

A NOTE ON DIRICHLET PROBLEM FOR SEMILINEAR SECOND ORDER ELLIPTIC EQUATION

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Abstract: In this paper, we study the Dirichlet problem for semilinear second order elliptic equation. By using the mountain pass lemma and the least action principle, we obtain the existence of weak solution to semilinear second order elliptic equation under some new conditions which are different from the previous ones.

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1 Introduction

We consider the following elliptic equation with Dirichlet boundary value condition

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$ and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

Many authors were interested in studying the existence of nontrivial solution of (1.1) via variational methods, for example, see [1–14].

Ambrosetti, Rabinowitz [1] established the existence of nontrivial solution of (1.1) by applying mountain pass lemma under the following conditions (also see [3]).

(f₁) There exist positive constants a, b and $s \in (0, \frac{n+2}{n-2})$ such that for $(x, u) \in \Omega \times \mathbb{R}$,

$$|f(x, u)| \leq a + b|u|^s;$$

(f₂) $\liminf_{u \rightarrow +\infty} \frac{f(x, u)}{u} > \lambda_1$ uniformly in $x \in \overline{\Omega}$;

(f₃) There exist positive constants $\mu > 2$ and $r > 0$ such that for $x \in \Omega$ and $|u| \geq r$,

$$0 < \mu F(x, u) \leq u f(x, u),$$

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where $F(x, u) = \int_0^u f(x, t)dt$ and λ_1 is the first eigenvalue of the Laplacian $(-\Delta)$ on Ω with zero Dirichlet boundary condition;

$$(f_4) \quad \limsup_{u \rightarrow 0} \frac{f(x, u)}{u} < \lambda_1 \text{ uniformly in } x \in \overline{\Omega}.$$

As is well known, (f_3) is so-called Ambrosetti-Rabinowitz condition (see [1]), (AR) for short, which guarantees that Palais-Smale sequence of the Euler-Lagrange functional is bounded. Actually, integrating (AR) it follows that $\liminf_{u \rightarrow +\infty} \frac{f(x, u)}{u} = +\infty$ uniformly in $x \in \overline{\Omega}$, and hence condition (f_2) can be eliminated in [3, Theorem 4.8.13]. Recently, Mavinga and Nkashama in [15] provided a new method to ensure that (PS) condition is satisfied. Motivated by above references, we in this paper prove the existence of solution for (1.1) by using mountain pass lemma, and replace (f_1) – (f_3) with

$(f'_1) \quad \lambda_1 < \liminf_{|u| \rightarrow \infty} \frac{f(x, u)}{u} \leq \limsup_{|u| \rightarrow \infty} \frac{f(x, u)}{u} < \lambda_2$ uniformly in $x \in \overline{\Omega}$, where λ_2 is the second eigenvalue of the Laplacian $(-\Delta)$ on Ω with zero Dirichlet boundary condition.

We also establish the existence of solution for (1.1) by the least action principle under the conditions:

$$(f'_2) \quad \limsup_{|u| \rightarrow \infty} \frac{f(x, u)}{u} < \lambda_1 \text{ uniformly in } x \in \overline{\Omega};$$

$$(f'_3) \quad \liminf_{u \rightarrow 0^+} \frac{f(x, u)}{u} > \lambda_1 \text{ uniformly in } x \in \overline{\Omega}.$$

Let the norm of u in Sobolev space $W_0^{1,2}(\Omega)$ be $\|u\|_{1,2} = (\int_{\Omega} |Du|^2 dx)^{\frac{1}{2}}$ and $\|u\|_2 = (\int_{\Omega} |u|^2 dx)^{\frac{1}{2}}$ stand for the usual L^2 -norm. In addition, the $W_0^{1,2}(\Omega)$ -inner product is defined as $[u, v] = \int_{\Omega} Du \cdot Dv dx$ and we denote the n -dimensional Lebesgue measure of Ω by $|\Omega|$. From [10] we know that under (f_1) the Euler-Lagrange functional

$$I(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} F(x, u) dx, \quad \forall u \in W_0^{1,2}(\Omega)$$

belongs to $C^1(W_0^{1,2}(\Omega), \mathbb{R})$ and

$$(I'(u), \varphi) = \int_{\Omega} Du \cdot D\varphi dx - \int_{\Omega} f(x, u)\varphi dx, \quad \forall u, \varphi \in W_0^{1,2}(\Omega).$$

Thus the critical points of I are the weak solutions to (1.1).

From [5] we have the following facts. The Laplacian $(-\Delta)$ on Ω with zero Dirichlet boundary condition has a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_j \leq \dots \rightarrow \infty$ as $j \rightarrow \infty$. The first eigenvalue

$$\lambda_1 = \min_{u \in W_0^{1,2}(\Omega), \|u\|_2 \neq 0} \frac{[u, u]}{\|u\|_2^2} = \min_{u \in W_0^{1,2}(\Omega), \|u\|_2 \neq 0} \frac{\|u\|_{1,2}^2}{\|u\|_2^2} \quad (1.2)$$

is simple and there exists an eigenfunction $\varphi_1 \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ (see [2, Theorem 1.16]) corresponding to λ_1 such that $\varphi_1(x) > 0$ in Ω and

$$\lambda_1 \|\varphi_1\|_2^2 = \|\varphi_1\|_{1,2}^2. \quad (1.3)$$

The eigenspace corresponding to λ_1 can be expressed as $V = \text{span}\{\varphi_1\} = \{t\varphi_1 : t \in \mathbb{R}\}$ and

$$\lambda_2 = \min_{u \in W_0^{1,2}(\Omega) \cap V^\perp, \|u\|_2 \neq 0} \frac{\|u\|_{1,2}^2}{\|u\|_2^2}. \quad (1.4)$$

Now we introduce some auxiliary results which will be need in the sequel.

Let E be a real Banach space. For $I \in C^1(E, \mathbb{R})$, we say I satisfies the Palais-Smale condition ((PS) for short) if any sequence $\{u_m\} \subset E$ for which $\{I(u_m)\}$ is bounded and $I'(u_m) \rightarrow 0$ as $m \rightarrow \infty$ possesses a convergent subsequence.

Lemma 1 [10] (Mountain pass lemma) Let $I \in C^1(E, \mathbb{R})$ satisfy (PS). Suppose $I(0) = 0$ and

- (I₁) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$, and
- (I₂) there is an $e \in E \setminus \overline{B}_\rho$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where $\Gamma = \{g \in C([0,1], E) \mid g(0) = 0, g(1) = e\}$.

Lemma 2 [11] (The least action principle) Suppose E is a reflexive Banach space and $I : E \rightarrow \mathbb{R}$ is coercive and (sequentially) weakly lower semi-continuous, that is, the following conditions are fulfilled:

- (i) $I(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$;
- (ii) For any $u \in E$, any sequence $\{u_m\}$ in E such that $u_m \rightarrow u$ weakly in E there holds

$$I(u) \leq \liminf_{m \rightarrow \infty} I(u_m).$$

Then I is bounded from below and attains its infimum which is a critical value if $I \in C^1(E, \mathbb{R})$.

2 Main Results and Proofs

Theorem 1 If (f₁') and (f₄) are satisfied, then (1.1) has at least one nontrivial weak solution $u \in W_0^{1,2}(\Omega)$.

Proof We know easily from (f₁') that (f₁) is fulfilled. By (f₁') we take $\varepsilon > 0$ such that $\lambda_1 + \varepsilon < \liminf_{|u| \rightarrow \infty} \frac{f(x,u)}{u} \leq \limsup_{|u| \rightarrow \infty} \frac{f(x,u)}{u} < \lambda_2 - \varepsilon$. Then there exists $r > 0$ such that for $|u| \geq r$,

$$\lambda_1 + \varepsilon \leq \frac{f(x,u)}{u} \leq \lambda_2 - \varepsilon, \quad \forall x \in \overline{\Omega}. \quad (2.1)$$

First we prove that the functional I satisfies (PS). By (2.1) we denote $\tau : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\tau(x, u) = \begin{cases} \frac{f(x,u)}{u}, & |u| \geq r, \\ \frac{f(x,r)+f(x,-r)}{2r^2}u + \frac{f(x,r)-f(x,-r)}{2r}, & |u| < r. \end{cases} \quad (2.2)$$

Since f is continuous, τ is continuous in $\overline{\Omega} \times \mathbb{R}$. It is easy to see from (2.1) that

$$\lambda_1 + \varepsilon \leq \tau(x, u) \leq \lambda_2 - \varepsilon, \quad \forall (x, u) \in \overline{\Omega} \times \mathbb{R}. \quad (2.3)$$

Define $l : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ by $l(x, u) = f(x, u) - \tau(x, u)u$. Then it follows from the continuity of f and τ that there exists a constant $k > 0$ such that

$$|l(x, u)| \leq k \quad (2.4)$$

for all $(x, u) \in \overline{\Omega} \times \mathbb{R}$.

Now suppose $\{u_m\} \subset W_0^{1,2}(\Omega)$ for which $\{I(u_m)\}$ is bounded and $\lim_{m \rightarrow \infty} I'(u_m) = 0$. Let $u_m = v_m + w_m$, where $v_m \in V, w_m \in X = V^\perp$.

Since $\lim_{m \rightarrow \infty} I'(u_m) = 0$, there exists $N > 0$ such that $(I'(u_m), w_m - v_m) \leq \varepsilon \|w_m - v_m\|_{1,2}$ for all $m \geq N$. According to the orthogonality of w_m and v_m in $W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} (I'(u_m), w_m - v_m) &= \int_{\Omega} Du_m \cdot D(w_m - v_m) dx - \int_{\Omega} f(x, u_m)(w_m - v_m) dx \\ &= \int_{\Omega} D(v_m + w_m) \cdot D(w_m - v_m) dx - \int_{\Omega} f(x, u_m)(w_m - v_m) dx \\ &= \int_{\Omega} |Dw_m|^2 dx - \int_{\Omega} |Dv_m|^2 dx - \int_{\Omega} l(x, u_m)w_m dx + \int_{\Omega} l(x, u_m)v_m dx \\ &\quad - \int_{\Omega} \tau(x, u_m)w_m^2 dx + \int_{\Omega} \tau(x, u_m)v_m^2 dx. \end{aligned}$$

Thus

$$\begin{aligned} &\|w_m\|_{1,2}^2 - \|v_m\|_{1,2}^2 - \int_{\Omega} \tau(x, u_m)w_m^2 dx + \int_{\Omega} \tau(x, u_m)v_m^2 dx \\ &\leq \varepsilon \|w_m - v_m\|_{1,2} + \int_{\Omega} l(x, u_m)w_m dx - \int_{\Omega} l(x, u_m)v_m dx. \end{aligned}$$

It follows from (2.3), (1.4) and (1.3) that

$$\begin{aligned} &\|w_m\|_{1,2}^2 - \|v_m\|_{1,2}^2 - \int_{\Omega} \tau(x, u_m)w_m^2 dx + \int_{\Omega} \tau(x, u_m)v_m^2 dx \\ &\geq \|w_m\|_{1,2}^2 - \|v_m\|_{1,2}^2 - (\lambda_2 - \varepsilon) \int_{\Omega} w_m^2 dx + (\lambda_1 + \varepsilon) \int_{\Omega} v_m^2 dx \\ &\geq \|w_m\|_{1,2}^2 - \|v_m\|_{1,2}^2 - \frac{\lambda_2 - \varepsilon}{\lambda_2} \|w_m\|_{1,2}^2 + \frac{\lambda_1 + \varepsilon}{\lambda_1} \|v_m\|_{1,2}^2 = \frac{\varepsilon}{\lambda_2} \|w_m\|_{1,2}^2 + \frac{\varepsilon}{\lambda_1} \|v_m\|_{1,2}^2, \end{aligned}$$

and from (2.4) and (1.2) that

$$\begin{aligned} &\varepsilon \|w_m - v_m\|_{1,2} + \int_{\Omega} l(x, u_m)w_m dx - \int_{\Omega} l(x, u_m)v_m dx \\ &\leq \varepsilon (\|w_m\|_{1,2} + \|v_m\|_{1,2}) + k \int_{\Omega} |w_m| dx + k \int_{\Omega} |v_m| dx \\ &\leq \varepsilon (\|w_m\|_{1,2} + \|v_m\|_{1,2}) + k |\Omega|^{\frac{1}{2}} \|w_m\|_2 + k |\Omega|^{\frac{1}{2}} \|v_m\|_2 \\ &\leq (\varepsilon + k |\Omega|^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}}) (\|w_m\|_{1,2} + \|v_m\|_{1,2}). \end{aligned}$$

Therefore,

$$\frac{\varepsilon}{\lambda_2} \|w_m\|_{1,2}^2 + \frac{\varepsilon}{\lambda_1} \|v_m\|_{1,2}^2 \leq (\varepsilon + k|\Omega|^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}}) (\|w_m\|_{1,2} + \|v_m\|_{1,2}).$$

By the orthogonality of w_m and v_m , we have

$$\frac{\varepsilon}{\lambda_2} \|u_m\|_{1,2}^2 = \frac{\varepsilon}{\lambda_2} (\|w_m\|_{1,2}^2 + \|v_m\|_{1,2}^2) \leq \frac{\varepsilon}{\lambda_2} \|w_m\|_{1,2}^2 + \frac{\varepsilon}{\lambda_1} \|v_m\|_{1,2}^2,$$

and $\|w_m\|_{1,2} + \|v_m\|_{1,2} \leq 2\|u_m\|_{1,2}$. Hence

$$\|u_m\|_{1,2}^2 \leq 2(\varepsilon + k|\Omega|^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}}) \frac{\lambda_2}{\varepsilon} \|u_m\|_{1,2},$$

which implies that $\{u_m\}$ is bounded in $W_0^{1,2}(\Omega)$. By [10, Proposition B.35], I satisfies (PS).

By means of mountain pass lemma, the rest of proof is similar to the proofs in [3, Theorem 4.8.13] and [2, Theorem 8.11].

Theorem 2 If (f_1) , (f'_2) and (f'_3) are satisfied, then (1.1) has at least one nontrivial weak solution $u \in W_0^{1,2}(\Omega)$.

Proof It follows from (f_1) and [10, Proposition B.10] that I is weakly lower semi-continuous. We will prove that I is coercive.

By (f'_2) and (f'_3) , we can take $0 < \varepsilon < \lambda_1$ and there exists $0 < r < R$ such that for $|u| > R$,

$$\frac{f(x, u)}{u} \leq \lambda_1 - \varepsilon, \forall x \in \overline{\Omega}, \quad (2.5)$$

and for $0 < u < r$,

$$f(x, u) \geq (\lambda_1 + \varepsilon)u, \forall x \in \overline{\Omega}. \quad (2.6)$$

From (2.5) we have that for $|u| > R$, $F(x, u) \leq \frac{1}{2}(\lambda_1 - \varepsilon)u^2, \forall x \in \overline{\Omega}$. For $|u| \leq R$, it is easy to see from (f_1) that $F(x, u) \leq aR + \frac{b}{s+1}R^{s+1} \triangleq C, \forall x \in \overline{\Omega}$. Then for every $(x, u) \in \overline{\Omega} \times \mathbb{R}$,

$$F(x, u) \leq \frac{1}{2}(\lambda_1 - \varepsilon)u^2 + C. \quad (2.7)$$

Thus for $u \in W_0^{1,2}(\Omega)$, we have from (2.7) and (1.2) that

$$I(u) \geq \frac{1}{2}\|u\|_{1,2}^2 - \frac{1}{2}(\lambda_1 - \varepsilon)\|u\|_2^2 - C|\Omega| \geq \frac{\varepsilon}{2\lambda_1}\|u\|_{1,2}^2 - C|\Omega|$$

and I is coercive. It follows from Lemma 2 that I has a critical point in $u \in W_0^{1,2}(\Omega)$ such that $I(u) = \inf_{v \in W_0^{1,2}(\Omega)} I(v)$.

Now we show that it is nontrivial. In fact, let $\varphi_1 \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ be the eigenfunction corresponding to λ_1 with $0 < \varphi_1(x) < r$ in Ω . Hence by (2.6) and (1.3) we have

$$\begin{aligned} I(\varphi_1) &= \frac{1}{2} \int_{\Omega} |D\varphi_1|^2 dx - \int_{\Omega} \left(\int_0^{\varphi_1} f(x, t) dt \right) dx \\ &\leq \frac{1}{2} \|\varphi_1\|_{1,2}^2 - \frac{1}{2}(\lambda_1 + \varepsilon) \|\varphi_1\|_2^2 = -\frac{\varepsilon}{2\lambda_1} \|\varphi_1\|_{1,2}^2 < 0. \end{aligned}$$

Therefore,

$$I(u) = \inf_{v \in W_0^{1,2}(\Omega)} I(v) \leq I(\varphi_1) < 0.$$

The proof is completed.

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关于二阶半线性椭圆方程的Dirichlet边值问题一个注记

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摘要: 本文研究二阶半线性椭圆方程的Dirichlet边值问题. 利用山路引理和最小作用原理, 获得了在新条件下具有Dirichlet边值问题的二阶半线性椭圆方程的弱解的存在性的结果.

关键词: 椭圆方程; 山路引理; 最小作用原理; 临界点

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