ESTIMATION OF GROWTH OF MEROMORPHIC SOLUTIONS OF SECOND ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we investigate the growth order of meromorphic solution of algebraic differential equations. By using normal family theory, we give an estimation of the growth order of meromorphic solutions of certain class of second order algebraic differential equations, which depend on the degrees of rational function coefficients of the equations, and generalize a result by Liao Liangwen and Yang Chungchun (2001).

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1 Introduction

Let $f(z)$ be a function meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions (see e.g. [1–3]). We denote the order of $f(z)$ by $\rho(f)$.

As one knows, it was one of the important topics to research the algebraic differential equation of Malmquist type. In 1913, Malmquist [4] gave a result for the first order algebraic differential equations. In 1933, Yosida [5] proved the Malmquist's theorem by using the Nevanlinna theory. In 1970s, Laine [6], Yang [7] and Hille [8] gave a generalization of Malmquist's theorem. Later, Steinmetz [9], Rieth [10] and He-Laine [11] all gave corresponding generalizations of Malmquist's theorem for the first order algebraic differential equations. In 1980, Gackstatter and Laine [12] gave a generalized result of Malmquist's theorem for some certain type of higher order algebraic differential equations. However, Malmquist type theorem for an arbitrary second order algebraic differential equation remains open. For a second

order algebraic differential equation

$$f'' = R(z, f, f'), \tag{1.1}$$

where R is a rational function in z, f and f', a classical and unsolved conjecture is the following.

Conjecture 1.1 (see [3]) If equation (1.1) has a transcendental meromorphic solution, then the equation can be reduced into the form

$$f'' = L_2(z, f)(f')^2 + L_1(z, f)f' + L_0(z, f), \tag{1.2}$$

where $L_i(z, f)$ $(i = 0, 1, 2)$ are rational functions in their variables.

In 2011, Gao, Zhang and Li [13] studied the problem of growth order of solutions of a type of non-linear algebraic differential equations. In 2001, Liao and Yang [14] considered the finite order of growth of the meromorphic solutions of equation (1.2) and obtained the following result.

Theorem A Let f be a meromorphic solution of equation (1.2). Further assume that $L_2(z, f) \not\equiv 0$ in equation (1.2) and has the form

$$L_2(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{a_n(z)f^n + a_{n-1}(z)f^{n-1} + \cdots + a_s(z)f^s}{b_m(z)f^m + b_{m-1}(z)f^{m-1} + \cdots + b_r(z)f^r},$$

where $a_i(z), b_j(z)$ $(s \leq i \leq n, r \leq j \leq m)$ are rational functions. If $m - n < 1$ or $r - s > 1$, then $\rho(f) < \infty$.

Remark The conditions $m - n < 1$ and $r - s > 1$ in Theorem A cannot be omitted simultaneously. Liao and Yang [14] gave a simple example to show it.

The paper is organized into 3 sections. After introduction some basic concepts and lemmas will be given in Section 2. In Section 3, we will give the main results.

2 Preliminaries

Let D be a domain in $\mathbb{C}$. We say that a family $\mathcal{F}$ of meromorphic functions in D is normal, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges locally uniformly by spherical distance to a meromorphic function $g(z)$ in D ($g(z)$ is permitted to be identically infinity). In this paper, we denote the spherical derivative of meromorphic function $f(z)$ by $f^\sharp(z)$, where

$$f^\sharp(z) := \frac{|f'(z)|}{1 + |f(z)|^2}$$

and define

$$A(r, f) = \frac{1}{\pi} \int \int_{|z| \leq r} [f^\sharp(z)]^2 dx dy.$$

For convenience, we still assume that $L_2(z, f) = \frac{P(z,f)}{Q(z,f)}$ and rewrite equation (1.2) into

$$Q(z, f)f'' = P(z, f)(f')^2 + M(z, f)f' + N(z, f), \tag{2.1}$$

where $M(z, f) = Q(z, f)L_1(z, f)$, $N(z, f) = Q(z, f)L_0(z, f)$, P, Q are defined as in Theorem A.

Let $H(z) = \frac{p(z)}{q(z)}$ be a rational function, where $p(z)$ and $q(z)$ are irreducible polynomials in z. Define the degree at infinity of $H(z)$ by

$$\deg_{z,\infty}(H) = \deg p(z) - \deg q(z).$$

We denote the largest number of the degrees at infinity of all the rational function coefficients in variable z concerning $L(z, f)$ by $\deg_{z,\infty} L(z, f)$. Denoting

$$\deg_{z,\infty} a = \max\{\deg_{z,\infty} P(z, f), \deg_{z,\infty} Q(z, f), \deg_{z,\infty} M(z, f), \deg_{z,\infty} N(z, f), 0\},$$

where $P(z, f), Q(z, f)$ are two polynomials in f with rational function coefficients, $M(z, f)$ and $N(z, f)$ are rational functions in variable z and f.

The following lemmas will be needed in the proof of our results. Lemma 2.1 is a result of Zalcman concerning normal families.

Lemma 2.1 (see [15]) Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc, α is a real number. Then $\mathcal{F}$ is not normal on the unit disc if and only if there exist, for each $-1 < \alpha < 1$,

    a) *a number* r, $0 < r < 1$;

    b) *a sequence points* $\{w_k\}$, $|w_k| < r$;

    c) *a sequence* $\{f_k\}_{k \in N} \subset \mathcal{F}$;

    d) *a positive sequence* $\{\rho_k\}$, $\rho_k \to 0$

such that $g_k(\zeta) := \rho_k^\alpha f_k(w_k + \rho_k \zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$. In particular, we may choose w_k and ρ_k properly such that

$$\rho_k \leq \frac{2}{[f_k^\sharp(w_k)]^{\frac{1}{1+|\alpha|}}}, \; f_k^\sharp(w_k) \geq f_k^\sharp(0).$$

The next lemma is a generalization of the Lemma 2 in [16] of Yuan et al.

Lemma 2.2   Let $f(z)$ be meromorphic in the complex plane, $\rho := \rho(f) > 2$, then for any positive constants $\varepsilon > 0$ and $0 < \lambda < (\frac{\rho-2}{2})\varepsilon$, there exist points $z_k \to \infty (k \to \infty)$, such that

$$\lim_{k \to \infty} \frac{(f^\sharp(z_k))^\varepsilon}{|z_k|^\lambda} = +\infty.$$

Proof   Suppose that the conclusion of Lemma 2.2 is not true, then there exist a positive number $M > 0$, such that for arbitrary $z \in \mathbb{C}$, we have

$$(f^\sharp(z))^\varepsilon \leq M|z|^\lambda. \tag{2.2}$$

By (2.2) we can get

$$A(t, f) = \frac{1}{\pi} \int\int_{|z| \leq t} [f^\sharp(z)]^2 dxdy \leq \frac{M^{\frac{2}{\varepsilon}}}{\pi} \int\int_{|z| \leq t} |z|^{\frac{2\lambda}{\varepsilon}} dxdy = O(|t|^{2+\frac{2\lambda}{\varepsilon}}).$$

Thus we obtain an estimation of Ahlfors-Shimizu characteristic function

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t} dt \leq O(|r|^{2+\frac{2\lambda}{\varepsilon}}).$$

Therefore, the order of $f(z)$ can be estimated as $\rho \leq 2 + \frac{2\lambda}{\varepsilon}$, namely, $\lambda \geq (\frac{\rho-2}{2})\varepsilon$. This is a contradiction with the choice of λ.

Lemma 2.3 (see [17]) Let $f(z)$ be holomorphic in the complex plane, $\sigma > -1$. If $f^\sharp(z) = O(r^\sigma)$, then $T(r,f) = O(r^{\sigma+1})$.

The result of Lemma 2.4 is more sharper than Lemma 2.2 when $f(z)$ is an entire function.

Lemma 2.4 Let $f(z)$ be holomorphic in the complex plane, $\rho := \rho(f) > 1$, then for any positive constants $\varepsilon > 0$ and $0 < \lambda < (\rho-1)\varepsilon$, there exist points $z_k \to \infty$, as $k \to \infty$, such that

$$\lim_{k\to\infty} \frac{(f^\sharp(z_k))^\varepsilon}{|z_k|^\lambda} = +\infty.$$

Proof Suppose that the conclusion of Lemma 2.4 is not true, then there exist a positive number $M > 0$, such that for arbitrary $z \in \mathbb{C}$, we have $(f^\sharp(z))^\varepsilon \leq M|z|^\lambda$, namely, $f^\sharp(z) = O(r^{\frac{\lambda}{\varepsilon}})$. By Lemma 2.3, we have

$$T(r,f) = O(r^{\frac{\lambda}{\varepsilon}+1}).$$

Therefore the order of $f(z)$ can be estimated as $\rho \leq 1 + \frac{\lambda}{\varepsilon}$, namely, $\lambda \geq (\rho-1)\varepsilon$. This is a contradiction with the choice of λ.

3 Main Results

We are now giving our main results as follows.

Theorem 3.1 Let f be a meromorphic solution of equation (2.1). Further assume that $\frac{P(z,f)}{Q(z,f)} \not\equiv 0$ in equation (2.1), $M(z,f) \not\equiv 0, N(z,f) \not\equiv 0$ are birational functions and have following forms

$$M(z,f) = \frac{c_{q_1}(z)f^{q_1} + \cdots + c_{t_1}(z)f^{t_1}}{d_{q_2}(z)f^{q_2} + \cdots + d_{t_2}(z)f^{t_2}}, \quad N(z,f) = \frac{e_{q_3}(z)f^{q_3} + \cdots + e_{t_3}(z)f^{t_3}}{u_{q_4}(z)f^{q_4} + \cdots + u_{t_4}(z)f^{t_4}},$$

where $c_{j_1}(z)$ $(t_1 \leq j_1 \leq q_1), d_{j_2}(z)$ $(t_2 \leq j_2 \leq q_2), e_{j_3}(z)$ $(t_3 \leq j_3 \leq q_3)$ and $u_{j_4}(z)$ $(t_4 \leq j_4 \leq q_4)$ are rational functions, $c_{t_1}(z) \not\equiv 0$, $d_{t_2}(z) \not\equiv 0$, $e_{t_3}(z) \not\equiv 0$ and $u_{t_4}(z) \not\equiv 0$, then

$$\rho(f) \leq 2 + \frac{2(1+\alpha)\deg_{z,\infty} a}{\alpha},$$

where $0 < \alpha < \min\{\frac{1}{|q_1-q_2-n|}, \frac{2}{|q_3-q_4-n-1|}, 1\}$ when $m - n < 1$ and $0 < \alpha < \min\{\frac{1}{|1+s+t_2-t_1|}, \frac{2}{|2+s+t_4-t_3|}, 1\}$ when $r - s > 1$.

Proof We assume that f is a meromorphic solution of equation (2.1). Next we discuss into two cases.

Case 1 $m - n < 1$. We choose α such that $0 < \alpha < \min\{\frac{1}{|q_1-q_2-n|}, \frac{2}{|q_3-q_4-n-1|}, 1\}$ and assume that

$$\rho(f) > 2 + \frac{2(1+\alpha)\deg_{z,\infty} a}{\alpha}.$$

By Lemma 2.2 we know that for $\frac{\alpha}{1+\alpha}$ and $0 < \lambda < (\frac{\rho-2}{2})\frac{\alpha}{1+\alpha}$, there exist points $z_k \to \infty$, as $k \to \infty$, such that

$$\lim_{k\to\infty} \frac{(f^\sharp(z_k))^{\frac{\alpha}{1+\alpha}}}{|z_k|^\lambda} = +\infty. \tag{3.1}$$

This implies that the family $\{f(z_k + z)\}_{k\in\mathbb{N}}$ is not normal at $z = 0$. Then by Lemma 2.1, there exist a sequence $\{\beta_k\}$ and a positive sequence $\{\rho_k\}$ such that

$$|z_k - \beta_k| < 1, \quad \rho_k \to 0, \tag{3.2}$$

and $g_k(\zeta) := \rho_k^\alpha f(\beta_k + \rho_k\zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$. In particular, we may choose β_k and ρ_k, such that

$$\rho_k \leq \frac{2}{f^\sharp(\beta_k)^{\frac{1}{1+\alpha}}}, \quad f^\sharp(\beta_k) \geq f^\sharp(z_k). \tag{3.3}$$

According to (3.1), (3.2) and (3.3), we can get the following conclusion.

For positive constant α and any constant $0 \leq \lambda < (\frac{\rho-2}{2})\frac{\alpha}{1+\alpha}$, we have

$$\lim_{k\to\infty} \beta_k^\lambda \rho_k^\alpha = 0. \tag{3.4}$$

Substituting $\beta_k + \rho_k\zeta$ for z in (2.1), we have

$$Q(\beta_k + \rho_k\zeta, \frac{g_k(\zeta)}{\rho_k^\alpha})\frac{g_k''(\zeta)}{\rho_k^{2+\alpha}}$$

$$= P(\beta_k + \rho_k\zeta, \frac{g_k(\zeta)}{\rho_k^\alpha})\frac{(g_k'(\zeta))^2}{\rho_k^{2(1+\alpha)}} + M(\beta_k + \rho_k\zeta, \frac{g_k(\zeta)}{\rho_k^\alpha})\frac{g_k'(\zeta)}{\rho_k^{1+\alpha}} + N(\beta_k + \rho_k\zeta, \frac{g_k(\zeta)}{\rho_k^\alpha}).$$

Noting $0 \leq \deg_{z,\infty} a < (\frac{\rho-2}{2})\frac{\alpha}{1+\alpha}$, by (3.4), we have

$$b_m(\beta_k + \rho_k\zeta)g_k^m(\zeta)g_k''(\zeta)\frac{1}{\rho_k^{2+(m+1)\alpha}} + o(\frac{1}{\rho_k^{2+(m+1)\alpha}})$$

$$= a_n(\beta_k + \rho_k\zeta)g_k^n(\zeta)g_k'(\zeta)^2\frac{1}{\rho_k^{2(1+\alpha)+n\alpha}} + o(\frac{1}{\rho_k^{2(1+\alpha)+n\alpha}})$$

$$+ o(\frac{1}{\rho_k^{1+2\alpha+(q_1-q_2)\alpha}}) + o(\frac{1}{\rho_k^{\alpha+(q_3-q_4)\alpha}}). \tag{3.5}$$

Multiplying $\rho_k^{2(1+\alpha)+n\alpha}\frac{1}{a_n(\beta_k+\rho_k\zeta)}$ on both sides of (3.5), and noting $m - n < 1$, $1 + n\alpha + (q_2-q_1)\alpha > 0$, and $2(1+\alpha) + n\alpha + (q_4-q_3)\alpha - \alpha > 0$, we can conclude from this, by letting $k \to \infty$, $g^n g'^2 \equiv 0$. Thus g is a constant, which is a contradiction. Therefore we have

$$\rho(f) \leq 2 + \frac{2(1+\alpha)\deg_{z,\infty} a}{\alpha}.$$

Case 2 $r - s > 1$. We choose α such that $0 < \alpha < \min\{\frac{1}{|1+s+t_2-t_1|}, \frac{2}{|2+s+t_4-t_3|}, 1\}$ and assume that

$$\rho(f) > 2 + \frac{2(1+\alpha)\deg_{z,\infty} a}{\alpha}.$$

Then there exist a sequence $\{\beta_k\}$ and a positive sequence $\{\rho_k\}$ satisfying

$$|z_k - \beta_k| < 1, \quad \rho_k \to 0$$

such that $h_k(\zeta) = \rho_k^{-\alpha} f(\beta_k + \rho_k \zeta)$ converges locally uniformly to a nonconstant meromorphic function $h(\zeta)$. By similar argument as in Case 1, we can obtain

$$h(\zeta)^s h'^2(\zeta) \equiv 0.$$

Hence h is a constant, which is a contradiction. Thus we have completed the proof of Theorem 3.1.

Similarly, from the proof of Theorem 3.1 and Lemma 2.4, we have

Corollary 3.2 Let f be an entire solution of equation (2.1). Further assume that $\frac{P(z,f)}{Q(z,f)} \not\equiv 0$ in equation (2.1), $M(z,f) \not\equiv 0, N(z,f) \not\equiv 0$ are birational functions and have the forms

$$M(z,f) = \frac{c_{q_1}(z)f^{q_1} + \cdots + c_{t_1}(z)f^{t_1}}{d_{q_2}(z)f^{q_2} + \cdots + d_{t_2}(z)f^{t_2}}, \quad N(z,f) = \frac{e_{q_3}(z)f^{q_3} + \cdots + e_{t_3}(z)f^{t_3}}{u_{q_4}(z)f^{q_4} + \cdots + u_{t_4}(z)f^{t_4}},$$

where $c_{j_1}(z)$ $(t_1 \leq j_1 \leq q_1), d_{j_2}(z)$ $(t_2 \leq j_2 \leq q_2), e_{j_3}(z)$ $(t_3 \leq j_3 \leq q_3)$ and $u_{j_4}(z)$ $(t_4 \leq j_4 \leq q_4)$ are rational functions, $c_{t_1}(z) \not\equiv 0, d_{t_2}(z) \not\equiv 0, e_{t_3}(z) \not\equiv 0, u_{t_4}(z) \not\equiv 0$, then

$$\rho(f) \leq 1 + \frac{1+\alpha}{\alpha} \deg_{z,\infty} a,$$

where $0 < \alpha < \min\left\{\frac{1}{|q_1 - q_2 - n|}, \frac{2}{|q_3 - q_4 - n - 1|}, 1\right\}$ when $m - n < 1$ and $0 < \alpha < \min\left\{\frac{1}{|1+s+t_2-t_1|}, \frac{2}{|2+s+t_4-t_3|}, 1\right\}$ when $r - s > 1$.

Remark In Theorem 3.1 and Corollary 3.2, if $m-n < 1$, $M(z,f) \equiv 0$ and $N(z,f) \not\equiv 0$, then for arbitrary $0 < \alpha < \min\{\frac{2}{|q_3-q_4-n-1|}, 1\}$, the results of Theorem 3.1 and Corollary 3.2 are also true. Similarly, if $m - n < 1, M(z,f) \not\equiv 0$ and $N(z,f) \equiv 0$, then we may choose any $0 < \alpha < \min\{\frac{1}{|q_1-q_2-n|}, 1\}$. If $r - s > 1, M(z,f) \equiv 0$ and $N(z,f) \not\equiv 0$, then we may choose any $0 < \alpha < \min\{\frac{2}{|2+s+t_4-t_3|}, 1\}$. If $r - s > 1, M(z,f) \not\equiv 0$ and $N(z,f) \equiv 0$, then we may choose any $0 < \alpha < \min\{\frac{1}{|1+s+t_2-t_1|}, 1\}$. If $M(z,f) = N(z,f) \equiv 0$, $m - n < 1$ or $r - s > 1$, then we may choose any $0 < \alpha < 1$.

Example There exists the entire function $f(z) = e^{z^n}$ $(n \geq 1)$ such that it is of order n and satisfies the following second-order differential equation

$$f'' = \frac{f^2 + f - 1}{f^2}(f')^2 - nz^{n-1}ff' + n(n-1)z^{n-2}f + n^2 z^{2(n-1)}, \tag{3.6}$$

where $\deg_{z,\infty} a = 2(n-1)$ and $0 < \alpha < 1$, then the order of any meromorphic solution f of equation (3.6) can be estimated as $\rho(f) \leq 2 + \frac{4(1+\alpha)(n-1)}{\alpha}$ and the order of any entire solution f of equation (3.6) can be estimated as $\rho(f) \leq 1 + \frac{2(1+\alpha)(n-1)}{\alpha}$ by Theorem 3.1 and Corollary 3.2, respectively. In particular, the estimation of growth order of entire solution is sharp when $n = 1$.

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二阶代数微分方程亚纯解的增长性估计

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摘要: 本文研究了代数微分方程亚纯解的增长级. 运用正规族理论, 给出了某类二阶代数微分方程亚纯解的增长级的一个估计, 该估计依赖于方程的有理函数系数. 推广了2001年廖良文与杨重骏的一个结果.

关键词: 代数微分方程; 亚纯解; 增长级

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