.2}$$

In [11], Truesdell gave a closed formula for the polypseudologarithms, as follows

$$\mathrm{Li}_{-n}(z) = \sum_{k=1}^{n} \frac{(-1)^{n+k} k! S(n,k) z}{(1-z)^{k+1}} \quad (n = 1, 2, \cdots), \tag{1.3}$$

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Biography: He Yuan (1982–), male, born at Neijiang, Sichuan, associate professor, major in number theory and its applications.

where $S(n,k)$ is the familiar Stirling numbers of the second kind. In [6], Eastham showed that there is no pure recurrence relation of the form

$$A_0(z)\mathrm{Li}_n(z)+A_1(z)\mathrm{Li}_{n-1}(z)+\cdots+A_r(z)\mathrm{Li}_{n-r}(z)=0, \tag{1.4}$$

where n is a positive integer, $r\geq n$ is allowed. The $A_n(z)$ are algebraic functions of z and $A_0(z)$ is not identically zero. More recently, Cvijović [5] discovered some similar ones for the polypseudologarithms to formula (1.3), and also established a new type closed formula for the polypseudologarithms in the following way

$$\mathrm{Li}_{-n}(z)=\frac{1}{2^{n+1}}\left[(-1)^{[\frac{n}{2}-1]}T(n,1)+\sum_{k=1}^{n+1}\frac{(-1)^{[\frac{n-k}{2}-1]}}{k}T(n+1,k)\left(\frac{1+z}{1-z}\right)^k\right], \tag{1.5}$$

where $[x]$ denotes the greatest integer $\leq x$ and $T(n,k)$ is the tangent numbers (of order k) or the higher order tangent numbers given by (see, e.g., [3])

$$\tan^k(t)=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!}\quad (k=1,2,\cdots). \tag{1.6}$$

Motivated by the work of Eastham and Cvijović, in this paper we perform a further investigation for the polylogarithm function at negative integral arguments, and establish some new recurrence formulae for this type function to state that there exist some explicit recurrence relations of form (1.4) for the polypseudologarithms by applying the generating function methods and Padé approximation techniques. And we accordingly consider some illustrative special cases as well as immediate consequences of the main results.

2 Padé Approximants

We begin by recalling the definition of Padé approximation to general series and their expression in the case of the exponential function. Let m,n be non-negative integers and let $\mathcal{P}_k$ be the set of all polynomials of degree $\leq k$. Given a function f with a Taylor expansion

$$f(u)=\sum_{k=0}^{\infty}c_kt^k \tag{2.1}$$

in a neighborhood of the origin, a Padé form of type (m,n) is a pair (P,Q) satisfying that

$$P=\sum_{k=0}^{m}p_kt^k\in\mathcal{P}_m,\quad Q=\sum_{k=0}^{n}q_kt^k\in\mathcal{P}_n\quad (Q\not\equiv 0) \tag{2.2}$$

and

$$Qf-P=\mathcal{O}(t^{m+n+1})\quad \text{as } t\to 0. \tag{2.3}$$

It is clear that every Padé form of type (m,n) for $f(t)$ always exists and obeys the same rational function. The uniquely determined rational function P/Q is called the Padé approximant of type (m,n) for $f(t)$, and is denoted by $[m/n]_f(t)$ or $r_{m,n}[f;t]$, see for example, [1, 2].

The study of Padé approximants to the exponential function was initiated by Hermite [7] and then continued by Padé [9]. Given a pair (m, n) of nonnegative integers, the Padé approximant of type (m, n) for e^t is the unique rational function

$$R_{m,n}(t) = \frac{P_{m,n}(t)}{Q_{m,n}(t)} \quad (P_{m,n} \in \mathcal{P}_m, Q_{m,n} \in \mathcal{P}_n, Q_{m,n}(0) = 1) \tag{2.4}$$

with the property that

$$e^t - R_{m,n}(t) = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \to 0. \tag{2.5}$$

Unlike Padé approximants to other functions, it is possible to determine explicit formulae for $P_{m,n}$ and $Q_{m,n}$ (see, e.g., [10, p.245])

$$P_{m,n}(t) = \sum_{k=0}^{m} \frac{m! \cdot (m+n-k)!}{(m+n)! \cdot (m-k)!} \cdot \frac{t^k}{k!}, \tag{2.6}$$

$$Q_{m,n}(t) = \sum_{k=0}^{n} \frac{n! \cdot (m+n-k)!}{(m+n)! \cdot (n-k)!} \cdot \frac{(-t)^k}{k!} \tag{2.7}$$

and

$$Q_{m,n}(t)e^t - P_{m,n}(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \int_0^1 x^n (1-x)^m e^{xt} \mathrm{d}x. \tag{2.8}$$

We here refer respectively to $P_{m,n}(t)$ and $Q_{m,n}(t)$ as the Padé numerator and denominator of type (m, n) for e^t. In next section, we shall use the above Padé approximation to the exponential function to establish some new recurrence formulae for the polylogarithm function at zero and negative integral arguments.

3 The Restatements of Results

In [4], Cvijović discovered some similar formulae to (1.3) by making use of the following generating functions for the polypseudologarithms (see, e.g., [11, 12])

$$\frac{1}{1-ze^t} = \sum_{n=1}^{\infty} \mathrm{Li}_{-n}(z) \frac{t^n}{n!}, \quad \frac{z}{e^t - z} = \sum_{n=0}^{\infty} (-1)^n \mathrm{Li}_{-n}(z) \frac{t^n}{n!} \tag{3.1}$$

and

$$\log(e^t - z) = \sum_{n=2}^{\infty} (-1)^{n-1} \mathrm{Li}_{-(n-1)}(z) \frac{t^n}{n!},$$

$$\frac{1}{2} \cdot \frac{1+ze^t}{1-ze^t} = \sum_{n=1}^{\infty} \mathrm{Li}_{-n}(z) \frac{t^n}{n!}. \tag{3.2}$$

We shall replace the exponential function e^t not by its Taylor expansion around $t = 0$ but by its Padé approximant in the generating function of the polypseudologarithms. We first rewrite the first formula of (3.1) as follows

$$(1-ze^t) \sum_{j=1}^{\infty} \mathrm{Li}_{-j}(z) \frac{t^j}{j!} = 1. \tag{3.3}$$

If we denote the right hand side of (2.8) by $S_{m,n}(t)$, the Padé approximant for the exponential function e^t can be expressed as

$$e^t = \frac{P_{m,n}(t) + S_{m,n}(t)}{Q_{m,n}(t)}. \tag{3.4}$$

We now apply (3.4) to (3.3) and then obtain

$$\big(Q_{m,n}(t) - zP_{m,n}(t) - zS_{m,n}(t)\big)\sum_{j=1}^{\infty}\mathrm{Li}_{-j}(z)\frac{t^j}{j!} = Q_{m,n}(t). \tag{3.5}$$

If we apply the exponential series $e^{xt} = \sum\limits_{k=0}^{\infty} x^k t^k/k!$ in the right hand side of (2.8), with the help of the familiar beta function, we get

$$\begin{aligned} S_{m,n}(t) &= (-1)^n \frac{t^{m+n+1}}{(m+n)!}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^1 x^{n+k}(1-x)^m \mathrm{d}x \\ &= \sum_{k=0}^{\infty}\frac{(-1)^n m!\cdot(n+k)!}{(m+n)!\cdot(m+n+k+1)!}\cdot\frac{t^{m+n+k+1}}{k!}. \end{aligned} \tag{3.6}$$

For convenience, we consider $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ of the coefficients of the polynomials $P_{m,n}(t), Q_{m,n}(t)$ and $S_{m,n}(t)$ such that

$$P_{m,n}(t) = \sum_{k=0}^{m} p_{m,n;k}t^k, \quad Q_{m,n}(t) = \sum_{k=0}^{n} q_{m,n;k}t^k, \quad S_{m,n}(t) = \sum_{k=0}^{\infty} s_{m,n;k}t^{m+n+k+1}. \tag{3.7}$$

Obviously, the coefficients $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ obey

$$p_{m,n;k} = \frac{m!\cdot(m+n-k)!}{(m+n)!\cdot k!\cdot(m-k)!}, \quad q_{m,n;k} = \frac{(-1)^k n!\cdot(m+n-k)!}{(m+n)!\cdot k!\cdot(n-k)!} \tag{3.8}$$

and

$$s_{m,n;k} = \frac{(-1)^n m!\cdot(n+k)!}{(m+n)!\cdot k!\cdot(m+n+k+1)!}, \tag{3.9}$$

respectively. If we apply (3.7) to (3.5), we obtain

$$\begin{aligned} &\Big(\sum_{k=0}^{n} q_{m,n;k}t^k\Big)\sum_{j=1}^{\infty}\mathrm{Li}_{-j}(z)\frac{t^j}{j!} - z\Big(\sum_{k=0}^{m} p_{m,n;k}t^k\Big)\sum_{j=1}^{\infty}\mathrm{Li}_{-j}(z)\frac{t^j}{j!} \\ - \;& z\Big(\sum_{k=0}^{\infty} s_{m,n;k}t^{m+n+k+1}\Big)\sum_{j=1}^{\infty}\mathrm{Li}_{-j}(z)\frac{t^j}{j!} = \sum_{k=0}^{n} q_{m,n;k}t^k \end{aligned} \tag{3.10}$$

from which and the familiar Cauchy product, we discover

$$\begin{aligned} &\sum_{l=1}^{\infty} t^l \sum_{\substack{k+j=l \\ k\geq 0, j\geq 1}} q_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!} - z\sum_{l=1}^{\infty} t^l \sum_{\substack{k+j=l \\ k\geq 0, j\geq 1}} p_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!} \\ - \;& z\sum_{l=1}^{\infty} t^l \sum_{\substack{k+j=l-m-n-1 \\ k\geq 0, j\geq 1}} s_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!} = \sum_{k=0}^{n} q_{m,n;k}t^k. \end{aligned} \tag{3.11}$$

Comparing the coefficients of t^l in (3.11) gives that for $1 \le l \le m+n$,

$$\sum_{\substack{k+j=l\\k\ge 0, j\ge 1}} q_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!} - z\sum_{\substack{k+j=l\\k\ge 0, j\ge 1}} p_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!} = q_{m,n;l}, \tag{3.12}$$

which together with (3.8) yields the following result.

Theorem 3.1 Let l, m, n be non-negative integers. Then for positive integer l with $\max(m,n) < l \le m+n$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(m+n-k)!\frac{\mathrm{Li}_{-(l-k)}(z)}{(l-k)!} = z\sum_{k=0}^{m}\binom{m}{k}(m+n-k)!\frac{\mathrm{Li}_{-(l-k)}(z)}{(l-k)!}. \tag{3.13}$$

We next discuss some special cases of Theorem 3.1. Setting $l = m+n$ in Theorem 3.1, we obtain that for positive integers m, n,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathrm{Li}_{-(m+k)}(z) = z\sum_{k=0}^{m}\binom{m}{k}\mathrm{Li}_{-(n+k)}(z). \tag{3.14}$$

It is obvious that the case $m = 1$ in (3.14) gives that for positive integer n,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathrm{Li}_{-(n+1-k)}(z) = z\mathrm{Li}_{-n}(z) + z\mathrm{Li}_{-(n+1)}(z) \tag{3.15}$$

and the case $n = 1$ in (3.14) arises

$$z\sum_{k=0}^{n}\binom{n}{k}\mathrm{Li}_{-(n+1-k)}(z) = \mathrm{Li}_{-(n+1)}(z) - \mathrm{Li}_{-n}(z) \quad (n \ge 1). \tag{3.16}$$

If we compare the coefficients of t^l in (3.11) for $l \ge m+n+1$, then

$$\begin{aligned}&\sum_{\substack{k+j=l\\k\ge 0, j\ge 1}} q_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!} - z\sum_{\substack{k+j=l\\k\ge 0, j\ge 1}} p_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!}\\ =\ & z\sum_{\substack{k+j=l-m-n-1\\k\ge 0, j\ge 1}} s_{m,n;k}\frac{\mathrm{Li}_{-j}(z)}{j!}.\end{aligned} \tag{3.17}$$

Hence applying (3.8) and (3.9) to (3.17) gives the following result.

Theorem 3.2 Let m, n be non-negative integers. Then for positive integer l with $l \ge m+n+1$,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(m+n-k)!\frac{\mathrm{Li}_{-(l-k)}(z)}{(l-k)!} - z\sum_{k=0}^{m}\binom{m}{k}(m+n-k)!\frac{\mathrm{Li}_{-(l-k)}(z)}{(l-k)!}\\ =\ & (-1)^n z\frac{m!\cdot n!}{l!}\sum_{k=1}^{l-m-n-1}\binom{l-m-1-k}{n}\binom{l}{k}\mathrm{Li}_{-k}(z).\end{aligned} \tag{3.18}$$

It follows that we show some special cases of Theorem 3.2. Taking $l = m + n + 1$ in Theorem 3.2, we obtain that for non-negative integers m, n,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{\mathrm{Li}_{-(m+k+1)}(z)}{m+k+1} - z\sum_{k=0}^{m}\binom{m}{k}\frac{\mathrm{Li}_{-(n+k+1)}(z)}{n+k+1} = 0. \tag{3.19}$$

In particular, the case $m = 0$ in (3.19) arises

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{\mathrm{Li}_{-(k+1)}(z)}{k+1} = \frac{z\mathrm{Li}_{-(n+1)}(z)}{n+1}. \tag{3.20}$$

More generally, by setting $m = 0$ and $l = n+r$ in Theorem 3.2, we get that for non-negative integer n and positive integer r,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k!\frac{\mathrm{Li}_{-(k+r)}(z)}{(k+r)!} - \frac{n!}{(n+r)!}z\mathrm{Li}_{-(n+r)}(z)\\ =\ &(-1)^n z\frac{n!}{(n+r)!}\sum_{k=1}^{r-1}\binom{n+r-k-1}{n}\binom{n+r}{k}\mathrm{Li}_{-k}(z).\end{aligned} \tag{3.21}$$

And the case $n = 0$ in (3.21) yields another recurrence formula to compute the values of the polypseudologarithms with $\mathrm{Li}_0(z) = z/(1-z)$:

$$(1-z)\mathrm{Li}_{-n}(z) = z\sum_{k=1}^{n-1}\binom{n}{k}\mathrm{Li}_{-k}(z) \quad (n \geq 2). \tag{3.22}$$

It becomes obvious that formulae (3.15), (3.16) and (3.22) mean that there exists pure recurrence relations of form (1.4) for the polypseudologarithms, respectively.

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关于polylogarithm函数新的循环公式

何 圆, 张家玲

(昆明理工大学理学院, 云南 昆明 650500)

摘要: 本文对polylogarithm函数在负整数点的情形作了进一步的研究. 利用生成函数方法及Padé估计技巧, 建立了此类函数的一些新的循环公式, 并给出了主要结果的一些特殊情况.

关键词: polylogarithm函数; 生成函数; Padé估计; 循环公式

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