Vol. 36 ( 2016 ) No. 5

# ENTROPY-ULTRA-BEE SCHEME WITH A NEW ENTROPY FUNCTION FOR LINEAR ADVECTION EQUATION

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**Abstract:** In this paper, we investigate the numerical solutions for linear advection equation. A new Entropy-Ultra-bee scheme is obtained by using physical entropy function. Numerical experiments show that the scheme has a very good quality in long-time numerical computation and has high resolution in the vicinity of continuities.

Keywords: linear advection equation; a new entropy function; Entropy-Ultra-bee scheme 2010 MR Subject Classification: 65M08; 35L65

Document code: A Article ID: 0255-7797(2016)05-0975-06

### 1 Introduction

The linear advection equation is of the form

$$u_t + au_x = 0, \tag{1.1}$$

where a is a constant. For any function U(u) differentiable with respect to u, it is not difficult to obtain

$$U(u)_t + aU(u)_x = 0, (1.2)$$

which show that (1.1) possesses infinitely many conservation laws, i.e., any smooth function of u, including u itself, is conservative.

In recent years, Mao and his co-workers developed a class of finite-volume schemes for linear advection equation, see [1–8]. As a scheme of the Godunov type, the schemes proceeded in the following three steps: reconstruction, evolution, cell-averaging. The entropy scheme was presented by Li and Mao in [3]. In [6], Cui and Mao revealed that the entropy scheme had an error self-canceling mechanism and showed that the scheme was super-convergent. However, when computing discontinuous solutions, the scheme produced spurious oscillations near discontinuities. To fix this problem, the authors of [3] used the TVD limiter of the Ultrabee scheme to control the reconstruction step in each cell. See [9, 10] or [1] for the Ultra-bee

<sup>\*</sup> Received date: 2015-07-07 Accepted date: 2015-11-03

Foundation item: Supported by National Natural Science Foundation of China (11201436).

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scheme. The resulted scheme was thus a combination of the entropy scheme and the Ultrabee scheme. In [3], Li and Mao chose  $U(u) = u^2$  as entropy function. As is known to all, physics Entropy is  $s = \log \frac{p}{\rho^{\gamma}}$ . As a preparation for the extension of the scheme to Euler system, it needs to choose a function related to the function log.

In this paper, we follow the idea of [3] and develop a Entropy-Ultra-bee scheme with a entropy function  $U(u) = u \log u$  for linear advection equation. We presents a number of numerical examples of linear advection equation, which show that the scheme has a very good quality in long-time numerical computation and has better resolution of continuities. The organization of the paper is as follows: Section 1 is the introduction. In Section 2, we describe the scheme. In Section 3, we present three numerical examples. Section 4 is the conclusion.

## 2 Description of Entropy-Ultra-Bee Scheme with the New Entropy Function

We will describe Entropy-Ultra-bee scheme with the new entropy function following [1]. We denote the cell size by h and use a Cartesian grid with grid cells  $(x_{j-1/2}, x_{j+1/2})$ 's centered at  $\{x_j\}$ 's, where  $x_{j\pm 1/2} = (j \pm 1/2)h$  and  $x_j = jh$ ,  $j = 0, \pm 1, \pm 2, \cdots$ . We use  $\tau$  to denote the time increment. The Entropy-Ultra-bee scheme is a Godunov type scheme. However, different from the other Godunov type schemes, it involves two numerical entities, the numerical solution  $u^n$ , which is a cell-average approximation to the true solution,

$$u_j^n \simeq \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n) dx, \qquad (2.1)$$

and the numerical entropy  $U^n$ , which is a cell-average approximation to an entropy of the true solution,

$$U_j^n \simeq \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} U(u(x, t_n)) dx, \qquad (2.2)$$

where U(u) is an entropy function of the solution.

For simplicity, we describe the scheme for (1.1) and (1.2) with a = 1. As a scheme of the Godunov type, Entropy-Ultra-bee scheme with the new entropy function proceeds in the following three steps.

i) Step-Reconstruction The solution is reconstructed in each cell as a step function involving two constant states,

$$R(x; u^{n}, U^{n}) = u_{j}^{n} + \begin{cases} -d_{j}^{n}, & x_{j-1/2} < x \le x_{j}, \\ +d_{j}^{n}, & x_{j} < x \le x_{j+1/2}, \end{cases}$$
(2.3)

where  $d_i^n$  is called the half step (HS) of the reconstruction. It is obvious that

$$\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} R(x; u^n, U^n) dx = u_j^n.$$
(2.4)

To compute the HS  $d_j^n$ , we need first to compute an entropy HS  $d_j^{n,e}$  by requiring

$$\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} U(R(x; u^n, U^n)) dx = U_j^n,$$
(2.5)

i.e., the entropy cell-average of the reconstructed solution is equal to the numerical entropy in the cell. Equation (2.5) turns out to be an equation for  $d_i^{n,e}$ ,

$$\frac{1}{2}(u_j^n - d_j^{n,e})\log(u_j^n - d_j^{n,e}) + \frac{1}{2}(u_j^n + d_j^{n,e})\log(u_j^n + d_j^{n,e}) = U_j^n.$$
(2.6)

The solvability of (2.6) was discussed in §2.2 in [1], which states that the equation will have two opposite roots with the same absolute value.

The Ultra-Bee HS is computed as

$$d_{j}^{n,ub} = \begin{cases} 0, & (u_{j}^{n} - u_{j-1}^{n})(u_{j+1}^{n} - u_{j}^{n}) < 0, \\ u_{j+1}^{n} - u_{j}^{n}, & |u_{j}^{n} - u_{j-1}^{n}| \ge \frac{\nu}{1-\nu} |u_{j+1}^{n} - u_{j}^{n}|, \\ \frac{1-\nu}{\nu} (u_{j}^{n} - u_{j-1}^{n}), & |u_{j}^{n} - u_{j-1}^{n}| < \frac{\nu}{1-\nu} |u_{j+1}^{n} - u_{j}^{n}|. \end{cases}$$
(2.7)

The HS is then taken as the one smaller in absolute value of  $d_j^{n,e}$  and  $d_j^{n,ub}$  with the sign of  $(u_{j+1}^n - u_{j-1}^n)$ ,

$$d_j^n = \operatorname{sgn}(u_{j+1}^n - u_{j-1}^n) \min\{|d_j^{n,e}|, |d_j^{n,ub}|\}.$$
(2.8)

ii) Evolution Solve the following initial value problem (IVP)

$$\begin{cases} v_t + v_x = 0, & t_n < t \le t_{n+1}, \\ v(x, t_n) = R(x; u^n, U^n). \end{cases}$$
(2.9)

The problem can be exactly solved as

$$v(x,t) = R(x - (t - t_n); u^n, U^n).$$
(2.10)

iii) Cell-averaging Compute the cell-averages of the numerical solution at  $t = t_{n+1}$  as

$$u_j^{n+1} = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} v(x, t_{n+1}) dx$$
(2.11)

and

$$U_j^{n+1} = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} U(v(x, t_{n+1})) dx.$$
(2.12)

We can compute  $u_j^{n+1}$  in the following flux form,

$$u_j^{n+1} = u_j^n - \lambda (\hat{f}_{j+1/2}^n - \hat{f}_{j-1/2}^n), \qquad (2.13)$$

where

$$\hat{f}_{j+1/2}^n = f(u^*(u_j^n + d_j^n, u_{j+1}^n - d_{j+1}^n)).$$
(2.14)

$$U_j^{n+1} = U_j^n - \lambda (\hat{F}_{j+1/2}^n - \hat{F}_{j-1/2}^n), \qquad (2.15)$$

where the numerical entropy flux  $\hat{F}_{j+1/2}^n$  is computed as

$$\hat{F}_{j+1/2}^n = U(u^*(u_j^n + d_j^n, u_{j+1}^n - d_{j+1}^n)).$$
(2.16)

Thus we complete the description of the scheme.

#### **3** Numerical Experiments

In this section, we will present three numerical examples to show the effectiveness of Entropy-Ultra-bee scheme with an entropy function  $U(u) = u \log u$  for linear advection equation (1.1) taking a = 1. The computational domain is [0, 1]. To show the accuracy of the scheme, numerical solutions are compared against the true solutions. The "new" stands for the numerical solution computed by the Entropy-Ultra-bee scheme with the entropy function  $U(u) = u \log u$ . We take  $\lambda = 0.45$ .

**Example 3.1** The initial value is

$$u(x,0) = \begin{cases} 1.0 + \exp\{-\frac{1}{1 - 16(x - \frac{1}{2})^2}\}, & \text{if } x \in (\frac{1}{4}, \frac{3}{4}), \\ 1.0, & \text{else} \end{cases}$$
(3.1)

with periodic boundary conditions at the two ends. This example is come from [11]. We conduct the simulation on a grid of 200 cells, up to t = 1, t = 10 and the numerical results are displayed in Fig. 3.1. It is clearly see that the results at t = 1 are in good agreement with the analytical solution. However, we notice a little defect near the corner at t = 10.



Fig. 3.1: Example 3.1, 200 cells, left: t = 1, and right: t = 10.

**Example 3.2** The initial conditions are

$$u_0(x) = 2.0 + \exp\{-100 \times (x - \frac{1}{2})^2\}\sin 80x, \quad \text{if } 0 \le x < 1 \quad (3.2)$$

with periodic boundary conditions at the two ends. This example is the Wave-packet problem, see [11]. This is a very challenge problem. We conduct the simulation on a grid of 200 cells, up to t = 1, t = 10 and the numerical results are displayed in Fig. 3.2. We can see that the scheme give good results for this problem. As to our knowledge, no scheme up to date has ever got qualified numerical result for this example with the same grid beyond the time t = 10.



Fig. 3.2: Example 3.2, 200 cells, left: t = 1, and right: t = 10.

**Example 3.3** The initial value is

$$u(x,0) = 1.0 + \exp\{-200(x-0.3)^2\} + \begin{cases} 1, & \text{if } 0.6 < x < 0.8, \\ 0, & \text{else.} \end{cases}$$
(3.3)

This example is come from [11]. We conduct the simulation on a grid of 200 cells, up to t = 1, t = 10 and the numerical results are displayed in Fig. 3.3. We see that the numerical solution agrees quite well with the exact one in the smooth region, all the spurious oscillations near discontinuities are eliminated and jumps are kept quite sharp with at most one transition point for each.



Fig. 3.3: Example 3.3, 200 cells, left: t = 1, and right: t = 10.

#### 4 Conclusion

In this paper, we present Entropy-Ultra-bee scheme with an entropy function U(u) = ulogu for linear advection equation. Numerical experiments show that the scheme has a very good quality in long-time numerical computation and has better resolution of continuities.

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## 带新熵函数的Entropy-Ultra-bee格式计算线性传输方程

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摘要:本文研究了线性传输方程的数值解法.利用物理的熵函数得到了一种新的Entropy-Ultrabee格式.数值实验表明该格式长时间计算效果非常好,在间断附近有比较高的分辨率. 关键词:线性发展方程;一个新的熵函数;Entropy-Ultra-bee格式

MR(2010)主题分类号: 65M08; 35L65 中图分类号: O241.82

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