

THE APPLICATION OF THE BASIN OF ATTRACTION TO THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THE SECOND ORDER PARABOLIC BOUNDARY VALUE PROBLEM

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Abstract: In this paper, a new sufficient condition of the existence and uniqueness of the second order parabolic boundary value problem is given by using the basin of attraction and the comparison theorem, which generalize some existed theorems.

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1 Introduction

We will study the parabolic operator

$$Lu = a_{ij}u_{x_i x_j} + b_i u_{x_i} - au - cu_t \quad (1.1)$$

acting on functions in $D = \Omega \times [0, T]$, where $a_{ij}(x, t) \in W_\infty^1(D)$, $b_i, a \in L_\infty(D)$, $c = c(x) \in W_\infty^1(\Omega)$ and Ω is a connected bounded subset of n -dimensional space.

Using a continuous method, Sigillito explored the solution for the heat equation, see [1]. Elcart and Sigillito derived an explicit coercivity inequality $\|u\| \leq \text{const}\|L_a u\|_0$ and gave a sufficient condition for the existence and uniqueness of solution to the second order parabolic, see [2].

Recently, in this area, the global diffeomorphism theorem was used to prove the existence and uniqueness of solutions of nonlinear differential equation of certain classes. In addition, many authors were extensively investigated this problem, see Mayer [3], Plastock [4], Radulescu and Radulescu [5], Shen Zuhe [6–7], Zampieri [8]. These theorems may be used for solving nonlinear systems of equation.

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Motivated by these results, we shall utilize an interesting tool, the attraction basin to give a new set of sufficient condition for the existence and uniqueness of the second order parabolic boundary value problems in this paper, which can be founded in Section 3. Using our approach it is easy to obtain results of Elcart and Sigillito. Moreover, the methods apply not only to this problem but also to other nonlinear differential equations.

2 Preliminaries

In this section, we will state some lemmas which are useful to our results. First, we introduce the basin of attraction.

Lemma 2.1 (see [8]) Let G, F be Banach spaces, D be an open subset of G , $x_0 \in D$ and $f : D \subset G \rightarrow F$ be a C^1 mapping and a local homeomorphism. Then for any $x \in D$, the path-lifting problem

$$\begin{cases} f(\gamma_x(t)) = f(x_0) + e^{-t}(f(x) - f(x_0)), & t \in R \cdots, \\ \gamma_x(0) = x, & \gamma_x(t) \in D \cdots \end{cases} \quad (2.1)$$

has a unique continuous solution $t \rightarrow \gamma_x(t)$ defined on the maximal open interval $I_x = (t_{x-}, t_{x+})$, $-\infty \leq t_{x-}, t_{x+} \leq +\infty$. Moreover, the set $\{(x, t) \in D \times R : t \in I_x\}$ is open in $D \times R$ and the mapping is $(x, t) \rightarrow \gamma_x(t)$ continuous.

Definition 2.1 [8] In the setting of Lemma 2.1, the basin of attraction of x_0 is the set

$$A = \{X \in D : t_x^+ = +\infty\}.$$

Theorem 2.1 [9] With the above setting, f is a global homeomorphism onto Y if and only if $\gamma_x(t)$ is defined on R for all $x \in A$, namely, $\gamma_x(t)$ can also be extended to $-\infty$.

Lemma 2.2 (see [8]) Let X be Banach space, $a, b \in R$ and $p : [a, b] \rightarrow X$ be a C^1 mapping on $[a, b]$. Then $\|p(t)\|$ has derivative $\|p(t)\|'$ almost everywhere and $\|p(t)\|' \leq \|p(t)\|$ for $a < t < b$.

Second, the following comparison theorem play an important role to prove the sufficient condition for the existence of a unique solution of problem (1.1).

Let E be an open (t, x) -set in R^2 and $g \in C[E, R]$. Consider the scalar differential equation with an initial condition

$$\begin{cases} u' = g(t, u) \cdots, \\ u(t_0) = u_0 \cdots. \end{cases} \quad (2.2)$$

Assume that there exists a sequence t_k such that $t_0 \leq t_k \rightarrow b$ as $k \rightarrow \infty$ and $u^0 = \lim_{k \rightarrow \infty} u(t_k)$ exists. If $g(t, u)$ is bounded on the intersection of E and a neighbourhood of (b, u^0) , then $\lim_{t \rightarrow b} u(t) = u^0$. If in addition, $g(b, u^0)$ is defined such that $g(t, u)$ is continuous at (b, u^0) , then $u(t)$ is continuously differentiable on $[t_0, b]$ and is a solution of (2.2) on $[t_0, b]$ (see [10]). In this case the solution $u(t)$ can be extended as a solution to the boundary of E .

Theorem 2.2 (Comparison theorem in [9]) With the above setting, suppose that $[t_0, t_0 + b)$ is the largest interval in which the maximal solution $r(t)$ of (2.2) exists. Let

$$m \in C[[t_0, t_0 + b), R], (t, m(t)) \in E \text{ for } t \in [t_0, t_0 + b), m(t_0) \leq u_0$$

and for a fixed Dini derivative

$$Dm(t) \leq g(t, m(t)), t \in [t_0, t_0 + b) \setminus T,$$

where T denotes an almost countable subset of $t \in [t_0, t_0 + b) \setminus T$, then

$$m(t) \leq r(t), t \in [t_0, t_0 + b).$$

3 Existence Theorem

Consider the boundary value problem

$$\begin{cases} a_{ij}u_{x_i x_j} + b_i u_{x_i} - cu_t - au = 0 \cdots, \\ u_t = 0, (x, t) \in \partial\Omega \times (0, T) \cdots, \end{cases} \quad (3.1)$$

where $u_t \in L_2([0, T], W_2^2(\Omega))$. Let $W_0(D)$ denote the Hilbert space with the norm

$$(\|u\|_{2,1})^2 = \nu^2 \int_D |D^2 u|^2 dx dt + \int_D (cu_t^2) dt,$$

where $|D^2 u|^2$ represents the sum of the squares of all the second derivatives with respect to space variables and ν is positive constant.

The following assumptions are needed later.

A1 The boundary of Ω is piecewise smooth with nonnegative mean curvature everywhere.

A2 $f : W_0(D) \rightarrow L_2(D)$ is continuous and a bounded function of t, x_1, \dots, x_n, u .

Define $S = \sup |b_i - (a_{ij})_{x_j}|$, $a_1 = \sup_{\Omega} a(x)$ and $a_0 = \inf_D a$, then $S < \sqrt{\lambda} \nu^2$ and $a_0 > \sqrt{\lambda} S - \lambda \nu^2$, where $\lambda = \inf \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} u^2 dx} > 0$ is the lowest eigenvalue of $-\Delta$ in Ω .

Elcart and Sigillito gave the following inequality in [2].

Lemma 3.1 If $u \in W_0$, then

$$\|u\|_{2,1} \leq C \|Lu\|_0 \cdots, \quad (3.2)$$

where

$$\begin{aligned} C^2 = & 336 + 204a_1^2(a_0 + \lambda \nu^2 - S\sqrt{\lambda})^{-2} + C_1[n^4 B^2 \nu^{-2} \\ & + 40\beta^2 + 82(2S + 2\mu\gamma_1 + \gamma_2)]. \end{aligned}$$

Denote $Mu = a_{ij}u_{x_i x_j} + b_i u_{x_i} - cu_t$, then M is the linear operator from $W_0(D)$ to $L_2(D)$. We may express (3.1) in the form

$$Fu = Mu - au = 0.$$

For $u, \phi \in W_0(D)$, we have

$$F'(u)(\phi) = M\phi - a_u(x, u(t, x))\phi.$$

Define

$$\delta(s) = \max_{\|x\| \leq s} \|F'(x)^{-1}\|.$$

Theorem 3.1 In the setting of the above, equation (1.1) exists a unique solution if the following conditions hold

- (1) $\inf_{\Omega \times R} a_u > -\lambda$;
- (2) for each, the maximum solution of the initial value problem

$$\begin{aligned} y'(t) &= \eta \delta(y(t)), t \in [0, c) \cdots, \\ y(0) &= 0 \end{aligned} \tag{3.3}$$

is defined on $[0, c)$ and there exists a sequence $t_n \rightarrow c$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} y(t_n) = y^*$ is finite.

Proof We have from (2.1) and Lemma 2.2 that

$$\begin{aligned} D \|\gamma_x(t)\| &\leq \|\gamma_x'(t)\| = \|F'(\gamma_x(t))^{-1}\| e^{-h} \|F(x) - F(x_0)\| \\ &\leq k e^{-h} \delta(\|\gamma_x(t)\|) \quad (k = \|F(x) - F(x_0)\|). \end{aligned}$$

By assumption A2, we know the maximum solution $y(t)$ of (3.3) is defined on $[0, c)$ and there exists a sequence $t_n \rightarrow c$ as $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} y(t_n) = y^*$$

is finite. It follows that $y(t)$ is continuous on $[0, c)$ and there is a constant M such that

$$|y(t)| \leq M, t \in [0, c].$$

By the comparison theorem, we have

$$\|\gamma_x(t)\| \leq |y(t)| \leq M, t \in [0, c].$$

From conditions A1, A2 and condition (1), since $\lambda = \inf \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} u^2 dx} > 0$ is the lowest eigenvalue of $-\Delta$ in Ω , it follows that for all $u \in W_0(D)$, zero is not an eigenvalue of

$M\phi - a_u(x, u(t, x))\phi$, so for every $u \in W_0(D)$, the operator $F'(u) = M - a_u I$ is invertible and F is a local homeomorphism from $W_0(D)$ onto $L_2(D)$, where I denotes the identical operator.

Then in view of Theorem 2.1, we need only show that for all $x \in A$, $\gamma_x(t)$ can also be extended to $-\infty$, namely, we need consider the problem in the opposite direction.

Let $g(-h) = \gamma_x(t)$, $t \in (a, 0]$, $h \in [0, -a]$, $a < 0$ for $t_1, t_2 \in (a, 0]$, we have

$$\begin{aligned} \|\gamma_x(t_1) - \gamma_x(t_2)\| &= \|g(-h_1) - g(-h_2)\| = \int_{-h_2}^{-h_1} \|g'(s)\| ds \\ &= \int_{-h_2}^{-h_1} \|F'(\gamma_x(s))^{-1}\| e^s \|F(x) - F(x_0)\| ds \\ &\leq \int_{-h_2}^{-h_1} \|\delta(\gamma_x(s))\| e^s \|F(x) - F(x_0)\| ds \\ &\leq k \int_{-h_2}^{-h_1} \|\delta(y(s))\| e^s ds \\ &\leq k\delta(M)e^{-a}|t_1 - t_2|. \end{aligned}$$

So $\gamma_x(t)$ is Lipschitz continuous on $(-a, 0]$, $\gamma_x(t)$ can also be extended to $-\infty$, the theorem is proved.

Elcart and Sigillito [2] studied the following initial-boundary value problem

$$\begin{aligned} a_{ij}u_{x_i x_j} + b_i u_{x_i} - cu_t &= f(x, t, u) \cdots, \\ u &= 0 \text{ for } t = 0 \text{ and } (x, t) \in \partial\Omega \times (0, T) \cdots, \end{aligned} \quad (3.4)$$

where $\partial\Omega \in C^2$ and f is continuous and has three derivatives with respect to u . Problem (3.4) may be formulated as an operator equation $Pu = 0$, where $Pu = Mu - f(x, u)$ is a mapping of $W_0(D)$ onto $L_2(D)$.

Corollary 3.1 Assume that f satisfies

$$(i) \inf_{\Omega \times R} f'_u > -\lambda;$$

(ii) uniformly in x , $\|f'_u\| = \omega(\|u\|)$, where ω is continuous map satisfying $\int_a^\infty \frac{dt}{\omega(t)} = \infty$, then there is a unique solution of the equation $Pu = 0$ in $W_0(D)$.

Proof Compare with equations (3.4) and (3.1), we have $f(x, u) = au$, so condition (1) of Theorem 3.1 is satisfied. Denote $\omega_1(t) = \alpha\omega(t) + \beta$, then $\int_a^\infty \frac{dt}{\omega_1(t)} = \infty$ and $\delta(s) = \max_{\|x\| \leq s} \|P'(x)^{-1}\|$. We have from Lemma 3.1

$$\|[P'(u)]^{-1}\| \leq \alpha \sup_{\Omega} |f'_u(x, u(x))| + \beta \cdots \quad (3.5)$$

for positive constant α, β , then $\delta(t) \leq \omega_1(t)$, and thus

$$\int_a^{+\infty} \frac{dt}{\delta(t)} \geq \int_a^{+\infty} \frac{dt}{\omega_1(t)} = +\infty \cdots \quad (3.6)$$

From problem (3.3), $\forall t > 0$, $\left| \int_0^t \frac{y'(r)}{\delta(y(r))} dr \right| = \eta t$. Let $y(r) = s$, we have $\int_{y(0)}^{y(t)} \frac{1}{\delta(s)} ds = \eta t$.

For equation (3.6), we have that $y(t)$ is bounded. Consequently, there exist a real sequence $\{t_n\}$: $t_n \rightarrow c$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} y(t_n) = y^*$ exists. The corollary is proved.

Remark Condition (ii) in Corollary 3.1 can be replaced with $f'_u = O(u)$, because $\int_a^\infty \frac{dt}{\omega(t)} = \infty$ holds. The result of Elcart and Sigillito in [2] becomes a special case of Theorem 3.1.

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吸引盆在二阶抛物线边值问题解存在唯一性中的应用

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摘要: 本文研究了二阶抛物线边值问题解的存在唯一性的问题. 利用吸引盆的方法和全局同胚理论, 推导出二阶抛物线边值问题解存在唯一性的一个充分条件, 从而推广了已经存在的一些定理.

关键词: 吸引盆; 全局同胚; 初值问题; 二阶抛物线边值问题

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