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SOME REMARKS ON GEOMETRIC INEQUALITIES FOR SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD OF QUASI-CONSTANT CURVATURE

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Abstract: In this paper, we study Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature. By using algebraic techniques, we establish Chen's general inequalities, Chen-Ricci inequalities and inequalities between the warping function and the squared mean curvature, which generalize several rusults of Özgür and Chen's.

Keywords: Chen's inequalities; Chen-Ricci inequalities; warped product; quasi-constant curvature

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1 Introduction

According to Chen [1], one of the most important problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Related with famous Nash embedding theorem [2], Chen introduced a new type of Riemannian invariants, known as δ-invariants [3, 4, 5]. The author's original motivation was to provide answers to a question raised by Chern concerning the existence of minimal isometric immersions into Euclidean space [6]. Therefore, Chen obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established inequalities for submanifolds in real space forms in terms of the sectional curvature, the scalar curvature and the squared mean curvature [7]. Later, he established general inequalities relating $\delta(n_1, \cdots, n_k)$ and the squared mean curvature for submanifolds in real space forms [8]. Similar inequalities also hold for Lagrangian submanifolds of complex space forms. In [9], Chen proved that, for any $\delta(n_1, \cdots, n_k)$, the equality case holds if and only if the Lagrangian submanifold is minimal. This interesting phenomenon inspired people to look for a more sharp inequality. In 2007, Oprea improved the inequality on $\delta(2)$ for Lagrangian submanifolds in complex space forms[10]. Recently, Chen and Dillen established general inequalities for Lagrangian

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submanifolds in complex space forms and provided some examples showing these new improved inequalities are best possible [11]. Such invariants and inequalities have many nice applications to several areas in mathematics [12].

Afterwards, many papers studied similar problems for different submanifolds in various ambient spaces, like complex space forms [13], Sasakian space forms [14], (κ,μ)-contact space forms [15], Lorentzian manifold [16], Euclidean space [17] and locally conformal almost cosymplectic manifolds [18].

This paper is organized as follows. In Section 2, the basic elements of the theory of δ-invariants are briefly presented. In Section 3, we establish general inequalities of δ-invariants for submanifolds of a Riemannian manifold of quasi-constant curvature [19], which generalize a result of paper [20]. In Section 4, we obtain an inequality between the Ricci curvature and the squared mean curvature for submanifolds of the ambient space by using a algebraic lemma. Finally, in Section 5, we establish inequalities between the warping function f (intrinsic structure) and the squared mean curvature (extrinsic structure) for warped product submanifolds $M_1 \times_f M_2$ in a Riemannian manifold of quasi-constant curvature, as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

2 Preliminaries

In [19], Chen and Yano introduced the notion of a Riemannian manifold (N,g) of quasi-constant curvature as a Riemannian manifold with the curvature tensor satisfying the condition

$$\begin{aligned}\overline{R}(X,Y,Z,W) = {}& a[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] + b[g(X,Z)T(Y)T(W) \\ & - g(X,W)T(Y)T(Z) + g(Y,W)T(X)T(Z) - g(Y,Z)T(X)T(W)],\end{aligned} \tag{2.1}$$

where a,b are scalar functions and T is a 1-form defined by

$$g(X,P) = T(X) \tag{2.2}$$

and P is a unit vector field. If $b=0$, it can be easily seen that the manifold reduces to a space of constant curvature.

Decomposing the vector field P on M uniquely into its tangent and normal components P^T and $P^\perp$, respectively, we have

$$P = P^T + P^\perp. \tag{2.3}$$

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold of quasi-constant curvature N^{n+p}. The Gauss equation is given by

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \tag{2.4}$$

for all $X,Y,Z,W \in TM$, where R and $\overline{R}$ are the curvature tensors of M and N^{n+p}, respectively, and h is the second fundamental form.

In N^{n+p} we choose a local orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$, such that, restricting to M^n, $e_1, \cdots, e_n$ are tangent to M^n. We write $h^r_{ij} = g(h(e_i, e_j), e_r)$. The mean curvature vector ζ is given by $\zeta = \sum\limits_{r=n+1}^{n+p} (\frac{1}{n} \sum\limits_{i=1}^{n} h^r_{ii}) e_r$, then the mean curvature H is given by $H = \parallel \zeta \parallel$.

Let $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of M^n is given by

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j). \tag{2.5}$$

Let L be an l-dimensional subspace of $T_x M$, $x \in M$, $l \geq 2$ and $\{e_1, \cdots, e_l\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the l-plane L by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq l} K(e_\alpha \wedge e_\beta). \tag{2.6}$$

For simplicity we put

$$\Psi(L) = \sum_{1 \leq i < j \leq l} [g(P^T, e_i)^2 + g(P^T, e_j)^2]. \tag{2.7}$$

For an integer $k \geq 0$ we denote by $S(n, k)$ the set of k-tuples $(n_1, \cdots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. We denote by $S(n)$ the set of unordered k-tuples with $k \geq 0$ for a fixed n. For each k-tuples $(n_1, \cdots, n_k) \in S(n)$, Chen defined a Riemannian invariant $\delta(n_1, \cdots, n_k)$ as follows [8]

$$\delta(n_1, \cdots, n_k)(x) = \tau(x) - S(n_1, \cdots, n_k)(x), \tag{2.8}$$

where $S(n_1, \cdots, n_k)(x) = \inf\{\tau(L_1) + \cdots + \tau(L_k)\}$, and $L_1, \cdots, L_k$ run over all k mutually orthogonal subspaces of $T_x M$ such that $\dim L_j = n_j$, $j \in \{1, \cdots, k\}$. For each $(n_1, \cdots, n_k) \in S(n)$, we put

$$c(n_1, \cdots, n_k) = \frac{n^2\big(n + k - 1 - \sum\limits_{j=1}^{k} n_j\big)}{2\big(n + k - \sum\limits_{j=1}^{k} n_j\big)}, \quad d(n_1, \cdots, n_k) = \frac{1}{2}[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)].$$

For a differentiable function f on M, the Laplacian $\triangle f$ of f is defined by

$$\triangle f = \sum_{i=1}^{n} [(\nabla_{e_i} e_i) f - e_i e_i f].$$

We shall use the following lemmas.

Lemma 2.1 [7] Let $a_1, a_2, \cdots, a_n, b$ be $(n+1)(n \geq 2)$ real numbers such that

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + b),$$

then $2a_1a_2 \geq b$, with the equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Lemma 2.2 Let $f(x_1, x_2, \cdots, x_n)$ be a function in R^n defined by

$$f(x_1, x_2, \cdots, x_n) = x_1 \sum_{i=2}^{n} x_i.$$

If $x_1 + x_2 + \cdots + x_n = 2\lambda$, then we have $f(x_1, x_2, \cdots, x_n) \leq \lambda^2$, with the equality holding if and only if $x_1 = x_2 + x_3 + \cdots + x_n = \lambda$.

Proof From $x_1 + x_2 + \cdots + x_n = 2\lambda$, we have $\sum\limits_{i=2}^{n} x_i = 2\lambda - x_1$. It follows that

$$f(x_1, x_2, \cdots, x_n) = x_1(2\lambda - x_1) = -(x_1 - \lambda)^2 + \lambda^2,$$

which represents Lemma 2.2 to prove.

3 Chen's General Inequalities

Theorem 3.1 If M^n $(n \geq 3)$ is a submanifold of a Riemannian manifold of quasi-constant curvature N^{n+p}, then we have

$$\delta(n_1, \cdots, n_k) \leq c(n_1, \cdots, n_k)H^2 + d(n_1, \cdots, n_k)a + b[(n-1) \parallel P^T \parallel^2 - \sum_{j=1}^{k} \Psi(L_j)] \quad (3.1)$$

for any k-tuples $(n_1, \cdots, n_k) \in S(n)$. The equality case of (3.1) holds at $x \in M^n$ if and only if there exist an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1}, \cdots, e_{n+p}\}$ of $T_x^{\perp}M$ such that the shape operators of M^n in N^{n+p} at x have the following forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \quad A_{e_r} = \begin{pmatrix} A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k^r & 0 \\ 0 & \cdots & 0 & \mu_r I \end{pmatrix}, \quad r = n+2, \cdots, n+p,$$

where $a_1, \cdots, a_n$ satisfy

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n$$

and each A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying $\mathrm{trace}(A_1^r) = \cdots = \mathrm{trace}(A_k^r) = \mu_r$, I is an identity matrix.

Remark 3.2 For $\delta(2)$, inequality (3.1) is due to Cihan Özgür [20, Theorem 3.1].

Proof Let $x \in M^n$ and $\{e_1, e_2, \cdots, e_n\}$ and $\{e_{n+1}, e_{n+2}, \cdots, e_{n+p}\}$ be orthonormal basis of T_xM^n and $T_x^{\perp}M^n$, respectively, such that the mean curvature vector ζ is in the direction of the normal vector to e_{n+1}. For convenience, we set

$$\begin{aligned}
&a_i = h_{ii}^{n+1}, \quad i = 1, 2, \cdots, n,\\
&b_1 = a_1, \quad b_2 = a_2 + \cdots + a_{n_1}, \quad b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \cdots,\\
&b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+n_2+\cdots+n_{k-1}+n_k},\\
&b_{k+2} = a_{n_1+\cdots+n_k+1}, \cdots, b_{\gamma+1} = a_n,\\
&\Delta_1 = \{1, \cdots, n_1\}, \cdots,\\
&\Delta_k = \{(n_1 + \cdots + n_{k-1}) + 1, \cdots, n_1 + \cdots + n_k\},\\
&\Delta_{k+1} = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k).
\end{aligned}$$

Let $L_1, \cdots, L_k$ be mutually orthogonal subspaces of T_xM with $\dim L_j = n_j$, defined by

$$L_j = \mathrm{Span}\{e_{n_1+\cdots+n_{j-1}+1}, \cdots, e_{n_1+\cdots+n_j}\}, \quad j = 1, \cdots, k.$$

From (2.4), (2.6) and (2.7) we have

$$\tau(L_j) = \frac{n_j(n_j-1)}{2}a + b\Psi(L_j) + \sum_{r=n+1}^{n+p} \sum_{\alpha_j<\beta_j} [h_{\alpha_j\alpha_j}^r h_{\beta_j\beta_j}^r - (h_{\alpha_j\beta_j}^r)^2], \tag{3.2}$$

$$2\tau = n(n-1)a + 2b(n-1) \parallel P^T \parallel^2 + n^2H^2 - \parallel h \parallel^2 . \tag{3.3}$$

We can rewrite (3.3) as $n^2H^2 = (\parallel h \parallel^2 + \eta)\gamma$, or equivalently,

$$(\sum_{i=1}^n h_{ii}^{n+1})^2 = \gamma[\sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^n (h_{ij}^r)^2 + \eta], \tag{3.4}$$

where

$$\eta = 2\tau - 2c(n_1, \cdots, n_k)H^2 - n(n-1)a - 2(n-1)b \parallel P^T \parallel^2, \tag{3.5}$$

$$\gamma = n + k - \sum_{j=1}^k n_j.$$

From (3.4) we deduce

$$(\sum_{i=1}^{\gamma+1} b_i)^2 = \gamma[\eta + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^n (h_{ij}^r)^2 - 2\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} - \cdots - 2\sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}],$$

where $\alpha_j, \beta_j \in \Delta_j$, for all $j = 1, \cdots, k$. Applying Lemma 2.1, we derive

$$\sum_{j=1}^k \sum_{\alpha_j<\beta_j} a_{\alpha_j}a_{\beta_j} \geq \frac{1}{2}[\eta + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^n (h_{ij}^r)^2],$$

it follows that

$$\sum_{j=1}^{k}\sum_{r=n+1}^{n+p}\sum_{\alpha_j<\beta_j}[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j}-(h^r_{\alpha_j\beta_j})^2]\geq\frac{\eta}{2}+\frac{1}{2}\sum_{r=n+1}^{n+p}\sum_{(\alpha,\beta)\notin\Delta_{k+1}}(h^r_{\alpha\beta})^2+\sum_{r=n+2}^{n+p}\sum_{\alpha_j\in\Delta_j}(h^r_{\alpha_j\alpha_j})^2 \geq\frac{\eta}{2}. \tag{3.6}$$

From (3.2) and (3.6) we have

$$\sum_{j=1}^{k}\tau(L_j)\geq\sum_{j=1}^{k}[\frac{n_j(n_j-1)}{2}a+b\Psi(L_j)]+\frac{1}{2}\eta. \tag{3.7}$$

Using (2.8), (3.5) and (3.7), we derive the desired inequality.

The equality case of (3.1) at a point $x\in M$ holds if and only if we have the equality in all the previous inequalities and also in Lemma 2.1, thus, the shape operators take the desired forms.

4 Chen-Ricci Inequalities

In [21], Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for any n-dimensional Riemannian submanifold of a real space form $R^m(c)$ of constant sectional curvature c as follows

Theorem 4.1 (see [21, Theorem 4]) Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then the following statements are true.

(1) For each unit vector $X\in T_pM$, we have

$$\|\zeta\|^2\geq\frac{4}{n^2}[\mathrm{Ric}(X)-(n-1)c]. \tag{4.1}$$

(2) If $\zeta(p)=0$, then a unit vector $X\in T_pM$ satisfies the equality case of (4.1) if and only if X belongs to the relative null space $N(p)$ given by

$$N(p)=\{X\in T_pM\mid h(X,Y)=0,\ \forall Y\in T_pM\}.$$

(3) The equality case of (4.1) holds for all unit vectors $X\in T_pM$ if and only if either p is a geodesic point or $n=2$ and p is an umbilical point.

Afterwards, many papers studied similar problems for different submanifolds in various ambient manifolds [22–24]. Thus, after putting an extra condition on the ambient manifold, like semi-symmetric metric connections in the case of real space forms [25] and curvature-like tensors in the case of a Riemannian manifold [26], one proves the results similar to that of Theorem 4.1.

In [20], Özgür obtained several Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature. However, he didn't establish an inequality between the clssical Ricci-curvature and the squared mean curvature. Under these circumstances it becomes necessary to give a theorem, which could present an inequality between the Ricci-curvature and the squared mean curvature for submanifolds in the ambient manifold.

Theorem 4.2 Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold of quasi-constant curvature N^{n+p}. For each unit vector X in T_xM we have

$$\mathrm{Ric}(X) \leq (n-1)a + (n-2)bg(P^T, X)^2 + b \parallel P^T \parallel^2 + \frac{n^2}{4}H^2. \tag{4.2}$$

The equality sign holds for any tangent vector X in T_xM if and only if either x is a totally geodesic point or $n=2$ and x is an umbilical point.

Remark 4.3 For $b=0$, inequality (4.2) is due to (4.1).

Remark 4.4 We should point out that our approach is different from Chen's.

Proof Let $x \in M^n$ and $\{e_1, e_2, \cdots, e_n\}$ and $\{e_{n+1}, e_{n+2}, \cdots, e_{n+p}\}$ be orthonormal basis of T_xM^n and $T_x^\perp M^n$, respectively, such that $X = e_1$. From equations (2.1), (2.2), (2.3) and (2.4) it follows that

$$R_{ijij} = a + b[g(P^T, e_i)^2 + g(P^T, e_j)^2] + \sum_{r=n+1}^{n+p} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{4.3}$$

Using (4.3) one derives

$$\begin{aligned} \mathrm{Ric}(X) &= \sum_{i=2}^{n} R_{1i1i} = (n-1)a + (n-1)bg(P^T, e_1)^2 \\ &\quad + b\sum_{i=2}^{n} g(P^T, e_i)^2 + \sum_{r=n+1}^{n+p} \sum_{i=2}^{n} [h_{11}^r h_{ii}^r - (h_{1i}^r)^2] \\ &\leq (n-1)a + (n-2)bg(P^T, X)^2 + b \parallel P^T \parallel^2 + \sum_{r=n+1}^{n+p} \sum_{i=2}^{n} h_{11}^r h_{ii}^r. \end{aligned} \tag{4.4}$$

Let us consider the quadratic forms $f_r : R^n \to R$, defined by

$$f_r(h_{11}^r, h_{22}^r, \cdots, h_{nn}^r) = \sum_{i=2}^{n} h_{11}^r h_{ii}^r.$$

We consider the problem $\max f_r$, subject to $\Gamma : h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r$, where k^r is a real constant.

From Lemma 2.2, we see that the solution $(h_{11}^r, h_{22}^r, \cdots, h_{nn}^r)$ of the problem in question must satisfy

$$h_{11}^r = \sum_{j=2}^{n} h_{jj}^r = \frac{k^r}{2}, \tag{4.5}$$

which implies

$$f_r \leq \frac{(k^r)^2}{4}. \tag{4.6}$$

From (4.4) and (4.6) we have

$$\begin{aligned}\mathrm{Ric}(X) &\leq (n-1)a+(n-2)bg(P^T,X)^2+b\parallel P^T\parallel^2+\sum_{r=n+1}^{n+p}\frac{(k^r)^2}{4}\\ &= (n-1)a+(n-2)bg(P^T,X)^2+b\parallel P^T\parallel^2+\frac{n^2}{4}H^2.\end{aligned}$$

Next, we shall study the equality case.

For each unit vector X at x, if the equality case of inequality (4.2) holds, from (4.4), (4.5) and (4.6) we have

$$h^r_{1i}=0,\quad i\neq 1,\ \forall\ r, \tag{4.7}$$

$$h^r_{11}+h^r_{22}+\cdots+h^r_{nn}-2h^r_{11}=0,\quad \forall r. \tag{4.8}$$

For any unit vector X at x, if the equality case of inequality (4.2) holds, noting that X is arbitrary, by computing $\mathrm{Ric}(e_j), j=2,3,\cdots,n$ and combining (4.7) and (4.8) we have

$$h^r_{ij}=0,\quad i\neq j,\quad \forall r;\quad h^r_{11}+h^r_{22}+\cdots+h^r_{nn}-2h^r_{ii}=0,\quad \forall i,r.$$

We can distinguish two cases:

(1) $n\neq 2,\ h^r_{ij}=0,\ i,j=1,2,\cdots,n,\ r=n+1,\cdots,n+p$ or

(2) $n=2,\ h^r_{11}=h^r_{22},\quad h^r_{12}=0,\ r=3,\cdots,2+p$.

The converse is trivial.

We immediately have the following

Corollary 4.5 Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold of quasi-constant curvature N^{n+p}. The equality case of inequality (4.2) holds for any tangent vector X of M^n if and only if either M^n is a totally geodesic submanifold in N^{n+p} or $n=2$ and M^n is a totally umbilical submanifold.

Corollary 4.6 If $\zeta(x)=0$, then a unit vector $X\in T_xM$ satisfies the equality case of (4.2) if and only if X belongs to the relative null space $N(x)$ given by

$$N(x)=\{X\in T_xM\mid h(X,Y)=0,\ \forall Y\in T_xM\}.$$

Proof Assume $\zeta(x)=0$. For each unit vector $X\in T_xM$, equality holds in (4.2) if and only if (4.5) and (4.7) hold. Then $h^r_{1i}=0,\ \forall i,r$, i.e., $X\in N(x)$.

5 Warped Product Submanifolds

Related with famous Nash embedding theorem[2], Chen established a general sharp inequality for wraped products in real space form [27]. Later, he studied warped products in complex hyperbolic spaces [28] and complex projective spaces [29], respectively. Afterwards, many papers studied similar prolems for different submanifolds in various ambient spaces [30–32]. In the present paper, we establish an inequality for warped product submanifolds of a Riemannian manifold of quasi-constant curvature.

The study of warped product manifolds was initiated by Bishop and O'Neill [33]. Following [33], we have

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1, where $\dim M_i = n_i$ $(i = 1, 2)$, $n_1 + n_2 = n$. The warped product of M_1 and M_2 is the Riemannian manifold $M_1 \times_f M_2 = (M_1 \times M_2, g)$, where $g = g_1 + f^2 g_2$. More explicitly, if vector fields X and Y tangent to $M_1 \times_f M_2$ at (x, y), then

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + f^2(x) g_2(\pi_{2*}X, \pi_{2*}Y),$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $M_1 \times_f M_2$ onto M_1 and M_2, respectively, and $*$ stands for derivative map.

For a warped product $M_1 \times_f M_2$, we denote by D_1 and D_2 the distributions given by the vectors tangent to leaves and fibres, respectively, where D_1 is obtained from the tangent vectors of M_1 via the horizontal lift and D_2 by tangent vectors of M_2 via the vertical lift.

Let $\phi : M^n = M_1 \times_f M_2 \to N^{n+p}$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold of quasi-constant curvature. Denote by h the second fundamental form of ϕ. Denote by $\mathrm{tr}h_1$ and $\mathrm{tr}h_2$ the trace of h restricted to M_1 and M_2, respectively. The immersion ϕ is called mixed totally geodesic if $h(X, Z) = 0$ for any X in D_1 and Z in D_2.

Since $M_1 \times_f M_2$ is a warped product, we have $\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z$ for any unit vector fields X, Z tangent to M_1, M_2, respectively. It follows that

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}[(\nabla_X X)f - X^2 f]. \tag{5.1}$$

We set $\| P^T \|^2_{M_1} = \sum\limits_{j=1}^{n_1} g(P^T, e_j)^2$, $\| P^T \|^2_{M_2} = \sum\limits_{s=n_1+1}^{n} g(P^T, e_s)^2$.

Theorem 5.1 Let $\phi : M_1 \times_f M_2 \to N^{n+p}$ be an isometric immersion of a warped product into a Riemannian manifold of quasi-constant curvature, then we have

$$\frac{\triangle f}{f} \leq \frac{n^2 H^2}{4n_2} + n_1 a + \frac{b}{n_2}[n_2 \| P^T \|^2_{M_1} + n_1 \| P^T \|^2_{M_2}], \tag{5.2}$$

where H^2 is the squared mean curvature of ϕ, and $\triangle$ is the Laplacian operator of M_1. The equality case of (5.2) holds if and only if ϕ is a mixed totally geodesic immersion with $\mathrm{tr}h_1 = \mathrm{tr}h_2$.

Proof In N^{n+p} we choose a local orthonormal frame $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$, such that $e_1, \cdots, e_{n_1}$ are tangent to M_1, $e_{n_1+1}, \cdots, e_n$ are tangent to M_2, e_{n+1} is parallel to the mean curvature vector ζ.

Using (5.1) and the definition of $\triangle f$, we get

$$\frac{\triangle f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s) \tag{5.3}$$

for each $s \in \{n_1 + 1, \cdots, n\}$.

Using (2.1), (2.3) and (2.4) we have

$$2\tau + \parallel h \parallel^2 - n^2H^2 = 2b(n-1) \parallel P^T \parallel^2 + (n^2-n)a. \tag{5.4}$$

We set

$$\delta = 2\tau - (n^2-n)a - 2b(n-1) \parallel P^T \parallel^2 - \frac{n^2}{2}H^2. \tag{5.5}$$

Then (5.4) can be written as

$$n^2H^2 = 2(\delta + \parallel h \parallel^2). \tag{5.6}$$

If we put $a_1 = h_{11}^{n+1}, a_2 = \sum\limits_{i=2}^{n_1} h_{ii}^{n+1}, a_3 = \sum\limits_{t=n_1+1}^{n} h_{tt}^{n+1}$, from (5.6) we have

$$\begin{aligned}(\sum_{i=1}^{3} a_i)^2 \quad &= \quad 2[\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\ne j\le n} (h_{ij}^{n+1})^2 \\ &\quad + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}].\end{aligned}$$

From Lemma 2.1 we get

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \ge \frac{\delta}{2} + \sum_{1\le i<j\le n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \tag{5.7}$$

with the equality holding if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{5.8}$$

From (5.3) we have

$$\begin{aligned}\frac{n_2\triangle f}{f} &= \tau - \sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t) \\ &= \tau - \frac{n_1(n_1-1)a}{2} - (n_1-1)b\sum_{j=1}^{n_1} g(P^T,e_j)^2 - \sum_{r=n+1}^{n+p}\sum_{1\le j<k\le n_1}[h_{jj}^r h_{kk}^r - (h_{jk}^r)^2] \\ &\quad - \frac{n_2(n_2-1)a}{2} - (n_2-1)b\sum_{s=n_1+1}^{n} g(P^T,e_s)^2 - \sum_{r=n+1}^{n+p}\sum_{n_1+1\le s<t\le n}[h_{ss}^r h_{tt}^r - (h_{st}^r)^2] \\ &= \tau - \frac{n(n-1)a}{2} + n_1n_2a - b[(n_1-1)\sum_{j=1}^{n_1} g(P^T,e_j)^2 + (n_2-1)\sum_{s=n_1+1}^{n} g(P^T,e_s)^2] \\ &\quad - \sum_{r=n+1}^{n+p}\sum_{1\le j<k\le n_1}[h_{jj}^r h_{kk}^r - (h_{jk}^r)^2] - \sum_{r=n+1}^{n+p}\sum_{n_1+1\le s<t\le n}[h_{ss}^r h_{tt}^r - (h_{st}^r)^2].\end{aligned} \tag{5.9}$$

Combing (5.7) and (5.9) we have

$$\begin{aligned}
\frac{n_2\triangle f}{f} &\leq \tau - \frac{n(n-1)a}{2} + n_1n_2a - b[(n_1-1)\sum_{j=1}^{n_1} g(P^T,e_j)^2 + (n_2-1)\sum_{s=n_1+1}^{n} g(P^T,e_s)^2] \\
&\quad - \frac{\delta}{2} - \sum_{1\leq j\leq n_1}\sum_{n_1+1\leq t\leq n}(h_{jt}^{n+1})^2 - \frac{1}{2}\sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\
&\quad - \sum_{r=n+2}^{n+p}\sum_{1\leq j<k\leq n_1}[h_{jj}^r h_{kk}^r - (h_{jk}^r)^2] - \sum_{r=n+2}^{n+p}\sum_{n_1+1\leq s<t\leq n}[h_{ss}^r h_{tt}^r - (h_{st}^r)^2]. \\
&= \tau - \frac{n(n-1)a}{2} + n_1n_2a - b[(n_1-1)\sum_{j=1}^{n_1} g(P^T,e_j)^2 + (n_2-1)\sum_{s=n_1+1}^{n} g(P^T,e_s)^2] \\
&\quad - \frac{\delta}{2} - \sum_{r=n+1}^{n+p}\sum_{1\leq j\leq n_1}\sum_{n_1+1\leq t\leq n}(h_{jt}^r)^2 - \frac{1}{2}\sum_{r=n+2}^{n+p}[(\sum_{j=1}^{n_1}h_{jj}^r)^2 + (\sum_{t=n_1+1}^{n}h_{tt}^r)^2] \\
&\leq \tau - \frac{n(n-1)a}{2} + n_1n_2a - b[(n_1-1)\sum_{j=1}^{n_1} g(P^T,e_j)^2 + (n_2-1)\sum_{s=n_1+1}^{n} g(P^T,e_s)^2] - \frac{\delta}{2} \\
&= \frac{n^2H^2}{4} + n_1n_2a - b[(n_1-1)\sum_{j=1}^{n_1} g(P^T,e_j)^2 \\
&\quad + (n_2-1)\sum_{s=n_1+1}^{n} g(P^T,e_s)^2] + b(n-1)\parallel P^T\parallel^2 \\
&= \frac{n^2H^2}{4} + n_1n_2a + b[n\parallel P^T\parallel^2 - n_1\sum_{j=1}^{n_1} g(P^T,e_j)^2 - n_2\sum_{s=n_1+1}^{n} g(P^T,e_s)^2],
\end{aligned} \tag{5.10}$$

which proves inequality.

Next, we shall study the equality case.

From (5.7) and (5.10) we know that the equality case of (5.2) holds if and only if

$$h_{jt}^r = 0, \quad 1\leq j\leq n_1, \quad n_1+1\leq t\leq n, \quad n+1\leq r\leq n+p, \tag{5.11}$$

$$\sum_{i=1}^{n_1} h_{ii}^r = \sum_{t=n_1+1}^{n} h_{tt}^r = 0, \quad n+2\leq r\leq n+p. \tag{5.12}$$

Obviously (5.11) is equivalent to $h(D_1,D_2)=0$, thus, the immersion ϕ is mixed totally geodesic. Further on, from (5.8) and (5.12), we have

$$\sum_{i=1}^{n_1} h_{ii}^r = \sum_{s=n_1+1}^{n} h_{ss}^r, \quad \forall r,$$

it follows that $\mathrm{tr}h_1 = \mathrm{tr}h_2$.

Remark 5.2 If $b=0$, inequality (5.2) is due to Chen [28, Theorem 1.4].

As applications of Theorem 5.1, we have

Corollary 5.3 Under the same assumption as in Theorem 5.1, if f is a harmonic function, there are no isometric minimal immersion of $M_1 \times_f M_2$ into N^{n+p} with $a < 0, b \leq 0$.

Corollary 5.4 Under the same assumption as in Theorem 5.1, if f is an eigenfunction of the Laplacian on M_1 with eigenvalue $\lambda > 0$, there are no isometric minimal immersion of $M_1 \times_f M_2$ into N^{n+p} with $a < 0, b \leq 0$.

Remark 5.5 In [34, Theorem 4.1], Ganchev and Mihova proved that a Riemannian manifold of quasi-constant curvature $N^{n+p}(n+p \geq 4)$ with $a < 0, b \neq 0$, can be locally ξ-isometric to a canal space-like hypersurface in the Minkowski space $\mathbb{R}_1^{n+p+1}$, ξ is a unit vector field on N^{n+p}.

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拟常曲率黎曼流形中子流形的几何不等式的一些注记

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摘要: 本文研究了拟常曲率黎曼流形中子流形的Chen不等式. 利用代数技巧, 建立了Chen 广义不等式、Chen-Ricci不等式和关于卷积函数和平均曲率平方的不等式, 推广了Özgür和Chen的一些结果.

关键词: Chen不等式; Chen-Ricci不等式; 卷积; 拟常曲率

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