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ON A NEW NONTRIVIAL ELEMENT INVOLVING THE THIRD PERIODICITY γ-FAMILY IN $\pi_* S$

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Abstract: In this paper, we discuss stable homotopy groups of spheres. By making a nontrivial secondary differential as geometric input in the Adams spectral sequence, the convergence of $h_0 g_n$ $(n > 3)$ in $\pi_* S$ is given. Furthermore, by the knowledge of Yoneda products, a new nontrivial element in $\pi_* S$ is detected. The scale of the nontrivial elements is expanded by our results.

Keywords: stable homotopy groups of spheres; Toda-Smith spectrum; Adams spectral sequence; May spectral sequence; Adams differential

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1 Introduction

Let S denote the sphere spectrum localized at p and p denote an odd prime. From [14], the homotopy group of n-dimensional sphere $\pi_{n+r} S^n$ $(r > 0)$ is a finite group. So the determination of $\pi_{n+r} S^n$ has become one of the central problems in algebraic topology.

Ever since the introduction of the Adams spectral sequence (ASS) in the late 1950's (see [1]), the study of the homotopy groups of spheres $\pi_* S$ was split into algebraic and geometric problems, including the computation of $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ and the detection which element of $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ can survive to $E_\infty^{*,*}$, here A is the mod p Steenrod algebra, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the E_2-term of the ASS. By [2],

$$E_2^{s,t} \cong \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s} S,$$

and the Adams differential is $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$.

In addition, we also have the Adams-Novikov spectral sequence (ANSS) [12, 13] based on the Brown-Peterson spectrum BP in the determination of $\pi_* S$.

Many wonderful results were obtained, however, it is still far from the total determination of $\pi_* S$. After the detection of $\eta_j \in \pi_{p^j q + pq - 2} S$ for $p = 2$, $j \neq 2$, by Mahowald in [11],

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which was represented by $h_1h_j \in \mathrm{Ext}_A^{2,p^jq+pq}(\mathbb{Z}_p,\mathbb{Z}_p)$, many nontrivial elements in π_*S were found. Please see references [5–9] for details. In recent years, the first author established several convergence of elements by an arithmatic method, see [16–18, 21].

In [5], Cohen made the nontrivial secondary Adams differential $d_2(h_i) = a_0b_{i-1}$ $(p > 2, i > 0)$ as geometric input, then, a nontrivial element $\xi_i \in \pi_{(p^{i+1}+1)q}S$ $(i \geqslant 0)$ is detected. In this paper, we also detect a new family in π_*S by geometric method, the only geometric input used in the proof is the secondary nontrivial differential given in [20].

The main result is obtained as follows.

Theorem 1.1 Let $3 \leqslant s < p-1$, $n > 3$, $p \geqslant 7$, then

$$0 \neq \widetilde{\gamma}_s h_0 g_n \in \mathrm{Ext}_A^{s+3,p^{n+1}q+2p^nq+sp^2q+(s-1)pq+(s-1)q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element of order p in $\pi_{p^{n+1}q+2p^nq+sp^2q+(s-1)pq+(s-1)q-6}S$.

The paper is organized as follows. After giving some necessary preliminaries and useful knowledge about the MSS in Section 2. The proof of Theorem 1.1 and some results on Ext groups will be given in Section 3.

2 Related Spectrum and the May Spectral Sequence

For the convenience of the reader, let us briefly indicate the necessary preliminaries in the proof of the propositions and theorems.

Let M be the Moore spectrum modulo an odd prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S\ .$$

Let α: $\sum^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} V(1) \xrightarrow{j'} \Sigma^{q+1}M.$$

Let β: $\sum^{(p+1)q} V(1) \to V(1)$ be the ν_2-mapping and $V(2)$ be the cofibre of β sitting in the cofibration

$$\Sigma^{(p+1)q}V(1) \xrightarrow{\beta} V(1) \xrightarrow{i''} V(2) \xrightarrow{j''} \Sigma^{(p+1)q+1}V(1).$$

Furthermore, γ: $\sum^{(p^2+p+1)q} V(2) \to V(2)$ is the ν_3-mapping and the γ-element $\gamma_s = jj'j''\gamma^s i''i'i$ is a nontrivial element in $\pi_{sp^2q+(s-1)pq+(s-2)q-3}S$, where $p \geqslant 7$ (see [15]).

From [19], we know that the third periodicity family γ_s is represented by the third Greek letter family element

$$\widetilde{\gamma}_s \in \mathrm{Ext}_A^{s,sp^2q+(s-1)pq+(s-2)q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS, which is represented by the element

$$s(s-1)(s-2)a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$$

in the May spectral sequence (MSS).

Let L be the cofibre of $\alpha_1 = j\alpha i$: $\sum^{q-1} S \to S$ given by the cofibration

$$\Sigma^{q-1}S \xrightarrow{\alpha_1} S \xrightarrow{\bar{i}} L \xrightarrow{\bar{j}} \Sigma^q S.$$

From [10], we can see that $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases

$$a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p),\ h_i \in \mathrm{Ext}_A^{1,p^i q}(\mathbb{Z}_p, \mathbb{Z}_p)(i \geqslant 0).$$

$\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\ \mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases

$$\alpha_2,\ a_0^2,\ a_0 h_i\ (i > 0),\ g_i\ (i \geqslant 0),\ k_i\ (i \geqslant 0),\ b_i\ (i \geqslant 0),\ \text{and}\ h_i h_j\ (j \geqslant i+2, i \geqslant 0),$$

whose internal degrees are

$$2q+1,\ 2,\ p^i q+1,\ 2p^i q + p^{i+1} q,\ 2p^{i+1} + p^i q,\ p^{i+1} q \text{ and } p^i q + p^j q,$$

respectively. Aikawa computed $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ by λ-algebra in [3].

In the following, recall the Adams resolution of some spectra related to S from [4]. Let

$$\begin{array}{ccccccc} \cdots \xrightarrow{\bar{\alpha}_2} & \Sigma^{-2}E_2 & \xrightarrow{\bar{\alpha}_1} & \Sigma^{-1}E_1 & \xrightarrow{\bar{\alpha}_0} & E_0 = S \\ & \downarrow{\scriptstyle \bar{b}_2} & & \downarrow{\scriptstyle \bar{b}_1} & & \downarrow{\scriptstyle \bar{b}_0} \\ & \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0 \end{array} \tag{2.1}$$

be the minimal Adams resolution of the sphere spectrum S which satisfies

(A) $E_s \xrightarrow{\bar{b}_s} KG_s \xrightarrow{\bar{c}_s} E_{s+1} \xrightarrow{\bar{a}_s} \Sigma E_s$ are cofibrations for all $s \geqslant 0$, which induce short exact sequences in $\mathbb{Z}_p$-cohomology

$$0 \longrightarrow H^* E_{s+1} \xrightarrow{\bar{c}_s^*} H^* KG_s \xrightarrow{\bar{b}_s^*} H^* E_s \longrightarrow 0.$$

(B) KG_s are the graded wedge sums of Eilenberg-Maclane spectrum $K\mathbb{Z}_p$ of type $\mathbb{Z}_p$.

(C) $\pi_t KG_s$ are the $E_1^{s,t}$-terms of the ASS,

$$(\bar{b}_s \bar{c}_{s-1})_* :\ \pi_t KG_{s-1} \longrightarrow \pi_t KG_s$$

are the $d_1^{s-1,t}$-differentials of the ASS, and $\pi_t KG_s \cong \mathrm{Ext}_{\mathrm{A}}^{\mathrm{s,t}}(\mathrm{\mathbb{Z}_p}, \mathrm{\mathbb{Z}_p})$. Then, an Adams resolution of an arbitrary spectrum V can be obtained by smashing V to (2.1).

Remark 2.1 In the ANSS, h_0 is a permanent cycle and converges to the corresponding homotopy element $i' i \alpha_1 (\alpha_1 = j\alpha i \in \pi_{q-1} S)$ in $\pi_{q-1} K$. Furthermore, if some suppositions on Ext groups are given, then there exists $\overline{w} \in \pi_{p^{n+1}q+2p^n q-2} K$ such that $i' i \xi = \alpha'' \cdot \overline{w}$ (mod $F^4 \pi_* K$) and $\overline{w}$ is represented by $(i' i)_*(g_n) \in \mathrm{Ext}_A^{2,p^{n+1}q+2p^n q}(H^*K, \mathbb{Z}_p)$ in the ASS, where $\xi \in \pi_{p^{n+1}q+2p^n q-4} S$ is the homotopy element which is represented by $h_0 l_n \in \mathrm{Ext}_A^{4,p^{n+1}q+2p^n q+q}$ $(\mathbb{Z}_p, \mathbb{Z}_p)$ in the ASS and $F^4 \pi_* K$ denotes the group consisting of all elements in $\pi_* K$ with filtration no less than 4.

To detect π_*S with the ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$. The most successful method for computing it is the MSS.

From [13], there is a MSS $\{E_r^{s,t,*}, d_r\}$, which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{i,j} \mid i > 0, j \geqslant 0) \otimes P(b_{i,j} \mid i > 0, j \geqslant 0) \otimes P(a_i \mid i \geqslant 0), \tag{2.2}$$

where $E(\ )$ denotes the exterior algebra, $P(\ )$ denotes the polynomial algebra, and

$$h_{i,j} \in E_1^{1,2(p^i-1)p^j,2i-1},\ b_{i,j} \in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)},\ a_i \in E_1^{1,2p^i-1,2i+1}.$$

One has d_r: $E_r^{s,t,M} \to E_r^{s+1,t,M-r}$ $(r \geqslant 1)$. If $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x)y + (-1)^s x d_r(y). \tag{2.3}$$

Furthermore, the May E_1-term is graded commutative in the sense that

$$\begin{aligned} a_m h_{n,j} &= h_{n,j} a_m, & h_{m,k} h_{n,j} &= -h_{n,j} h_{m,k}, \\ a_m b_{n,j} &= b_{n,j} a_m, & h_{m,k} b_{n,j} &= b_{n,j} h_{m,k}, \\ a_m a_n &= a_n a_m, & b_{m,n} b_{i,j} &= b_{i,j} b_{m,n}. \end{aligned}$$

The first May differential d_1 is given by

$$\begin{cases} d_1(h_{i,j}) = -\sum\limits_{0<k<i} h_{i-k,k+j} h_{k,j}, \\ d_1(a_i) = -\sum\limits_{0<k<i} h_{i-k,k}, a_k, \\ d_1(b_{i,j}) = 0. \end{cases} \tag{2.4}$$

For each element $x \in E_1^{s,t,*}$, if we denote dim $x = s$, deg $x = t$, we have

$$\begin{cases} \dim h_{i,j} = \dim a_i = 1,\ \ \dim b_{i,j} = 2, \\ \deg h_{i,j} = 2(p^i-1)p^j = (p^{i+j-1} + \cdots + p^j)q, \\ \deg b_{i,j} = 2(p^i-1)p^{j+1} = (p^{i+j} + \cdots + p^{j+1})q, \\ \deg a_i = 2p^i - 1 = (p^{i-1} + \cdots + 1)q + 1, \\ \deg a_0 = 1. \end{cases} \tag{2.5}$$

Remark 2.2 Any positive integer t can be expressed uniquely as $t = q(c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0) + e$, where $0 \leqslant c_i < p$ $(0 \leqslant i < n)$, $0 < c_n < p$, $0 \leqslant e < q$.
Then, it is easy to get the following result from [16].

Proposition 2.3 In the MSS, we have $E_1^{s,t,*} = 0$ for some j $(0 \leqslant j \leqslant n)$, $s < c_j$, where s is also a positive integer with $0 < s < p$.

3 Some Adams E_2-Terms

In this section, we mainly give some important results about Adams E_2-terms. At the end, the proof of Theorem 1.1 will be given.

Proposition 3.1 Let $3 \leqslant s < p-1,\ n > 3,\ p \geqslant 7$, then

$$0 \neq \widetilde{\gamma}_s h_0 g_n \in \mathrm{Ext}_A^{s+3,p^{n+1}q+2p^nq+sp^2q+(s-1)pq+(s-1)q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Proof Consider the structure of $E_1^{s+2,t,*}$ in the MSS, where $t = p^{n+1}q + 2p^nq + sp^2q + (s-1)pq + (s-1)q + s - 3$. Due to $3 \leqslant s < p-1$, then $5 \leqslant s+2 < p+1$.

Case 1 $5 \leqslant s+2 < p$. Let $h = x_1x_2\cdots x_m$ be the generator of $E_1^{s+2,t,*}$, where x_i is one of a_k, $h_{i,j}$ or $b_{u,z}$, $0 \leqslant k \leqslant n+2$, $0 < i+j \leqslant n+2$, $0 < u+z \leqslant n+1$, $i>0$, $j \geqslant 0$, $u > 0$, $z \geqslant 0$.

Assume that $\deg x_i = q(c_{i,n+1}p^{n+1} + \cdots + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_k$ or $e_i = 0$, then

$$\begin{aligned}\deg h &= \sum_{i=1}^m \deg x_i = q((\sum_{i=1}^m c_{i,n+1})p^{n+1} + (\sum_{i=1}^m c_{i,n})p^n + \cdots + (\sum_{i=1}^m c_{i,0})) + (\sum_{i=1}^m e_i)\\ &= q(p^{n+1} + 2p^n + sp^2 + (s-1)p + (s-1)) + s - 3,\\ \dim h &= \sum_{i=1}^m \dim x_i = s+2.\end{aligned}$$

Note that $\dim x_i = 1$ or 2, we can see that $m \leqslant s+2 < p$ from $\sum\limits_{i=1}^m \dim x_i = s+2$. By the fact that $c_{i,j} = 0$ or 1, $e_i = 0$ or 1, $m \leqslant s+2 < p$, we have

$$\begin{aligned}&\sum_{i=1}^m e_i = s-3, \sum_{i=1}^m c_{i,0} = s-1, \sum_{i=1}^m c_{i,1} = s-1, \sum_{i=1}^m c_{i,2} = s,\\ &\sum_{i=1}^m c_{i,3} = \cdots = \sum_{i=1}^m c_{i,n-1} = 0, \sum_{i=1}^m c_{i,n} = 2, \sum_{i=1}^m c_{i,n+1} = 1.\end{aligned}$$

From the above results, we can see that $b_{1,n}b_{1,n-1}h_{1,n}$, $h_{2,n}h_{1,n}$, $h_{2,n}b_{1,n-1}$, $b_{2,n-1}h_{1,n}$, $b_{1,n-1}b_{2,n-1}$, $b_{1,n}b_{1,n-1}^2$, $h_{1,n+1}b_{1,n-1}^2$ and $h_{1,n+1}b_{1,n-1}h_{1,n}$ are contained in the x_i. By the commutativity of $E_1^{*,*,*}$, we can denote

$$\begin{aligned}
h_1 &= x_1x_2\cdots x_{m-3}b_{1,n}h_{1,n}b_{1,n-1}, & h_1' &= x_1x_2\cdots x_{m-3} \in E_1^{s-3,t',*};\\
h_2 &= x_1x_2\cdots x_{m-3}b_{1,n}b_{1,n-1}^2, & h_2' &= x_1x_2\cdots x_{m-3} \in E_1^{s-4,t',*};\\
h_3 &= x_1x_2\cdots x_{m-3}h_{1,n+1}b_{1,n-1}^2, & h_3' &= x_1x_2\cdots x_{m-3} \in E_1^{s-3,t',*};\\
h_4 &= x_1x_2\cdots x_{m-2}h_{2,n}h_{1,n}, & h_4' &= x_1x_2\cdots x_{m-2} \in E_1^{s,t',*};\\
h_5 &= x_1x_2\cdots x_{m-2}h_{2,n}b_{1,n-1}, & h_5' &= x_1x_2\cdots x_{m-2} \in E_1^{s-1,t',*};\\
h_6 &= x_1x_2\cdots x_{m-2}b_{2,n-1}h_{1,n}, & h_6' &= x_1x_2\cdots x_{m-2} \in E_1^{s-1,t',*};\\
h_7 &= x_1x_2\cdots x_{m-2}b_{2,n-1}b_{1,n-1}, & h_7' &= x_1x_2\cdots x_{m-2} \in E_1^{s-2,t',*};\\
h_8 &= x_1x_2\cdots x_{m-3}h_{1,n+1}h_{1,n}b_{1,n-1}, & h_8' &= x_1x_2\cdots x_{m-3} \in E_1^{s-2,t',*},
\end{aligned}$$

where $t' = sp^2q + (s-1)pq + (s-1)q + s - 3$.

We list all the possibilities of h_i' in the following table $(i = 1,2,\cdots,8)$, thus h doesn't exist in this case.

Table 1: the possibilities of h'_i

The possibility	Analysis	The existence of h'_i
h'_1	$s-3<\sum_{i=1}^{m-3} c_{i,2}=s$	Nonexistent
h'_2	$s-4<\sum_{i=1}^{m-3} c_{i,2}=s$	Nonexistent
h'_3	$s-3<\sum_{i=1}^{m-3} c_{i,2}=s$	Nonexistent
h'_4	$h'_4=a_3^{s-3}h_{2,0}^2h_{1,0}=0$	Nonexistent
h'_5	$s-1<\sum_{i=1}^{m-2} c_{i,2}=s$	Nonexistent
h'_6	$s-1<\sum_{i=1}^{m-2} c_{i,2}=s$	Nonexistent
h'_7	$s-2<\sum_{i=1}^{m-2} c_{i,2}=s$	Nonexistent
h'_8	$s-2<\sum_{i=1}^{m-3} c_{i,2}=s$	Nonexistent

Case 2 If $s+2=p$, then $E_1^{s+2,t'',*}=E_1^{p,t'',*}$, where $t''=p^{n+1}q+2p^nq+(p-2)p^2q+(p-3)pq+(p-3)q+p-5$. Let $h=x_1x_2\cdots x_r$ be the generator of $E_1^{p,t'',*}$, and assume that

$$\deg x_i=q(c_{i,n+1}p^{n+1}+c_{i,n}p^n+\cdots+c_{i,1}p+c_{i,0})+e_i,$$

where $c_{i,j}=0$ or 1, $e_i=1$ if $x_i=a_{k_i}$ or $e_i=0$, then

$$\begin{aligned}\deg h&=\sum_{i=1}^{r}\deg x_i=q((\sum_{i=1}^{r}c_{i,n+1})p^{n+1}+(\sum_{i=1}^{r}c_{i,n})p^n+\cdots+(\sum_{i=1}^{r}c_{i,0}))+(\sum_{i=1}^{r}e_i)\\&=q(p^{n+1}+2p^n+(p-2)p^2+(p-3)p+(p-3))+p-5,\\ \dim h&=\sum_{i=1}^{r}\dim x_i=p.\end{aligned}$$

We claim that $\sum_{i=1}^{r}c_{i,0}$, $\sum_{i=1}^{r}c_{i,1}$ and $\sum_{i=1}^{r}c_{i,2}$ are impossible to constitute p. The reason is the following: if $\sum_{i=1}^{r}c_{i,0}=p$, because of $\sum_{i=1}^{r}e_i=p-5$, then

$$q\Big((\sum_{i=1}^{r}c_{i,n+1})p^{n+1}+(\sum_{i=1}^{r}c_{i,n})p^n+\cdots+(\sum_{i=1}^{r}c_{i,0})\Big)+(\sum_{i=1}^{r}e_i)=\sum_{i=1}^{r}e_i\ (\mathrm{mod}\ p),$$

this contradicts to $q(p^{n+1}+2p^n+(p-2)p^2+(p-3)p+(p-3))+p-5=(p-3)q+p-5\ (\mathrm{mod}\ p)$. For the same reason, $\sum_{i=1}^{r}c_{i,1}$ and $\sum_{i=1}^{r}c_{i,2}$ are impossible to constitute p.

From $\dim x_i=1$ or 2 and $\sum_{i=1}^{r}\dim x_i=p$, we can see that $r\leqslant p$. By Remark 2.2 and

$r \leqslant p$, $c_{i,j} = 0$ or 1, $e_i = 0$ or 1, we have

$$\sum_{i=1}^{r} e_i = p-5,\ \sum_{i=1}^{r} c_{i,0} = p-3,\ \sum_{i=1}^{r} c_{i,1} = p-3,\ \sum_{i=1}^{r} c_{i,2} = p-2,$$
$$(\sum_{i=1}^{r} c_{i,3})p^3 + \cdots + (\sum_{i=1}^{r} c_{i,n})p^n + (\sum_{i=1}^{r} c_{i,n+1})p^{n+1} = p^{n+1} + 2p^n, \tag{3.1}$$

so

$$(\sum_{i=1}^{r} c_{i,3}) + \cdots + (\sum_{i=1}^{r} c_{i,n})p^{n-3} + (\sum_{i=1}^{r} c_{i,n+1})p^{n-2} = p^{n-2} + 2p^{n-3}. \tag{3.2}$$

Thus $p \mid \sum\limits_{i=1}^{r} c_{i,3}$. Note that $c_{i,3} = 0$ or 1, $r \leqslant p$, it is known that $\sum\limits_{i=1}^{r} c_{i,3} = 0$ or p.

Case 2.1 When $\sum\limits_{i=1}^{r} c_{i,3} = 0$, we have

$$(\sum_{i=1}^{r} c_{i,4})p + \cdots + (\sum_{i=1}^{r} c_{i,n})p^{n-3} + (\sum_{i=1}^{r} c_{i,n+1})p^{n-2} = p^{n-2} + 2p^{n-3}.$$

Case 2.1.1 When $n > 4$, we claim that $\sum\limits_{i=1}^{r} c_{i,4} = 0$. Otherwise, if $\sum\limits_{i=1}^{r} c_{i,4} = p$, then $r = p$. So $\dim x_i = 1 (1 \leqslant i \leqslant p)$ and $\deg x_i =$ (higher terms) $+ p^4 q +$ (lower terms). Because of $\sum\limits_{i=1}^{r} e_i = p-5$, $\deg a_k \equiv 1 (\mathrm{mod}\ q)$, $\dim h_{l,j} \equiv 0 (\mathrm{mod}\ q)$ and $\dim b_{u,z} \equiv 0 (\mathrm{mod}\ q)$, there exist factors $a_{j_1} a_{j_2} \cdots a_{j_{p-5}}$ among the generators x_i $(j_i \geqslant 5, 1 \leqslant i \leqslant p-5)$. Thus, $\sum\limits_{i=1}^{r} c_{i,3} \geqslant p-5$, which contradicts to $\sum\limits_{i=1}^{r} c_{i,3} = 0$, so $\sum\limits_{i=1}^{r} c_{i,4} = 0$. By induction on j, we can get $\sum\limits_{i=1}^{r} c_{i,j} = 0$ $(5 \leqslant j \leqslant n-1)$, $\sum\limits_{i=1}^{r} c_{i,n} = 2$, $\sum\limits_{i=1}^{r} c_{i,n+1} = 1$.

Case 2.1.2 When $n = 4$, it is easy to get $\sum\limits_{i=1}^{r} c_{i,4} = 2$ and $\sum\limits_{i=1}^{r} c_{i,5} = 1$.

From the above discussion of Case 2.1.1 and Case 2.1.2, similarly to Case 1, we can see that $b_{1,n}b_{1,n-1}h_{1,n}$, $h_{2,n}h_{1,n}$, $h_{2,n}b_{1,n-1}$, $b_{2,n-1}h_{1,n}$, $b_{1,n-1}b_{2,n-1}$, $b_{1,n}b_{1,n-1}^2$, $h_{1,n}b_{1,n-1}h_{1,n+1}$ and $h_{1,n+1}b_{1,n-1}^2$ are contained in the x_i, so h is impossible to exist.

Case 2.2 When $\sum\limits_{i=1}^{r} c_{i,3} = p$, then $r = p$. We get $\dim x_i = 1$ from $\dim h = p$, then $h = x_1 x_2 \cdots x_p$, $x_i \in E(h_{i,j} \mid i > 0, j \geqslant 0) \otimes P(a_k \mid k \geqslant 0)$.

Case 2.2.1 When $n > 4$, we get

$$p \cdot p^3 + (\sum_{i=1}^{r} c_{i,4})p^4 + \cdots + (\sum_{i=1}^{r} c_{i,n})p^n + (\sum_{i=1}^{r} c_{i,n+1})p^{n+1} = p^{n+1} + 2p^n,$$

that is $(1 + \sum\limits_{i=1}^{r} c_{i,4}) + (\sum\limits_{i=1}^{r} c_{i,5})p + \cdots + (\sum\limits_{i=1}^{r} c_{i,n+1})p^{n-3} = p^{n-3} + 2p^{n-4}$, thus $p \mid (1 + \sum\limits_{i=1}^{r} c_{i,4})$, so $\sum\limits_{i=1}^{r} c_{i,4} = p-1$ from $c_{i,4} = 0$ or 1 and $r = p$. By induction on j, we can get

$$\sum_{i=1}^{r} c_{i,j} = p-1 (4 \leqslant j \leqslant n-1),\ \sum_{i=1}^{r} c_{i,n} = 1,\ \sum_{i=1}^{r} c_{i,n+1} = 1.$$

By the reason of degree and the Proposition 2.3, h is impossible to exist.

Case 2.2.2 When $n=4$, we know that $\sum\limits_{i=1}^{r} c_{i,4}=1$, $\sum\limits_{i=1}^{r} c_{i,5}=1$ from (3.2), then

$$\deg h = q(p^5+2p^4+(p-2)p^2+(p-3)p+p-3)+(p-5).$$

By the reason of degree and the Proposition 2.3, h is impossible to exist.

From the above discussion, for $5\leqslant s+2<p+1$, $E_1^{s+2,t,*}=0$, so $E_r^{s+2,t,*}=0$ $(r\geqslant 2)$. It is known that $h_{2,n}h_{1,n}$, $h_{1,n}, a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}\in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to g_n, h_n, $\widetilde{\gamma}_s\in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ $(n\geqslant 0)$, respectively, so $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,n}h_{2,n}h_{1,0}\in E_1^{s+3,t,*}$ is a permanent cycle in the MSS and converges nontrivially to $\widetilde{\gamma}_s h_0 g_n\in \mathrm{Ext}_A^{s+3,t}(\mathbb{Z}_p,\mathbb{Z}_p)$. Note that $E_r^{s+2,t,*}=0$ $(r\geqslant 1)$, thus the permanent cycle is not d_r-boundary and converges nontrivially to $\widetilde{\gamma}_s h_0 g_n\in \mathrm{Ext}_A^{s+3,t}(\mathbb{Z}_p,\mathbb{Z}_p)$, that is, when $5\leqslant s+2<p+1$, $0\neq \widetilde{\gamma}_s h_0 g_n\in \mathrm{Ext}_A^{s+3,t}(\mathbb{Z}_p,\mathbb{Z}_p)$.

Proposition 3.2 Let $3\leqslant s<p-1$, $n>3$, $p\geqslant 7$, $2\leqslant r<s+3$, then

$$\mathrm{Ext}_A^{s+3-r,p^{n+1}q+2p^nq+sp^2q+(s-1)pq+(s-1)q+s-r-2}(\mathbb{Z}_p,\mathbb{Z}_p)=0.$$

Proof We only need to prove that $E_1^{s+3-r,t,*}=0$ in the MSS, where $t=p^{n+1}q+2p^nq+sp^2q+(s-1)pq+(s-1)q+s-r-2$. Let $h=x_1x_2\cdots x_m$ be the generator of $E_1^{s+3-r,t,*}$, where x_i is a_k, $h_{i,j}$ or $b_{u,z}$, $0\leqslant k\leqslant n+2$, $0<i+j\leqslant n+2$, $0<u+z\leqslant n+1$, $i>0$, $j\geqslant 0$, $u>0$, $z\geqslant 0$.

Assume that $\deg x_i=q(c_{i,n+1}p^{n+1}+c_{i,n}p^n+\cdots+c_{i,0})+e_i$, where $c_{i,j}=0$ or 1, $e_i=1$ if $x_i=a_{k_i}$ or $e_i=0$, then

$$\begin{aligned}\deg h&=\sum_{i=1}^{m}\deg x_i=q((\sum_{i=1}^{m}c_{i,n+1})p^{n+1}+(\sum_{i=1}^{m}c_{i,n})p^n+\cdots+(\sum_{i=1}^{m}c_{i,0}))+(\sum_{i=1}^{m}e_i)\\&=q(p^{n+1}+2p^n+sp^2+(s-1)p+(s-1))+s-r-2,\\ \dim h&=\sum_{i=1}^{m}\dim x_i=s+3-r.\end{aligned}$$

Note that $\dim x_i=1$ or 2, we can see that $m\leqslant s+3-r\leqslant s+1<p$. We claim that $s-r-2\geqslant 0$, otherwise, $p>\sum\limits_{i=1}^{m}e_i=q+(s-r-2)\geqslant q-5>p$. Because of $c_{i,j}=0$ or 1, $e_i=0$ or 1 and $r<p$, we have

$$\begin{aligned}&\sum_{i=1}^{m}e_i=s-r-2,\sum_{i=1}^{m}c_{i,0}=s-1,\quad \sum_{i=1}^{m}c_{i,1}=s-1,\quad \sum_{i=1}^{m}c_{i,2}=s,\\&\sum_{i=1}^{m}c_{i,3}=\cdots=\sum_{i=1}^{m}c_{i,n-1}=0,\qquad \sum_{i=1}^{m}c_{i,n}=2,\qquad \sum_{i=1}^{m}c_{i,n+1}=1.\end{aligned}$$

From the above results, we can see that $b_{1,n}b_{1,n-1}h_{1,n}$, $h_{2,n}h_{1,n}$, $h_{2,n}b_{1,n-1}$, $b_{2,n-1}h_{1,n}$, $b_{1,n-1}b_{2,n-1}$, $b_{1,n}b_{1,n-1}^2$, $h_{1,n}b_{1,n-1}h_{1,n+1}$ and $h_{1,n+1}b_{1,n-1}^2$ are contained in the x_i. By the commutativity of $E_1^{*,*,*}$, we can denote

$$\begin{aligned}
&h_1 = x_1x_2\cdots x_{m-3}b_{1,n}h_{1,n}b_{1,n-1}, && h_1^{'} = x_1x_2\cdots x_{m-3} \in E_1^{s-2-r,t(r),*};\\
&h_2 = x_1x_2\cdots x_{m-3}b_{1,n}b_{1,n-1}^2, && h_2^{'} = x_1x_2\cdots x_{m-3} \in E_1^{s-3-r,t(r),*};\\
&h_3 = x_1x_2\cdots x_{m-3}h_{1,n+1}b_{1,n-1}^2, && h_3^{'} = x_1x_2\cdots x_{m-3} \in E_1^{s-2-r,t(r),*};\\
&h_4 = x_1x_2\cdots x_{m-2}h_{2,n}h_{1,n}, && h_4^{'} = x_1x_2\cdots x_{m-2} \in E_1^{s+1-r,t(r),*};\\
&h_5 = x_1x_2\cdots x_{m-2}h_{2,n}b_{1,n-1}, && h_5^{'} = x_1x_2\cdots x_{m-2} \in E_1^{s-r,t(r),*};\\
&h_6 = x_1x_2\cdots x_{m-2}b_{2,n-1}h_{1,n}, && h_6^{'} = x_1x_2\cdots x_{m-2} \in E_1^{s-r,t(r),*};\\
&h_7 = x_1x_2\cdots x_{m-2}b_{2,n-1}b_{1,n-1}, && h_7^{'} = x_1x_2\cdots x_{m-2} \in E_1^{s-1-r,t(r),*};\\
&h_8 = x_1x_2\cdots x_{m-3}h_{1,n+1}h_{1,n}b_{1,n-1}, && h_8^{'} = x_1x_2\cdots x_{m-3} \in E_1^{s-1-r,t(r),*},
\end{aligned}$$

where $t(r) = sp^2q + (s-1)pq + (s-1)q + s - 2 - r$.

For h_1', $s-2-r < \sum\limits_{i=1}^{m-3} c_{i,2} = s$, by Proposition 2.3, we get that h_1' is impossible to exist. For the same reason, h_i' $(i = 2, 3, \cdots 8)$ are impossible to exist. So we have $E_1^{s+3-r,t,*} = 0$, that is $\mathrm{Ext}_A^{s+3-r,t}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$.

Proposition 3.3 Let $p \geqslant 7$, $tq = p^{n+1}q + 2p^nq$, $n > 3$, then

(1)

$$\begin{aligned}
&\mathrm{Ext}_A^{4,tq+rq+u}(\mathbb{Z}_p, \mathbb{Z}_p) = 0\ (r = 2, 3, 4,\ u = -1, 0 \text{ or } r = 3, 4,\ u = 1);\\
&\mathrm{Ext}_A^{4,tq+q}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{h_0l_n\};\ \ \mathrm{Ext}_A^{4,tq}(\mathbb{Z}_p, \mathbb{Z}_p) = 0;\\
&\mathrm{Ext}_A^{4,tq+2q+1}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{\tilde{\alpha}_2 g_n\}, a_0^2 g_n \neq 0.
\end{aligned}$$

(2)

$$\begin{aligned}
&\mathrm{Ext}_A^{5,tq+rq+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0\ \ (r = 1, 3, 4);\ \ \mathrm{Ext}_A^{5,tq+rq}(\mathbb{Z}_p, \mathbb{Z}_p) = 0\ \ (r = 2, 3);\\
&\mathrm{Ext}_A^{5,tq+2q+1}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{\tilde{\alpha}_2\};\ \ \mathrm{Ext}_A^{5,tq+2}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{a_0^2 l_n\};\\
&\mathrm{Ext}_A^{5,tq+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0.
\end{aligned}$$

Proof (1) Consider the second degrees (mod $p^{n+1}q$) of the generators in the E_1-terms of the MSS, where $0 \leqslant j \leqslant n+1$,

$$\begin{aligned}
\deg h_{s,j} &= (p^{s+j-1} + \cdots + p^j)q \ (\mathrm{mod}\, p^{n+1}q),\ 0 \leqslant j < s+j-1 < n+1,\\
&= (p^n + \cdots + p^j)q \ (\mathrm{mod}\, p^{n+1}q),\ 0 \leqslant j < s+j-1 = n+1;\\
\deg b_{s,j-1} &= (p^{s+j-1} + \cdots + p^j)q \ (\mathrm{mod}\, p^{n+1}q),\ 1 \leqslant j < s+j-1 < n+1,\\
&= (p^n + \cdots + p^j)q \ (\mathrm{mod}\, p^{n+1}q),\ 1 \leqslant j < s+j-1 = n+1;\\
\deg a_{j+1} &= (p^j + \cdots + 1)q + 1 \ (\mathrm{mod}\, p^{n+1}q),\ 0 \leqslant j < n+1,\\
&= (p^n + \cdots + 1)q + 1 \ (\mathrm{mod}\, p^{n+1}q),\ j = n+1.
\end{aligned}$$

For the second degree $k = tq + rq + u$ $(0 \leqslant r \leqslant 4, -1 \leqslant u \leqslant 2) = 2p^nq + rq + u$ (mod $p^{n+1}q$), and excluding the factor which has second degree $\geqslant tq + pq$, we can get that the possibility of the factor of the generators in $E_1^{w,tq+rq+u,*}$ $(4 \leqslant w \leqslant 5)$ are $a_0,\ a_1,\ h_{1,0},\ h_{1,n+1},\ h_{1,n},\ h_{2,n},$

$b_{1,n},\ b_{1,n-1}$ and $b_{2,n-1}$. Thus from the degree we know that

$$\begin{aligned}&E_1^{4,tq+rq+u,*}=0\ (r=3,4,u=1);\\&E_1^{4,tq+rq+u,*}=0\ (r=2,3,4,u=-1,0);\\&E_1^{4,tq,*}\cong\mathbb{Z}_p\{b_{1,n-1}b_{2,n-1}\};\\&E_1^{4,tq+q,*}\cong\mathbb{Z}_p\{h_{1,0}h_{1,n}b_{2,n-1},\ h_{2,n}b_{1,n-1}h_{1,0}\};\\&E_1^{4,tq+2q+1,*}\cong\mathbb{Z}_p\{a_1h_{1,0}h_{1,n}h_{2,n}\}.\end{aligned}$$

In the MSS, note that $d_r(xy)=d_r(x)y+(-1)^s xd_r(y)$ $(x\in E_1^{s,t,*}, y\in E_1^{s',t',*})$. Since $d_1(b_{1,n-1}h_{2,n}h_{1,0})\neq 0$, then $E_r^{4,tq+q}=\mathbb{Z}_p\{b_{2,n-1}h_{1,n}h_{1,0}\}$ $(r\geqslant 2)$. Moreover, $h_{2,n}h_{1,n}h_{1,0}$ is permanent cycle in the MSS which converges to $h_0g_n\in \mathrm{Ext}_A^{3,*}(\mathbb{Z}_p,\mathbb{Z}_p)$, then $d_r(E_r^{3,tq+q,*})=0$ for $r\geqslant 1$, so that $b_{2,n-1}h_{1,n}h_{1,0}$ is not d_r-boundary and it converges nontrivially to h_0l_n.

In addition, we say that $\mathrm{Ext}_A^{4,tq}(\mathbb{Z}_p,\mathbb{Z}_p)=0$, since $E_1^{4,tq,*}\cong\mathbb{Z}_p\{b_{1,n-1}\ b_{2,n-1}\}$, where $b_{1,n-1}$ converges to b_{n-1}, while in the $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$, there is no element in relation to $b_{2,n-1}\in E_1^{2,p^{n+1}q+p^nq,*}$.

(2) Similarly, due to the reason of the degree, we can get the following results

$$\begin{aligned}&E_1^{5,tq+q+1,*}\cong\mathbb{Z}_p\{a_1b_{1,n-1}b_{2,n-1},\ a_0h_{1,0}h_{2,n}b_{1,n-1},\ a_0h_{1,0}b_{2,n-1}h_{1,n}\};\\&E_1^{5,tq+rq+1,*}=0\ (r=3,4);\\&E_1^{5,tq+rq,*}=0\ (r=2,3);\\&E_1^{5,tq+2q+1,*}\cong\mathbb{Z}_p\{a_1h_{1,0}b_{2,n-1}h_{1,n},\ h_{2,n}b_{1,n-1}h_{1,0}a_1\};\\&E_1^{5,tq+2,*}\cong\mathbb{Z}_p\{a_0^2h_{1,n}b_{2,n-1},\ a_0^2h_{2,n}b_{1,n-1}\};\\&E_1^{5,tq+1,*}\cong\mathbb{Z}_p\{a_0b_{1,n-1}b_{2,n-1},\ a_0h_{1,n-1}h_{1,n}h_{1,n+1}\}.\end{aligned}$$

The generators of $E_1^{5,tq+q+1,*}$ in the MSS all die, this is because that

$$\begin{aligned}&d_1(a_1b_{1,n-1}b_{2,n-1})=-a_0h_{1,0}b_{1,n-1}b_{2,n-1}\neq 0,\\&a_0h_{1,0}h_{2,n}b_{1,n-1}=-d_1(a_1h_{2,n}b_{1,n-1})\end{aligned}$$

and

$$a_0h_{1,0}b_{2,n-1}h_{1,n}=-d_1(a_1h_{2,n-1}h_{1,n}).$$

So we have $\mathrm{Ext}_A^{5,tq+q+1}(\mathbb{Z}_p,\mathbb{Z}_p)=0$. In addition, with a similar proof of (1), we know that $d_rE_r^{4,tq+2,*}=0$. So the generator of $E_1^{5,tq+2,*}$ in the MSS converges to $a_0^2l_n$.

Since $d_1(h_{2,n}b_{1,n-1}h_{1,0}a_1)\neq 0$, then $E_r^{5,tq+2q+1}=\mathbb{Z}_p\{b_{2,n-1}h_{1,n}h_{1,0}a_1\}$ for $r\geqslant 2$. Moreover, $h_{2,n}h_{1,n}h_{1,0}a_1$ is a permanent cycle in the MSS which converges to $\tilde{\alpha}_2g_n(a_1h_{1,0}$ is a permanent cycle in the MSS and converges to $\widetilde{\alpha}_2\in\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p))$, then $d_r(E_r^{4,tq+2q+1})=0$ for $r\geqslant 1$. Thus $b_{2,n-1}h_{1,n}h_{1,0}a_1$ is not d_r-boundary and converges nontrivially to $\widetilde{\alpha}_2l_n$.

Since a_0, $b_{1,n-1}$, $h_{1,n}$ and $h_{1,n+1}$ are all permanent cycles in the MSS and converge to a_0, b_{n-1}, h_n and h_{n+1}, respectively, it is easy to get that $a_0b_{1,n-1}h_{1,n}h_{1,n+1}$ is a permanent cycle in the MSS and converges to $a_0b_{n-1}h_nh_{n+1}$ which equals $0\in\mathrm{Ext}_A^{5,tq+1}(\mathbb{Z}_p,\mathbb{Z}_p)$ by

$h_nh_{n+1}=0$. Furthermore, we have $d_{2p-1}(b_{2,n-1})=b_{1,n}h_{1,n}-b_{1,n-1}h_{1,n+1}$ from [10], then $d_{2p-1}(a_0b_{2,n-1}b_{1,n-1})\neq 0$ and so $\mathrm{Ext}_A^{5,tq+1}(\mathbb{Z}_p,\mathbb{Z}_p)=0$.

Theorem 3.4 Let $p\geqslant 7$, $n>3$, then

$$h_0g_n\in \mathrm{Ext}_A^{3,p^{n+1}q+2p^nq+q}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a permanent cycle in the ASS, and converges to a nontrivial element in $\pi_{p^{n+1}q+2p^nq+q-3}S$.

Proof From [20, Theorem 1.1], there is a nontrivial differential $d_2(g_n)=a_0l_n(n\geqslant 1)$ in the ASS, the elements g_n and l_n are called a pair of a_0-related elements. The condition of Theorem A in [7] can be established by the $\mathbb{Z}_p$-bases of $\mathrm{Ext}_A^{s,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ $(s\leqslant 3)$ in [10] and Proposition 3.3 in the above. Furthermore, we have $\kappa\cdot(\alpha_1)_L=(1_{E_4}\wedge p)f$ with $f\in[\sum^{tq+q}L,E_4]$ (see [7], 9.2.34), then $(1_{E_4}\wedge i)\kappa\cdot(\alpha_1)_L=0$. Thus

$$(1_{E_4}\wedge 1_L\wedge i)(\kappa\wedge 1_L)\phi=(1_{E_4}\wedge 1_L\wedge i)(\kappa\wedge 1_L)((\alpha_1)_L\wedge 1_L)\widetilde{i''}=0,$$

where $\widetilde{i''}\in\pi_qL\wedge L$ such that $((\alpha_1)_L\wedge 1_L)\widetilde{i''}=\phi$. It can be easily proved that $(\kappa\wedge 1_L)\phi=(\bar{c}_3\wedge 1_L)\sigma\phi$, where $\sigma\phi\in\pi_{tq+2q}(KG_3\wedge L)$ is a d_1-cycle which represents $(\phi)_*(\sigma)\in\mathrm{Ext}_A^{3,tq+2q}(H^*L,\mathbb{Z}_p)$. Thus

$$(\bar{c}_3\wedge 1_{L\wedge M})(1_{KG_3}\wedge i)\sigma\phi=0.$$

So we can get that $(1_L\wedge i)_*\phi_*(g_n)\in\mathrm{Ext}_A^{3,p^{n+1}q+2p^nq+q}(H^*L\wedge M,\mathbb{Z}_p)$ is a permanent cycle in the ASS. Then Theorem 3.4 will be concluded by Theorem C in [7], here $\phi\in[\sum^{2q-1}S,L]$, $\kappa\in\pi_{tq+1}E_4$.

The Proof of Theorem 1.1 From Theorem 3.4, $h_0g_n\in\mathrm{Ext}_A^{3,p^{n+1}q+2p^nq+q}(\mathbb{Z}_p,\mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\varphi\in\pi_{p^{n+1}q+2p^nq+q-3}S$ for $n>3$.

Consider the following composition of mappings

$$\begin{aligned}\widetilde{f}\ \ &\Sigma^{p^{n+1}+2p^nq+q-3}S\xrightarrow{\varphi}S\xrightarrow{i''i'i}V(2)\xrightarrow{\gamma^s}\\ &\Sigma^{-s(p^2+p+1)q}V(2)\xrightarrow{jj'j''}\Sigma^{-s(p^2+p+1)q+(p+1)q+q}S\ ,\end{aligned}$$

because φ is represented by h_0g_n in the ASS, then the above $\widetilde{f}$ is represented by

$$\widetilde{g}=(jj'j'')_*(\gamma^s)_*(i''i'i)_*(h_0g_n)=(jj'j''\gamma^si''i'i)_*(h_0g_n)$$

in the ASS. Furthermore, we know that $\gamma_s=jj^{'}j^{''}\gamma^si''i'i\in\pi_*S$ is represented by $\widetilde{\gamma}_s$ in the ASS. By using the Yoneda products, we know that the composition

$$\begin{aligned}&\mathrm{Ext}_A^{0,0}(\mathbb{Z}_p,\mathbb{Z}_p)\xrightarrow{(i''i'i)_*}\mathrm{Ext}_A^{0,0}(H^*V(2),\mathbb{Z}_p)\\ &\xrightarrow{(jj'j'')_*(\gamma^s)_*}\mathrm{Ext}_A^{s,(sp^2+(s-1)p+(s-2))q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p)\end{aligned}$$

is a multiplication (up to nonzero scalar) by

$$\widetilde{\gamma}_s \in \mathrm{Ext}_A^{s,sp^2q+(s-1)pq+(s-2)q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Hence, the composite map $\widetilde{f}$ is represented (up to nonzero scalar) by

$$\widetilde{\gamma}_s h_0 g_n \in \mathrm{Ext}_A^{s+3,p^{n+1}q+2p^nq+sp^2q+(s-1)pq+(s-1)q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS.

From Proposition 3.1, we see that $\widetilde{\gamma}_s h_0 g_n \neq 0$. Moreover, from Proposition 3.2, it follows that $\widetilde{\gamma}_s h_0 g_n$ can not be hit by any differential in the ASS. Thus $\widetilde{\gamma}_s h_0 g_n$ survives nontrivially to a homotopy element in π_*S.

References

[1] Adams J F. On the structure and application of Steenrod algebra[J]. Comm. Math. Helv., 1958, 32(1): 180–214.

[2] Adams J F. Stable homotopy and generalized homotopy[M]. Chicago: Univ. Chicago Press, 1974.

[3] Aikawa T. 3-dimensional cohomology of the mod p Steenrod algebra[J]. Math. Scand., 1980, 47: 91–115.

[4] Cohen R, Goeress P. Secondary cohomology operations that detect homotopy class[J]. Topology, 1984, 23(2): 177–194.

[5] Cohen R. Odd primary infinite families in the stable homotopy theory[M]. Providence RI: AMS, 1981.

[6] Lin J K. Third periodicity families in the stable homotopy of spheres[J]. J. P. J. Geom. Topology, 2003, 3(3): 179–219.

[7] Lin J K. Adams spectral sequence and stable homotopy groups of spheres (in Chinese)[M]. Beijing: Science Press, 2007, 163–185.

[8] Liu X G. A new family represented by $b_1g_0\tilde{\gamma}_s$ in the stable homotopy of spheres[J]. Sys. Sci. Math., 2006, 26(2): 129–136.

[9] Lee C N. Detection of some elements in the stable homotopy groups of spheres[J]. Math. Z., 1996, 222(2): 231–246.

[10] Liulevicius A. The factorizations of cyclic reduced powers by secondary cohomology operations[M]. Providence RI: AMS, 1962.

[11] Mahowald M. A new infinite family in π_*S[J]. Topology, 1977, 16(3): 249–256.

[12] Miller H R, Ravenel D C, Wilson W S. Periodic phenomena in the Adams-Novikov spectral sequence[J]. Ann. Math., 1977, 106(3): 469–516.

[13] Ravenel D C. Complex cobordism and stable homotopy groups of spheres[M]. New York: Academic Press, 1986.

[14] Serre J P. Groupes d′ homotopie et classes de groupes abeliens[J]. Ann. Math., 1953, 58(2): 258–294.

[15] Toda H. On spectra realizing exterior parts of the Steenrod algebra[J]. Topology, 1971, 10(1): 53–65.

[16] Wang Y Y. A new familiy of filtration $s + 5$ in the stable homotopy groups of spheres[J]. Acta. Math. Sci., 2008, 28(2): 321–332.

[17] Wang Y Y, Wang J L. A new family of elements in the stable homotopy groups of spheres[J]. J. Math., 2015,35(2): 294–306.

[18] Wang Y Y, Wang J B. The nontrivility of ζ_n-related elements in the stable homotopy groups of sphere[J]. Math. Scand., 2015, 117: 304–319.

[19] Wang X J, Zheng Q B. The convergence of $\widetilde{\alpha}_s^{(n)} h_0 h_k$ in Adams spectral sequence[J]. Sci. China Math., 1998, 41(6): 622–628.

[20] Zhao H, Wang X J. Two nontrivial differentials in the Adams spectral sequence[J]. Chinese Ann. Math., 2008, 29A(1): 557–566.

[21] Zhong L N, Wang Y Y. Detection of a nontrivial product in the stable homotopy groups of spheres[J]. Alge. Geom. Top., 2013, 13: 3009–3029.

球面稳定同伦群中第三周期γ类非平凡新元素

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摘要: 本文研究了球面稳定同伦群的问题. 以Adams谱序列中的第二非平凡微分为几何输入, 给出了球面稳定同伦群中$h_0 g_n (n > 3)$的收敛性. 同时, 由Yoneda乘积的知识, 发掘了球面稳定同伦群中的一个非平凡新元素. 非平凡元素的范围将被我们的结果进一步扩大.

关键词: 球面稳定同伦群; Toda-Smith谱; Adams谱序列; May谱序列; Adams微分

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