Vol. 37 (2017)
No. 3

数 学 杂 志
J. of Math. (PRC)

具有时滞的比率型三种群捕食模型的分支分析

徐昌进 [1], 廖茂新 [2]

(1. 贵州财经大学贵州经济系统仿真重点实验室, 贵州 贵阳 550004)

(2. 南华大学数理学院, 湖南 衡阳 421001)

摘 要: 本文研究了一类具有时滞的比率型三种群捕食模型. 通过分析该模型的特征方程, 证明了该模型在正平衡点的稳定性. 通过选择时滞 τ 为分支参数, 得到了当时滞 τ 通过一系列的临界值时, Hopf 分支产生. 应用中心流形和规范型理论, 得到了关于确定 Hopf 分支特性的计算公式. 最后进行数值模拟验证了我们所得结果的正确性. 所得结果是对前人工作的补充.

关键词: 捕食系统; Hopf 分支; 稳定性; 时滞; 比率型

MR(2010) **主题分类号**: 34K18; 34K20 **中图分类号**: O175.13

文献标识码: A **文章编号**: 0255-7797(2017)03-0533-16

1 引言

在种群动力学中, 对捕食者和食饵之间的动力学行为的研究已经长时期并且将继续成为生态学和数学生态学研究的重要课题. 近年的研究表明, 在很多情况下, 特别当捕食者捕食食物时, 就必须得和其他物种进行分享和竞争食物, 因此一个更符合实际的捕食模型应该是具有比率型的模型. 关于有比率型种群捕食模型研究已有大量工作, 诸多学者取得了很多成果 [1–11]. 2003 年, 谭德君 [12] 研究了具有时滞和基于比率的三种群捕食模型

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)\left[a_1 - a_{11}x_1(t) - \frac{a_{12}x_2(t)}{m_{12}x_2(t)+x_1(t)} - \frac{a_{13}x_3(t)}{m_{13}x_3(t)+x_1(t)}\right], \\ \frac{dx_2(t)}{dt} = x_2(t)\left[-a_2 + \frac{a_{21}x_1(t-\tau_1)}{m_{12}x_2(t-\tau_1)+x_1(t-\tau_1)}\right], \\ \frac{dx_3(t)}{dt} = x_3(t)\left[-a_3 + \frac{a_{31}x_1(t-\tau_2)}{m_{13}x_3(t-\tau_2)+x_1(t-\tau_2)}\right] \end{cases} \tag{1.1}$$

的持续生存和无时滞时的全局渐近稳定性, 其中 $x_i(i=1,2,3)$ 分别表示捕食者, 食饵种群的密度, $a_i, a_{ij}, m_{ij}(i,j=1,2,3)$ 均为正常数, $\tau_i(i=1,2)$ 为非负常数. 详细的生物学意义可参见文献 [13]. 2007 年, 黄建科和吴筱宁 [14] 进一步研究了模型 (1.1) 的全局渐近稳定性, 推广了文献 [12] 的结果, 同时也研究了该系统的持久性. Song 和 Zou [15] 讨论了一类具有比率型的捕食模型的分支问题. Shi 和 Li [16] 分析了类具有比率型的捕食模型的全局渐近稳定性. Bai 等学者 [17] 考虑了一类具有比率型的随机捕食模型的动力学行为. 具体相关研究可参考文献 [18–22].

我们知道在很多生物现象中, 生物种群的周期性行为经常出现, 因此为了能更清晰全面地理解该捕食系统的动力学行为, 本文将研究系统 (1.1) 的周期现象, 具体而言, 本文将以时

*收稿日期: 2015-03-19 接收日期: 2015-05-18

基金项目: 国家自然科学基金项目 (11261010) 和贵州省优秀科技教育人才省长资金项目 (黔省专合 (2012) 53 号); 125 重大科技专项资金项目 (黔教合重大专项字 [2012]011 号).

作者简介: 徐昌进 (1970–), 男, 侗族, 湖南怀化, 教授, 主要研究方向: 泛函微分方程理论及其应用.

滞为参数, 考察因时滞的变化导致系统 (1.1) 出现 Hopf 分支, 得到了系统 (1.1) 的平凡解稳定的条件, Hopf 分支产生的条件及确定 Hopf 分支方向和分支周期解的稳定性的计算公式. 为了研究简化, 假设 $\tau_1=\tau_2=\tau$, 于是得到下面的系统

$$\begin{cases} \frac{dx_1(t)}{dt}=x_1(t)\left[a_1-a_{11}x_1(t)-\frac{a_{12}x_2(t)}{m_{12}x_2(t)+x_1(t)}-\frac{a_{13}x_3(t)}{m_{13}x_3(t)+x_1(t)}\right], \\ \frac{dx_2(t)}{dt}=x_2(t)\left[-a_2+\frac{a_{21}x_1(t-\tau)}{m_{12}x_2(t-\tau)+x_1(t-\tau)}\right], \\ \frac{dx_3(t)}{dt}=x_3(t)\left[-a_3+\frac{a_{31}x_1(t-\tau)}{m_{13}x_3(t-\tau)+x_1(t-\tau)}\right], \end{cases} \tag{1.2}$$

这里 $x_i(i=1,2,3)$ 分别表示捕食者, 食饵种群的密度, a_i,a_{ij},m_{ij} $(i,j=1,2,3)$ 均为正常数, τ 为非负常数.

本文作如下安排: 第二节研究了 (1.2) 式的平凡解的稳定性和 Hopf 分支的存在性; 第三节讨论了 Hopf 分支方向及其分支周期解的稳定性的清晰的计算公式; 第四节进行数值模拟验证理论分析的正确性.

2 平衡点的稳定性和 Hopf 分支的存在性

由文献 [23] 知, 系统的持续生存必然蕴含着正平衡点的存在, 我们设系统 (1.2) 的正平衡点为 $E^*(x_1^*,x_2^*,x_3^*)$. 令 $\bar{x}_i=x_i-x_i^*(i=1,2,3)$, 仍记 $\bar{x}_i$ 分别为 x_i, 则 (1.2) 式化为

$$\begin{cases} \frac{dx_1(t)}{dt}=p_1x_1(t)+p_2x_2(t)+p_3x_3(t)+p_4x_1^2(t)+p_5x_2^2(t)+p_6x_3^2(t) \\ \qquad +p_7x_1(t)x_2(t)+p_8x_1(t)x_3(t)+p_9x_1^2(t)x_2(t) \\ \qquad +p_{10}x_2^2(t)x_1(t)+p_{11}x_3^2(t)x_1(t)+\text{h.o.t.}, \\ \frac{dx_2(t)}{dt}=q_1x_1(t-\tau)+q_2x_2(t-\tau)+q_3x_1^2(t-\tau)+q_4x_2^2(t-\tau) \\ \qquad +q_5x_1(t-\tau)x_2(t-\tau)+q_6x_1^3(t-\tau)+q_7x_2^3(t-\tau)+q_8x_1^2(t-\tau)x_2(t-\tau) \\ \qquad +q_9x_1(t-\tau)x_2^2(t-\tau)+\text{h.o.t.}, \\ \frac{dx_3(t)}{dt}=r_1x_1(t-\tau)+r_2x_3(t-\tau)+r_3x_1^2(t-\tau)+r_4x_3^2(t-\tau) \\ \qquad +r_5x_1(t-\tau)x_3(t-\tau)+r_6x_1^3(t-\tau)+r_7x_3^3(t-\tau)+r_8x_1^2(t-\tau)x_3(t-\tau) \\ \qquad +r_9x_1(t-\tau)x_3^2(t-\tau)+\text{h.o.t.}, \end{cases} \tag{2.1}$$

其中 p_i $(i=1,2,\cdots,11); q_j,r_j$ $(j=1,2,\cdots,9)$ 的表达式参见附录 A. 系统 (2.1) 的线性部分为

$$\begin{cases} \frac{dx_1(t)}{dt}=p_1x_1(t)+p_2x_2(t)+p_3x_3(t), \\ \frac{dx_2(t)}{dt}=q_1x_1(t-\tau)+q_2x_2(t-\tau), \\ \frac{dx_3(t)}{dt}=r_1x_1(t-\tau)+r_2x_3(t-\tau). \end{cases} \tag{2.2}$$

(2.2) 式的特征方程为

$$\lambda^3+m_2\lambda^2+m_1\lambda+(n_2\lambda^2+n_1\lambda+n_0)e^{-\lambda\tau}+(s_1\lambda+s_0)e^{-2\lambda\tau}=0, \tag{2.3}$$

其中

$$m_1=-p_3r_1,m_2=-p_1,n_0=p_3q_2r_1,n_1=p_1(r_2+q_2)-p_2q_2,$$
$$n_2=-(r_2+q_2),s_0=p_2q_1r_2-p_1q_2r_2,s_1=q_2r_2.$$

当 $\tau=0$ 时, 方程 (2.3) 变为

$$\lambda^3+(m_2+n_2)\lambda^2+(m_1+n_1+s_1)\lambda+n_0+s_0=0, \tag{2.4}$$

根据 Routh-Hurwitz 准则, 如果下列条件

$$\text{(H1)}\ \ n_0+s_0>0, (m_2+n_2)(m_1+n_1+s_1)>n_0+s_0$$

成立, 则方程 (2.4) 的所有根都具有负实部. 方程的两边同时乘以 $e^{\lambda\tau}$ 得

$$(\lambda^3+m_2\lambda^2+m_1\lambda)e^{\lambda\tau}+(n_2\lambda^2+n_1\lambda+n_0)+(s_1\lambda+s_0)e^{-\lambda\tau}=0, \tag{2.5}$$

若 iω 为 (2.5) 式的解, 把 $\lambda=\mathrm{i}\omega$ 代入 (2.5) 式并分离实部和虚部得

$$\left\{\begin{array}{l}(s_0-m_1\omega+\omega^3)\cos\omega\tau+s_1\omega\sin\omega\tau=(m_2+n_1s_1+n_2s_0)\omega^2-s_0n_0,\\ (m_1\omega+s_1\omega-\omega^2)\cos\omega\tau-(s_0+m_2\omega^2)\omega\sin\omega\tau=-n_1\omega.\end{array}\right. \tag{2.6}$$

于是

$$\begin{aligned}\cos\omega\tau &= \frac{[(m_2+n_1s_1+n_2s_0)\omega^2-s_0n_0](s_0+m_2\omega^2)-n_1s_1\omega^2}{(s_0-m_1\omega+\omega^3)(s_0+m_2\omega^2)+s_1\omega(m_1\omega+s_1\omega-\omega^2)},\\ \sin\omega\tau &= \frac{[(m_2+n_1s_1+n_2s_0)\omega^2-s_0n_0](m_1\omega+s_1\omega-\omega^2)}{(s_0-m_1\omega+\omega^3)(s_0+m_2\omega^2)+s_1\omega(m_1\omega+s_1\omega-\omega^2)}\\ &\quad+\frac{n_1\omega(s_0-m_1\omega+\omega^3)}{(s_0-m_1\omega+\omega^3)(s_0+m_2\omega^2)+s_1\omega(m_1\omega+s_1\omega-\omega^2)}.\end{aligned}$$

从而由 $\sin^2\omega\tau+\cos^2\omega\tau=1$, 得

$$m_2^2\omega^{10}+k_8\omega^8+k_7\omega^7+k_6\omega^6+k_5\omega^5+k_4\omega^4+k_3\omega^3+k_2\omega^2+k_1\omega+k_0=0, \tag{2.7}$$

其中 k_i $(i=0,1,2,\cdots,8)$ 的表达式参见附录 B.

假设方程 (2.7) 有正根, 不失一般性, 假设方程有 10 个正根, 记为 $\omega_k(k=1,2,\cdots,10)$. 于是由 (2.6) 式得

$$\tau_k^{(j)}=\frac{1}{\omega_k}\left[\arccos\frac{[(m_2+n_1s_1+n_2s_0)\omega_k^2-s_0n_0](s_0+m_2\omega_k^2)-n_1s_1\omega_k^2}{(s_0-m_1\omega_k+\omega_k^3)(s_0+m_2\omega_k^2)+s_1\omega_k(m_1\omega_k+s_1\omega_k-\omega_k^2)}+2j\pi\right], \tag{2.8}$$

这里 $k=1,2,\cdots,10; j=0,1,2,\cdots$. 记

$$\tau_0=\tau_{k0}^{(0)}=\min_{k\in\{1,2,\cdots,10\}}\{\tau_k^{(0)}\}. \tag{2.9}$$

设 $\lambda(\tau)=\alpha(\tau)+i\omega(\tau)$ 为 (2.5) 式的满足 $\alpha(\tau_k^{(j)})=0,\omega(\tau_k^{(j)})=\omega_k$ 的根, 其中 $\tau_k^{(j)}$ 由 (2.8) 式定义. 把 $\lambda(\tau)$ 代入 (2.5) 式, 两端对 τ 求导, 整理得

$$\left[\frac{d\lambda(\tau)}{d\tau}\right]^{-1}=\frac{(3\lambda^2+2m_2\lambda+m_1)e^{\lambda\tau}+(2n_2\lambda+n_1)}{\lambda\left[(n_2\lambda^2+n_1\lambda+n_0)+2(s_1\lambda+s_0)e^{-\lambda\tau}\right]}-\frac{\tau}{\lambda}. \tag{2.10}$$

于是

$$
\begin{aligned}
\left[\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}\right]^{-1}\Big|_{\tau=\tau_k^{(j)}} &= \mathrm{Re}\left\{\frac{(3\lambda^2+2m_2\lambda+m_1)e^{\lambda\tau}+(2n_2\lambda+n_1)}{\lambda\left[(n_2\lambda^2+n_1\lambda+n_0)+2(s_1\lambda+s_0)e^{-\lambda\tau}\right]}\right\}_{\tau=\tau_k^{(j)}} \\
&= \frac{A_1A_3-A_2A_4}{A_1^2+A_2^2},
\end{aligned}
$$

这里

$$
\begin{aligned}
A_1 &= 2\left[s_0\sin\omega_k\tau_k^{(j)}-s_1\omega_k\cos\omega_k\tau_k^{(j)}\right], \\
A_2 &= \omega_k\left[n_0-n_2\omega_k^2+2s_0\cos\omega_k\tau_k^{(j)}+2s_1\omega_k\sin\omega_k\tau_k^{(j)}\right], \\
A_3 &= (m_1-3\omega_k^2)\cos\omega_k\tau_k^{(j)}-2m_2\omega_k\sin\omega_k\tau_k^{(j)}+n_1, \\
A_4 &= 2n_2\omega_k.
\end{aligned}
$$

为了得到本文的主要结果, 假设

$$
\text{(H2)}\ \ \mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|_{\tau=\tau_k^{(j)}}\neq 0.
$$

由上述讨论和 Hale [24] 的第 11 章的定理 1.1 可得

定理 2.1 对系统 (1.2), 假设条件 (H1) 和 (H2) 成立, 则

(i) $\tau\in[0,\tau_0)$, 其正平衡点是渐近稳定的;

(ii) $\tau>\tau_0$, 其正平衡点是不稳定的;

(iii) $\tau=\tau_k^{(j)}$ $(k=1,2,3,\cdots,10;k=1,2,\cdots)$ 是 Hopf 分支值, 其中 $\tau_k^{(j)}$ 由 (2.8) 式定义.

3 Hopf 分支方向及稳定性

本节利用中心流形理论和和规范型方法 [25] 给出系统 (1.2) 的 Hopf 分支方向, 分支周期解的稳定性等计算公式.

为方便, 令 $t=s\tau, x_i(s\tau)=\tilde{x}_i(s)\,(i=1,2,3)$, 仍记 $\tilde{x}_i(s)$ 为 $x_i(s)$. $\tau=\tau_k^{(j)}+\mu,\mu\in R,\tau_k^{(j)}$ 由 (2.8) 式定义, 且仍记 $t=s$, 则系统 (1.2) 等价于系统

$$
\left\{
\begin{aligned}
\tfrac{dx_1(t)}{dt} &= (\tau_k^{(j)}+\mu)p_1x_1(t)+p_2x_2(t)+p_3x_3(t)+p_4x_1^2(t)+p_5x_2^2(t)+p_6x_3^2(t) \\
&\quad +p_7x_1(t)x_2(t)+p_8x_1(t)x_3(t)+p_9x_1^2(t)x_2(t) \\
&\quad +p_{10}x_2^2(t)x_1(t)+p_{11}x_3^2(t)x_1(t)+\text{h.o.t.}, \\
\tfrac{dx_2(t)}{dt} &= (\tau_k^{(j)}+\mu)q_1x_1(t-1)+q_2x_2(t-1)+q_3x_1^2(t-1)+q_4x_2^2(t-1) \\
&\quad +q_5x_1(t-1)x_2(t-1)+q_6x_1^3(t-1)+q_7x_2^3(t-1)+q_8x_1^2(t-1)x_2(t-1) \\
&\quad +q_9x_1(t-1)x_2^2(t-1)+\text{h.o.t.}, \\
\tfrac{dx_3(t)}{dt} &= (\tau_k^{(j)}+\mu)r_1x_1(t-1)+r_2x_3(t-1)+r_3x_1^2(t-1)+r_4x_3^2(t-1) \\
&\quad +r_5x_1(t-1)x_3(t-1)+r_6x_1^3(t-1)+r_7x_3^3(t-1)+r_8x_1^2(t-1)x_3(t-1) \\
&\quad +r_9x_1(t-1)x_3^2(t-1)+\text{h.o.t.}.
\end{aligned}
\right. \tag{3.1}
$$

(3.1) 式的线性部分为

$$\begin{cases} \frac{dx_1(t)}{dt} = (\tau_k^{(j)} + \mu)p_1x_1(t) + p_2x_2(t) + p_3x_3(t), \\ \frac{dx_2(t)}{dt} = (\tau_k^{(j)} + \mu)q_1x_1(t-1) + q_2x_2(t-1), \\ \frac{dx_3(t)}{dt} = (\tau_k^{(j)} + \mu)r_1x_1(t-1) + r_2x_3(t-1). \end{cases} \tag{3.2}$$

(3.1) 的右端的非线性部分为

$$h = (\tau_k + \mu)\begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}, \tag{3.3}$$

其中

$$\begin{aligned} f_1(t) &= p_4x_1^2(t) + p_5x_2^2(t) + p_6x_3^2(t) + p_7x_1(t)x_2(t) + p_8x_1(t)x_3(t) \\ &\quad + p_9x_1^2(t)x_2(t) + p_{10}x_2^2(t)x_1(t) + p_{11}x_3^2(t)x_1(t) + \text{h.o.t.}, \\ f_2(t) &= q_3x_1^2(t-1) + q_4x_2^2(t-1) + q_5x_1(t-1)x_2(t-1) + q_6x_1^3(t-1) \\ &\quad + q_7x_2^3(t-1) + q_8x_1^2(t-1)x_2(t-1) + q_9x_1(t-1)x_2^2(t-1) + \text{h.o.t.}, \\ f_3(t) &= r_3x_1^2(t-1) + r_4x_3^2(t-1) + r_5x_1(t-1)x_3(t-1) + r_6x_1^3(t-1) \\ &\quad + r_7x_3^3(t-1) + r_8x_1^2(t-1)x_3(t-1) + r_9x_1(t-1)x_3^2(t-1) + \text{h.o.t.}. \end{aligned}$$

对 $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C([-1,0], R^3)$, 记

$$L_\mu\varphi = B_1\varphi(0) + B_2\varphi(-1), \quad h(\mu,\varphi) = (\tau_k + \mu)\begin{pmatrix} h_1(\mu,\varphi) \\ h_2(\mu,\varphi) \\ h_3(\mu,\varphi) \end{pmatrix},$$

这里

$$\begin{aligned} h_1(\mu,\varphi) &= p_4\varphi_1^2(0) + p_5\varphi_2^2(0) + p_6\varphi_3^2(0) + p_7\varphi_1(0)\varphi_2(0) + p_8\varphi_1(0)\varphi_3(0) \\ &\quad + p_9\varphi_1^2(0)\varphi_2(0) + p_{10}\varphi_2^2(0)\varphi_1(0) + p_{11}\varphi_3^2(0)\varphi_1(0) + \text{h.o.t.}, \\ h_2(\mu,\varphi) &= q_3\varphi_1^2(-1) + q_4\varphi_2^2(-1) + q_5\varphi_1(-1)\varphi_2(-1) + q_6\varphi_1^3(-1) \\ &\quad + q_7\varphi_2^3(-1) + q_8\varphi_1^2(-1)\varphi_2(-1) + q_9\varphi_1(-1)\varphi_2^2(-1) + \text{h.o.t.}, \\ h_3(\mu,\varphi) &= r_3\varphi_1^2(-1) + r_4\varphi_3^2(-1) + r_5\varphi_1(-1)\varphi_3(-1) + r_6\varphi_1^3(-1) \\ &\quad + r_7\varphi_3^3(-1) + r_8\varphi_1^2(-1)x_3(-1) + r_9\varphi_1(-1)\varphi_3^2(-1) + \text{h.o.t.}, \\ B_1 &= (\tau_k^{(j)} + \mu)\begin{pmatrix} p_1 & p_2 & p_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_2 = (\tau_k^{(j)} + \mu)\begin{pmatrix} 0 & 0 & 0 \\ q_2 & q_2 & 0 \\ r_1 & 0 & r_2 \end{pmatrix}. \end{aligned}$$

于是由 Riesz 表示定理, 存在分量为有界变差函数的二阶矩阵 $\eta(\theta,\mu) : [-1,0] \to R^{3^2}$. 使对任意的 $\varphi \in C([-1,0], R^3)$, 有

$$L_\mu\varphi = \int_{-1}^{0} d\eta(\theta,\mu)\varphi(\theta).$$

事实上, 只要取

$$\eta(\theta,\mu)=\begin{cases}-B_2, & \theta=-1,\\ 0, & -1<\theta<0,\\ B_1, & \theta=0\end{cases}$$

即可. 对 $\varphi\in C([-1,0],R^3)$, 定义

$$A(\mu)\varphi=\begin{cases}\frac{d\varphi(\theta)}{d\theta}, & -1\le\theta<0,\\ \int_{-1}^{0}d\eta(s,\mu)\varphi(s), & \theta=0,\end{cases}\quad R\varphi=\begin{cases}0, & -1\le\theta<0,\\ h(\mu,\varphi), & \theta=0.\end{cases}$$

于是 (3.1) 式可写成如下形式

$$\dot{x_t}=A(\mu)x_t+Rx_t, \tag{3.4}$$

其中 $x=(x_1,x_2,x_3)^T$. 对 $\alpha\in C([0,1],(R^3)^*)$, 定义

$$A^*\alpha(s)=\begin{cases}-\frac{d\alpha(s)}{ds}, & s\in(0,1],\\ \int_{-1}^{0}d\eta^T(t,0)\varphi(-t), & s=0.\end{cases}$$

对 $\varphi\in C([-1,0],R^3)$ 和 $\psi\in C([0,1],(R^3)^*)$, 定义双线性积

$$\langle\psi,\varphi\rangle=\overline{\psi}(0)\varphi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\psi^T(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi,$$

这里 $\eta(\theta)=\eta(\theta,0)$, 则算子 $A=A(0)$ 与 A^* 是共轭算子.

由第 2 节的讨论及变换 $t=s\tau$ 可知, $\pm\mathrm{i}\omega_k\tau_k^{(j)}$ 是算子 $A=A(0)$ 的特征值, 且 A 的其它的特征值都具有严格负实部, 从而 $\pm\mathrm{i}\omega_k\tau_k^{(j)}$ 也是算子 A^* 的特征值, 于是有

引理 3.1 $q(\theta)=(1,\alpha,\beta)^Te^{\mathrm{i}\omega_k\tau_k^{(j)}\theta}$ 是算子 $A(0)$ 关于 $\omega_k\tau_k^{(j)}\mathrm{i}$ 的特征向量, $q^*(s)=D(1,\alpha^*,\beta^*)^Te^{i\omega_k\tau_k^{(j)}s}$ 是算子 A^* 关于 $-\omega_k\tau_k^{(j)}\mathrm{i}$ 的特征向量并且 $\langle q^*,q\rangle=1,\langle q^*,\overline{q}\rangle=0$, 其中

$$\begin{aligned}\alpha &= -\frac{q_1e^{-\mathrm{i}\omega_k\tau_k^{(j)}}}{\mathrm{i}\omega_k-q_2e^{-\mathrm{i}\omega_k\tau_k^{(j)}}},\beta=-\frac{r_1}{\mathrm{i}\omega_k-r_2e^{-\mathrm{i}\omega_k\tau_k^{(j)}}},\\ \alpha^* &= -\frac{p_2}{q_2e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+\mathrm{i}\omega_k},\beta^*=-\frac{p_3}{r_3e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+\mathrm{i}\omega_k},\\ D &= \frac{1}{1+\bar{\alpha}\alpha^*+\bar{\beta}\beta^*+(\bar{q_1}\alpha^*+r_1\beta^*+\bar{q_2}\bar{\alpha}\alpha^*+r_1+r_2\bar{\beta}\beta^*)e^{\mathrm{i}\omega_k\tau_k^{(j)}}}.\end{aligned}$$

上述引理的证明直接计算即可, 故省略.

令 $z(t)=\langle q^*,x_t\rangle$, 其中 x_t 是方程 (3.4) 当 $\mu=0$ 时的解, 则由文 [25] 有

$$\frac{dz(t)}{dt}=\mathrm{i}\tau_k^{(j)}\omega_kz(t)+\bar{q^*}(0)\bar{f}(z,\bar{z}), \tag{3.5}$$

其中 $\bar{f}(z,\bar{z})=h[0,W(z,\bar{z})+2\mathrm{Re}\{zq\}]$, $W(z,\bar{z})=x_t-2\mathrm{Re}\{zq\}$. 而

$$W(z,\bar{z})=W_{20}\frac{z^2}{2}+W_{11}z\bar{z}+W_{02}\frac{\bar{z}^2}{2}+\cdots. \tag{3.6}$$

再把 (3.5) 式改写为

$$\frac{dz(t)}{dt} = \mathrm{i}\tau_k^{(j)}\omega_k z(t) + g(z,\bar{z}),$$

其中

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.7}$$

又

$$W^{'} = x_t^{'} + z^{'}q - \bar{z}^{'}\bar{q}. \tag{3.8}$$

将 (3.4), (3.5) 式代入 (3.8) 式得

$$\begin{aligned} W^{'} &= \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -1 \le \theta < 0, \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\} + \bar{f}, & \theta = 0, \end{cases} \\ &\overset{\text{def}}{=} AW + H(z,\bar{z},\theta), \end{aligned} \tag{3.9}$$

其中

$$H(z,\bar{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \tag{3.10}$$

且 $W^{'} = W_z z^{'} + W_{\bar{z}}\bar{z}^{'} = AW + H(z,\bar{z},\theta)$. 将上述相应的级数展式代入, 比较系数得

$$(A - 2\mathrm{i}\tau_k\omega_k)W_{20} = -H_{20}(\theta), \tag{3.11}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{3.12}$$

注意到 $x_t(\theta) = (x_{1t}(\theta), x_{3t}(\theta), x_{3t}(\theta)) = W(z,\bar{z},\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ 和 $q(\theta) = (1,\alpha,\beta)^T e^{\mathrm{i}\theta\omega_k\tau_k^{(j)}}$, 有

$$\begin{aligned}
x_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3), \\
x_{2t}(0) &= z\alpha + \bar{z}\bar{\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3), \\
x_{3t}(0) &= z\beta + \bar{z}\bar{\beta} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + O(|z,\bar{z})|^3, \\
x_{1t}(-1) &= ze^{-\mathrm{i}\omega_k\tau_k^{(j)}} + \bar{z}e^{\mathrm{i}\omega_k\tau_k^{(j)}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3), \\
x_{2t}(-1) &= z\alpha e^{-\mathrm{i}\omega_k\tau_k^{(j)}} + \bar{z}\bar{\alpha}e^{\mathrm{i}\omega_k\tau_k^{(j)}} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3), \\
x_{3t}(-1) &= z\beta e^{-\mathrm{i}\omega_k\tau_k^{(j)}} + \bar{z}\bar{\beta}e^{\mathrm{i}\omega_k\tau_k^{(j)}} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + O(|z,\bar{z}|^3).
\end{aligned}$$

于是

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)\bar{f}(z,\bar{z}) \\
&= \bar{D}\tau_k^{(j)}(1,\bar{\alpha^*},\bar{\beta}^*)\begin{pmatrix} p_4x_{1t}^2(0) + p_5x_{2t}^2(0) + p_6x_{3t}^2(0) + p_7x_{1t}(0)x_{2t}(0) + p_8x_{1t}(0)x_{3t}(0) \\ +p_9x_{1t}^2(0)x_{2t}(0) + p_{10}x_{2t}^2(0)x_{1t}(0) + p_{11}x_{3t}^2(0)x_{1t}(0) + \text{h.o.t.} \\ q_3x_{1t}^2(-1) + q_4x_{2t}^2(-1) + q_5x_{1t}(-1)x_{2t}(-1) + q_6x_{1t}^3(-1) \\ +q_7x_{2t}^3(-1) + q_8x_{1t}^2(-1)x_{2t}(-1) + q_9x_{1t}(-1)x_{2t}^2(-1) + \text{h.o.t.} \\ r_3x_{1t}^2(-1) + r_4x_{3t}^2(-1) + r_5x_{1t}(-1)x_{3t}(-1) + r_6x_{1t}^3(-1) \\ +r_7x_{3t}^3(-1) + r_8x_{1t}^2(-1)x_{3t}(-1) + r_9x_{1t}(-1)x_{3t}^2(-1) + \text{h.o.t.} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \bar{D}\tau_k^{(j)}\Big\{ \Big[(p_4+p_5\alpha^2+p_6\beta^2+p_7\alpha+p_8\beta)+\alpha^*(q_3+q_4\alpha^2+q_5\alpha)e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +\beta^*(r_3+r_4\beta^2+q_5\beta)e^{-2\mathrm{i}\omega_k\tau_k^{(j)}}\Big]z^2+\Big[(2p_4+2p_5|\alpha|^2+2p_6|\beta|^2+2p_7\mathrm{Re}\{\alpha\}) \\
& +\alpha^*(2q_3+2q_4|\alpha|^2+2q_5\mathrm{Re}\{\alpha\})+\beta^*(2r_3+2r_4|\alpha|^2+2r_5\mathrm{Re}\{\alpha\})\Big]z\bar{z} \\
& +\Big[(p_4+p_5\bar{\alpha}^2+p_6\bar{\beta}^2+p_7\bar{\alpha}+p_8\bar{\beta})+\alpha^*(q_3+q_4\bar{\alpha}^2+q_5\bar{\alpha})e^{2\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +\beta^*(r_3+r_4\bar{\beta}^2+r_5\bar{\beta})e^{2\mathrm{i}\omega_k\tau_k^{(j)}}\Big]\bar{z}^2+\Big[p_4(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0))+p_5(W_{20}^{(2)}(0)+2W_{11}^{(2)}(0)) \\
& +p_6(W_{20}^{(3)}(0)+2W_{11}^{(3)}(0))+p_7(W_{11}^{(2)}(0)+\alpha W_{11}^{(1)}(0)+\frac{1}{2}W_{20}^{(2)}(0)+\frac{1}{2}\bar{\alpha}W_{20}^{(1)}(0)) \\
& +p_8(\beta W_{11}^{(1)}(0)+W_{11}^{(3)}(0)+\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)+\frac{1}{2}W_{20}^{(3)}(0))+2p_9\mathrm{Re}\{\alpha\}+p_{10}(\alpha^2+2|\alpha|^2) \\
& +p_{11}(\beta^2+2|\beta|^2)+\alpha^*(q_3(2W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+W_{20}^{(1)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}}) \\
& +q_4(2\alpha W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+W_{20}^{(2)}(-1)\bar{\alpha}e^{2\mathrm{i}\omega_k\tau_k^{(j)}})+q_5(W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +\frac{1}{2}W_{20}^{(2)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}}+\alpha W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}})+2q_6e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+2q_7\beta^2\bar{\beta}e^{-\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +2q_8\mathrm{Re}\{\alpha\}e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+q_9(\alpha^2+|\alpha|^2)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+\beta^*(q_3(2W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +W_{20}^{(1)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}})+q_4(2\alpha W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+W_{20}^{(2)}(-1)\bar{\alpha}e^{2\mathrm{i}\omega_k\tau_k^{(j)}}) \\
& +q_5(W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+\frac{1}{2}W_{20}^{(2)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}}+\alpha W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}) \\
& +2q_6e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+2q_7\beta^2\bar{\beta}e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+2q_8\mathrm{Re}\{\alpha\}e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+q_9(\alpha^2+|\alpha|^2)e^{-\mathrm{i}\omega_k\tau_k^{(j)}})\Big]z^2\bar{z}+\cdots\Big\}.
\end{aligned}
$$

与 (3.7) 式比较得

$$
\begin{aligned}
g_{20} \;=\; & 2\bar{D}\tau_k^{(j)}\Big[(p_4+p_5\alpha^2+p_6\beta^2+p_7\alpha+p_8\beta)+\alpha^*(q_3+q_4\alpha^2+q_5\alpha)e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +\beta^*(r_3+r_4\beta^2+q_5\beta)e^{-2\mathrm{i}\omega_k\tau_k^{(j)}}\Big], \\
g_{11} \;=\; & \bar{D}\tau_k^{(j)}\big[(2p_4+2p_5|\alpha|^2+2p_6|\beta|^2+2p_7\mathrm{Re}\{\alpha\})+\alpha^*(2q_3+2q_4|\alpha|^2+2q_5\mathrm{Re}\{\alpha\}) \\
& +\beta^*(2r_3+2r_4|\alpha|^2+2r_5\mathrm{Re}\{\alpha\})\big], \\
g_{02} \;=\; & 2\bar{D}\tau_k^{(j)}\Big[(p_4+p_5\bar{\alpha}^2+p_6\bar{\beta}^2+p_7\bar{\alpha}+p_8\bar{\beta})+\alpha^*(q_3+q_4\bar{\alpha}^2+q_5\bar{\alpha})e^{2\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +\beta^*(r_3+r_4\bar{\beta}^2+r_5\bar{\beta})e^{2\mathrm{i}\omega_k\tau_k^{(j)}}\Big], \\
g_{21} \;=\; & 2\bar{D}\tau_k^{(j)}\Big[p_4(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0))+p_5(W_{20}^{(2)}(0)+2W_{11}^{(2)}(0))+p_6(W_{20}^{(3)}(0) \\
& +2W_{11}^{(3)}(0))+p_7(W_{11}^{(2)}(0)+\alpha W_{11}^{(1)}(0)+\frac{1}{2}W_{20}^{(2)}(0)+\frac{1}{2}\bar{\alpha}W_{20}^{(1)}(0)) \\
& +p_8(\beta W_{11}^{(1)}(0)+W_{11}^{(3)}(0)+\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)+\frac{1}{2}W_{20}^{(3)}(0))+2p_9\mathrm{Re}\{\alpha\} \\
& +p_{10}(\alpha^2+2|\alpha|^2)+p_{11}(\beta^2+2|\beta|^2)+\alpha^*(q_3(2W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}} \\
& +W_{20}^{(1)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}})+q_4(2\alpha W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+W_{20}^{(2)}(-1)\bar{\alpha}e^{2\mathrm{i}\omega_k\tau_k^{(j)}})
\end{aligned}
$$

$$
\begin{aligned}
&+q_5(W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+\frac{1}{2}W_{20}^{(2)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}}+\alpha W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}})\\
&+2q_6e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+2q_7\beta^2\bar{\beta}e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+2q_8\mathrm{Re}\{\alpha\}e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+q_9(\alpha^2+|\alpha|^2)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}\\
&+\beta^*(q_3(2W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+W_{20}^{(1)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}})+q_4(2\alpha W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}\\
&+W_{20}^{(2)}(-1)\bar{\alpha}e^{2\mathrm{i}\omega_k\tau_k^{(j)}})+q_5(W_{11}^{(2)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+\frac{1}{2}W_{20}^{(2)}(-1)e^{\mathrm{i}\omega_k\tau_k^{(j)}}\\
&+\alpha W_{11}^{(1)}(-1)e^{-\mathrm{i}\omega_k\tau_k^{(j)}})+2q_6e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+2q_7\beta^2\bar{\beta}e^{-\mathrm{i}\omega_k\tau_k^{(j)}}+2q_8\mathrm{Re}\{\alpha\}e^{-\mathrm{i}\omega_k\tau_k^{(j)}}\\
&+q_9(\alpha^2+|\alpha|^2)e^{-\mathrm{i}\omega_k\tau_k^{(j)}})\Big].
\end{aligned}
$$

下面来确定 $W_{11}(\theta), W_{20}(\theta)$.

由 (3.9) 式, 对 $\theta\in[-1,0)$,

$$H(z,\bar{z},\theta)=-\bar{q^*}(0)\bar{f}q(\theta)-q^*(0)\bar{f}\bar{q}(\theta)=-g(z,\bar{z})q(\theta)-\bar{g}(z,\bar{z})\bar{q}(\theta). \tag{3.13}$$

比较 (3.10) 和 (3.13) 式的同次幂系数, 得

$$H_{20}=-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta), \tag{3.14}$$

$$H_{11}=-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta). \tag{3.15}$$

由 (3.10), (3.14) 式及 A 的定义, 得

$$\dot{W}_{20}(\theta)=2\mathrm{i}\omega_k\tau_k^{(j)}W_{20}(\theta)+g_{20}q(\theta)+\bar{g_{02}}\bar{q}(\theta). \tag{3.16}$$

注意到 $q(\theta)=(1,\alpha,\beta)^Te^{\mathrm{i}\omega_k\tau_k^{(j)}\theta}$ 得

$$W_{20}(\theta)=\frac{\mathrm{i}g_{20}}{\omega_k\tau_k^{(j)}}q(0)e^{\mathrm{i}\omega_k\tau_k^{(j)}\theta}+\frac{i\bar{g}_{02}}{3\omega_k\tau_k^{(j)}}\bar{q}(0)e^{-i\omega_k\tau_k^{(j)}\theta}+E_1e^{2i\omega_k\tau_k^{(j)}\theta}, \tag{3.17}$$

其中 $E_1=(E_1^{(1)},E_1^{(2)},E_1^{(3)})^T\in R^3$ 为常向量. 同理, 由 (3.10), (3.15) 式及 A 的定义, 得

$$W_{11}(\theta)=-\frac{\mathrm{i}g_{11}}{\omega_k\tau_k^{(j)}}q(0)e^{\mathrm{i}\omega_k\tau_k^{(j)}\theta}+\frac{i\bar{g}_{11}}{\omega_k\tau_k^{(j)}}\bar{q}(0)e^{-i\omega_k\tau_k^{(j)}\theta}+E_2, \tag{3.18}$$

其中 $E_2=(E_2^{(1)},E_2^{(2)},E_1^{(3)})^T\in R^3$ 为常向量. 现在只需求 (3.17) 式中的 E_1, (3.18) 式中的 E_2. 由 (3.11) 和 (3.12) 式及 A 的定义有

$$\int_{-1}^0 d\eta(\theta)W_{20}(\theta)=2\mathrm{i}\omega_k\tau_k^{(j)}W_{20}(0)-H_{20}(0), \tag{3.19}$$

$$\int_{-1}^0 d\eta(\theta)W_{11}(\theta)=-H_{11}(0), \tag{3.20}$$

其中 $\eta(\theta)=\eta(0,\theta)$. 由 (3.9) 和 (3.10) 式得

$$H_{20}(0)=-g_{20}q(0)-\bar{g_{02}}\bar{q}(0)+2\tau_k^{(j)}(H_1,H_2,H_3)^T, \tag{3.21}$$

$$H_{11}(0)=-g_{11}q(0)-\bar{g_{11}}(0)\bar{q}(0)+2\tau_k^{(j)}(P_1,P_2,P_3)^T, \tag{3.22}$$

这里

$$\begin{aligned} H_1 &= p_4+p_5\alpha^2+p_6\beta^2+p_7\alpha+p_8\beta, H_2=q_3+q_4\alpha^2+q_5\alpha)e^{-2\mathrm{i}\omega_k\tau_k^{(j)}},\\ H_3 &= (r_3+r_4\beta^2+q_5\beta)e^{-2\mathrm{i}\omega_k\tau_k^{(j)}},\\ P_1 &= p_4+p_5|\alpha|^2+p_6|\beta|^2+p_7\mathrm{Re}\{\alpha\}, P_2=q_3+q_4|\alpha|^2+q_5\mathrm{Re}\{\alpha\},\\ P_3 &= r_3+r_4|\alpha|^2+r_5\mathrm{Re}\{\alpha\}. \end{aligned}$$

注意到

$$\left(\mathrm{i}\omega_k\tau_k^{(j)}I-\int_{-1}^{0}e^{\mathrm{i}\theta\omega_k\tau_k^{(j)}\theta}d\eta(\theta)\right)q(0)=0, \tag{3.23}$$

$$\left(-\mathrm{i}\omega_k\tau_k^{(j)}I-\int_{-1}^{0}e^{-\mathrm{i}\omega_k\tau_k^{(j)}\theta}d\eta(\theta)\right)\bar{q}(0)=0. \tag{3.24}$$

把 (3.17) 和 (3.21) 式代入 (3.19) 式得

$$\left(2\mathrm{i}\omega_k\tau_k^{(j)}I-\int_{-1}^{0}e^{2\mathrm{i}\omega_k\tau_k^{(j)}\theta}d\eta(\theta)\right)E_1=2\tau_k^{(j)}(H_1,H_2,H_3)^T. \tag{3.25}$$

也就是

$$\left(2\mathrm{i}\omega_kI-B_1-B_2e^{-2\mathrm{i}\omega_k\tau_k^{(j)}}\right)E_1=2\tau_k^{(j)}(H_1,H_2,H_3)^T, \tag{3.26}$$

从而

$$\begin{pmatrix} 2\mathrm{i}\omega_0-p_1 & -p_2 & -p_3\\ -q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 2\mathrm{i}\omega_0-q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 0\\ -r_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 0 & 2\mathrm{i}\omega_0-r_2e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} \end{pmatrix}\begin{pmatrix}E_1^{(1)}\\E_1^{(2)}\\E_1^{(3)}\end{pmatrix}=2\begin{pmatrix}H_1\\H_2\\H_3\end{pmatrix}. \tag{3.27}$$

于是 $E_1^{(1)}=\frac{\Delta_{11}}{\Delta_1}$, $E_1^{(2)}=\frac{\Delta_{12}}{\Delta_1}$, $E_1^{(3)}=\frac{\Delta_{13}}{\Delta_1}$, 这里

$$\begin{aligned} \Delta_1 &= \det\begin{pmatrix} 2\mathrm{i}\omega_0-p_1 & -p_2 & -p_3\\ -q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 2\mathrm{i}\omega_0-q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 0\\ -r_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 0 & 2\mathrm{i}\omega_0-r_2e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} \end{pmatrix},\\ \Delta_{11} &= 2\det\begin{pmatrix} H_1 & -p_2 & -p_3\\ H_2 & 2\mathrm{i}\omega_0-q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 0\\ H_3 & 0 & 2\mathrm{i}\omega_0-r_2e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} \end{pmatrix},\\ \Delta_{12} &= 2\det\begin{pmatrix} 2\mathrm{i}\omega_0-p_1 & H_1 & -p_3\\ -q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & H_2 & 0\\ -r_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & H_3 & 2\mathrm{i}\omega_0-r_2e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} \end{pmatrix},\\ \Delta_{13} &= 2\det\begin{pmatrix} 2\mathrm{i}\omega_0-p_1 & -p_2 & H_1\\ -q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 2\mathrm{i}\omega_0-q_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & H_2\\ -r_1e^{-2\mathrm{i}\omega_k\tau_k^{(j)}} & 0 & H_3 \end{pmatrix}. \end{aligned}$$

类似地, 把 (3.18) 和 (3.22) 式代入 (3.20) 式, 得

$$\left(\int_{-1}^{0} d\eta(\theta)\right) E_2 = 2(P_1, P_2)^T. \tag{3.28}$$

于是

$$(B+C)E_2 = 2(-P_1, -P_2, -p_3)^T. \tag{3.29}$$

也就是

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & 0 \\ r_1 & 0 & r_2 \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = 2 \begin{pmatrix} -P_1 \\ -P_2 \\ -P_3 \end{pmatrix}. \tag{3.30}$$

从而

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},$$

其中

$$\begin{aligned} \Delta_2 &= \det \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & 0 \\ r_1 & 0 & r_2 \end{pmatrix}, \quad \Delta_{21} = 2\det \begin{pmatrix} -P_1 & p_2 & p_3 \\ -P_2 & q_2 & 0 \\ -P_3 & 0 & r_2 \end{pmatrix}, \\ \Delta_{22} &= 2\det \begin{pmatrix} p_1 & -P_1 & p_3 \\ q_1 & -P_2 & 0 \\ r_1 & -P_3 & r_2 \end{pmatrix}, \quad \Delta_{23} = 2\det \begin{pmatrix} p_1 & p_2 & -P_1 \\ q_1 & q_2 & -P_2 \\ r_1 & 0 & -P_3 \end{pmatrix}. \end{aligned}$$

根据 (3.17) 和 (3.18) 式, 可以计算 g_{21} 的值, 于是可以计算如下数值

$$\begin{aligned} c_1(0) &= \frac{\mathrm{i}}{2\omega_k \tau_k^{(j)}} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_k^{(j)})\}}, \\ \beta_2 &= 2\mathrm{Re}(c_1(0)), \\ T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2 \mathrm{Im}\{\lambda'(\tau_k^{(j)})\}}{\omega_k \tau_k^{(j)}}. \end{aligned}$$

定理 3.1　假设条件 (H1) 和 (H2) 成立, 则 $\mu = 0$ 是系统 (1.2) 的 Hopf 分支值. 分支方向由 μ 确定, 如果 $\mu > 0$ ($\mu < 0$), 则系统 (1.2) 有超临界分支 (次临界分支); 分支周期解稳定性由 β_2 确定, 如果 $\beta_2 > 0$ ($\beta_2 < 0$), 分支周期解是稳定的 (不稳定的); 分支周期解的周期由 T_2 确定, 如果 $T_2 > 0$ ($T_2 < 0$), 分支周期解的周期是增加的 (减小的).

4 数值模拟

在这一节, 取不同的时滞 τ 对系统 (1.2) 进行数值模拟, 考虑下列系统

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)\left[0.5 - 0.2x_1(t) - \frac{3x_2(t)}{12x_2(t)+x_1(t)} - \frac{0.3x_3(t)}{10x_3(t)+x_1(t)}\right], \\ \frac{dx_2(t)}{dt} = x_2(t)\left[-0.2 + \frac{0.82x_1(t-\tau_1)}{12x_2(t-\tau_1)+x_1(t-\tau_1)}\right], \\ \frac{dx_3(t)}{dt} = x_3(t)\left[-0.2 + \frac{0.5x_1(t-\tau_2)}{10x_3(t-\tau_2)+x_1(t-\tau_2)}\right]. \end{cases} \tag{4.1}$$

显然, 系统 (4.1) 有正平衡点 $E^*(1.465, 0.375, 0.22)$ 并且满足定理 2.1 的条件, 经计算得 $\tau_0 = 8.5, \lambda'(\tau_0) = 0.1275 - 0.7182\mathrm{i}$, 从而, $c_1(0) = -12.0332 - 21.2541\mathrm{i}, \mu_2 > 0, \beta_2 < 0, T_2 > 0$. 故得到当 $\tau < \tau_0 = 8.5$ 时, 正平衡点 $E^*(1, 0.5)$ 是渐近稳定的, 当 τ 通过临界值 $\tau_0 = 8.5$ 时, 正平衡点 $E^*(1, 0.5)$ 失去稳定性, Hopf 分支产生, 由 $\mu_2 > 0, \beta_2 < 0$ 知 Hopf 分支为超临界的, 分支方向为 $\tau > \tau_0$ 且从 $E^*(1.465, 0.375, 0.22)$ 附近产生的 Hopf 分支周期解是稳定的, 其数值模拟见图 1–2.

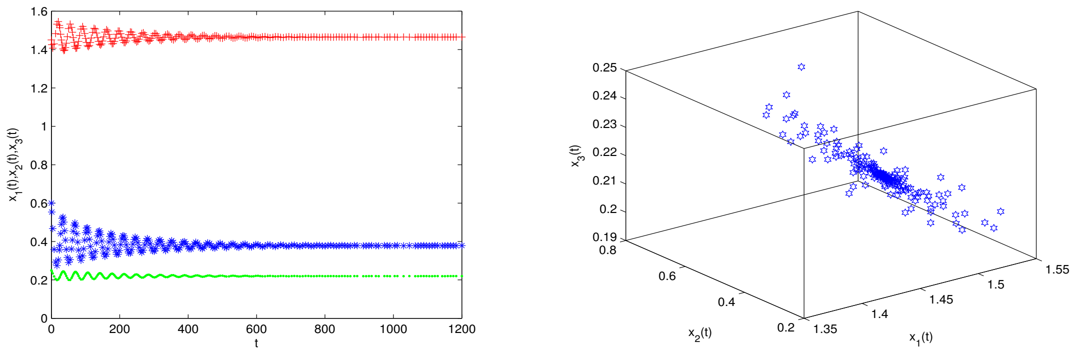

图 1: 当 $\tau = 8.2 < \tau_0 \approx 8.5$ 时, 系统 (4.1) 的波形图和相图, 正平衡点 $E^*(1.465, 0.375, 0.22)$ 是渐近稳定的, 初值为 (1.45,0.65,0.25).

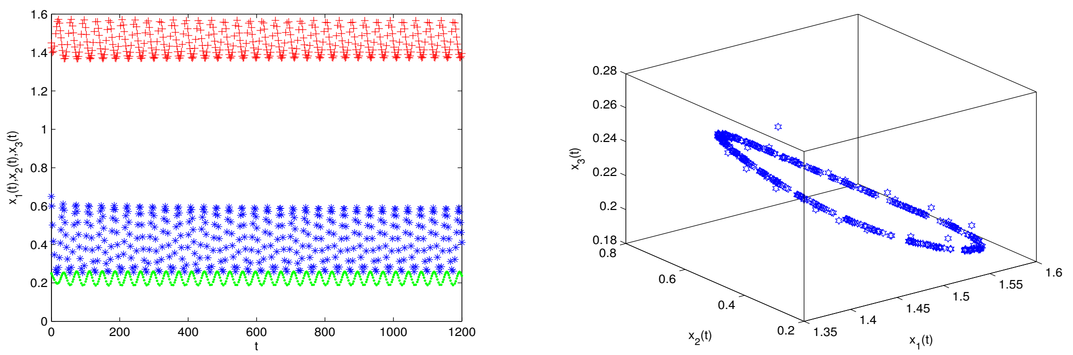

图 2: 当 $\tau = 8.8 > \tau_0 \approx 8.5$ 时, 系统 (4.1) 的波形图和相图, 平衡点 $E^*(1.465, 0.375, 0.22)$ 处 Hopf 分支产生, 初值为 (1.45,0.65,0.25).

参 考 文 献

[1] Xu Rui, Gan Qintao, Ma Zhien. Stability and bifurcation analysis on a ratio-dependent predator-prey model with time delay[J]. J. Comput. Appl. Math., 2009, 230(1): 187–203.

[2] Xu Changjin, Tang Xianhua, Liao Maoxin. Stability and bifurcation analysis of a delayed predator-prey model of prey dispersal in two-patch environments[J]. Appl. Math. Comput., 2010, 216(10): 2920–2936.

[3] Duque C, Kiss K, Lizana M. On the dynamics of an n-dimensional ratio-dependent predator-prey system with diffusion[J]. Appl. Math. Comput., 2009, 208: 98–105.

[4] Dai Binxiang, Li Ying, Luo Zhengguo. Multiple periodic solutions for impulsive Gause-type ratio-dependent predator-prey systems with non-monotonic numerical responses[J]. Appl. Math. Comput., 2011, 217(18): 7478–7487.

[5] Xiao Min, Cao Jinde. Hopf bifurcation and non-hyperbolic equilibrium in a ratio-dependent predator-prey model with linear harvesting rate: analysis and computation[J]. Math. Comp. Model., 2009, 50(3-4): 360–379.

[6] Zhang Zhengqiu, Hou Zhenting. Existence of four positive periodic solutions for a ratio-dependent predator-prey system with multiple exploited (or harvesting) terms[J]. Nonl. Anal.: Real World Appl., 2010, 11(3): 1560–1571.

[7] Ding Xiaoquan, Wang Fangfang. Positive periodic solution for a semi-ratio-dependent predator-prey system with diffusion and time delays[J]. Nonl. Anal.: Real World Appl., 2008, 9(2): 239–249.

[8] Song Yongli, Peng Yahong, Wei Junjie. Bifurcations for a predator-prey system with two delays[J]. J. Math. Anal. Appl., 2008, 337(1): 466–479.

[9] Xiao Dongmei, Li Wenxue, Han Maoan. Dynamics in a ratio-dependent predator-prey model with predator harvesting[J]. J. Math. Anal. Appl., 2006, 324(1): 14–29.

[10] Yuan Sanling, Song Yongli. Stability and Hopf bifurcations in a delayed Leslie-Gower predator-prey system[J]. J. Math. Anal. Appl., 2009, 355: 82–100.

[11] Hu Haijun, Huang Lihong. Stability and Hopf bifurcation in a delayed predator-prey system with stage structure for prey[J]. Nonl. Anal.: Real World Appl., 2010, 11(4): 2757–2769.

[12] 谭德君. 基于比率的三种群捕食系统的持续生存 [J]. 生物数学学报, 2003, 18(1): 50–56

[13] 徐瑞, 陈兰荪. 具有时滞和基于比率的三种群捕食系统的持久性与稳定性 [J]. 系统科学与数学, 2001, 21(2): 204–212.

[14] 黄建科, 吴筱宁. 比率型 3 种群捕食模型的持久性与全局渐近稳定性 [J]. 信息工程大学学报, 2007, 8(4): 396–402.

[15] Song Yongli, Zou Xinfu. Bifurcation analysis of a diffusive ratio-dependent predator-prey model[J]. Nonl. Dyn., 2014, 78(1): 49–70.

[16] Shi Hongbo, Li Yan. Global asymptotic stability of a diffusive predator-prey model with ratio-dependent functional response[J]. Appl. Math. Comput., 2015, 250: 71–77.

[17] Bai Ling, Li Jingshi, Zhang Kai, Zhao Wenju. Analysis of a stochastic ratio-dependent predator-prey model driven by Lévy noise[J]. Appl. Math. Comput., 2014, 233: 480–493.

[18] Song Yongli, Peng Yahong, Zou Xinfu. Persistence, stability and Hopf bifurcation in a diffusive ratio-dependent predator-prey model with delay[J]. Intern. J. Bifur. Chaos, 2014, 24(7): 1450093.

[19] Fan Yonghong, Wang Linlin. Multiplicity of periodic solutions for a delayed ratio-dependent predator-prey model with monotonic functional response and harvesting terms[J]. Appl. Math. Comput., 2014, 244: 878–894.

[20] Banerjee M, Zhang Lai. Influence of discrete delay on pattern formation in a ratio-dependent prey-predator model[J]. Chaos, Solitons Fractals, 2014, 67: 73–81.

[21] Liu Xixian, Li Biwen, Chen Boshan. Global stability for a predator-prey model with disease in the prey[J]. J. Math., 2015, 35(1): 85–94.

[22] Li Xiaojuan. Global solutions for the ratio-dependent food-chain model with cross-diffusion[J]. J. Math., 2015, 35(2): 267–280.

[23] Huston V. The existence of an equilibrium for permanence systems[J]. Rocky Mountain J. Math., 1990, 20(4): 1033-1040.

[24] Hale J, Lunel S V. Introduction to functional differential equations[M]. New York: Springer-Verlag, 1993.

[25] Hassard B, Kazarinoff N, Wan Y H. Theory and application of Hopf bifurcation[M]. London Math. Soc. Lec. Notes, Series 41, Cambridge: Cambridge University Press, 1981.

BIFURCATION ANALYSIS FOR A THREE-SPECIES RATIO-DEPENDENT PREDATOR-PREY MODEL WITH DELAY

XU Chang-jin[1], LIAO Mao-xin[2]

(*1.Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, China*)

(*2.School of Mathematics and Physics, University of South China, Hengyang 421001, China*)

Abstract: In this paper, a class of three-species ratio-dependent predator-prey model is investigated. By analyzing the characteristic equation of the model, the stability at the positive equilibrium is proved. By choosing the delay τ as a bifurcation parameter, we show that Hopf bifurcation can occur when the delay τ passes a sequence of critical values. Meanwhile, using the center manifold theory and normal form approach, we derive the formulae for determining the properties of Hopf bifurcating periodic orbit, such as the direction of Hopf bifurcation, the stability of Hopf bifurcating periodic solution and the periodic of Hopf bifurcating periodic solution. Finally, numerical simulations are carried out to illustrate the analytical results. Our results complement some previously known ones.

Keywords: predator-prey system; Hopf bifurcation; stability; delay; ratio-dependent

2010 MR Subject Classification: 34K18; 34K20

附录 A

$$
\begin{aligned}
p_1 &= \left[\frac{a_{12}x_2^*}{(m_{12}x_2^*+x_1^*)^2}+\frac{a_{13}x_3^*}{(m_{13}x_3^*+x_1^*)^2}-a_{11}\right]x_1^*,\\
p_2 &= \left[\frac{a_{12}m_{12}x_2^*}{(m_{12}x_2^*+x_1^*)^2}-\frac{a_{12}}{m_{12}x_2^*+x_1^*}\right]x_1^*,\\
p_3 &= \left[\frac{a_{13}m_{13}x_3^*}{(m_{13}x_3^*+x_1^*)^2}+\frac{a_{13}}{m_{13}x_3^*+x_1^*}\right]x_1^*,\\
p_4 &= \left[\frac{a_{12}x_2^*}{(m_{12}x_2^*+x_1^*)^2}+\frac{a_{13}x_3^*}{(m_{13}x_3^*+x_1^*)^2}-a_{11}\right]\left[\frac{a_{13}x_3^*}{(m_{13}x_3^*+x_1^*)^3}-\frac{a_{12}x_2^*}{(m_{12}x_2^*+x_1^*)^2}\right],\\
p_5 &= \left[\frac{a_{12}m_{12}}{(m_{12}x_2^*+x_1^*)^2}-\frac{a_{12}}{m_{12}^2x_2^*+x_1^*}\right]x_1^*, p_6=\left[\frac{a_{13}m_{13}}{(m_{13}x_2^*+x_1^*)^2}+\frac{a_{13}}{(m_{13}^2x_3^*+x_1^*)^3}\right]x_1^*,\\
p_7 &= \left[\frac{a_{12}-2a_{12}x_2^*}{(m_{13}x_2^*+x_1^*)^2}\right]x_1^*+\left[\frac{a_{12}m_{12}x_2^*}{(m_{12}x_2^*+x_1^*)^2}-\frac{a_{12}}{m_{12}x_2^*+x_1^*}\right],\\
p_8 &= \left[\frac{a_{13}m_{13}x_3^*}{(m_{13}x_3^*+x_1^*)^2}+\frac{a_{13}}{m_{13}x_3^*+x_1^*}\right]\left[\frac{a_{13}-2a_{12}m_{13}x_3^*}{(m_{13}x_3^*+x_1^*)^3}\right]x_1^*,
\end{aligned}
$$

$$
\begin{aligned}
p_9 &= \left[\frac{3a_{12}m_{12}x_2^*}{(m_{12}x_2^*+x_1^*)^3}-\frac{a_{12}}{(m_{12}x_2^*+x_1^*)^2}\right]\left[\frac{a_{12}-2a_{12}x_2^*}{(m_{13}x_2^*+x_1^*)^2}\right]x_1^*,\\
p_{10} &= \left[\frac{3a_{12}m_{12}^2x_2^*}{(m_{12}x_2^*+x_1^*)^3}-\frac{2a_{12}}{(m_{12}x_2^*+x_1^*)^2}\right]\left[\frac{a_{12}m_{12}}{(m_{12}x_2^*+x_1^*)^2}-\frac{a_{12}}{m_{12}^2x_2^*+x_1^*}\right]x_1^*,\\
p_{11} &= \left[\frac{3a_{13}m_{13}^2x_3^*}{(m_{13}x_3^*+x_1^*)^4}-\frac{2a_{13}}{(m_{13}x_3^*+x_1^*)^3}\right]\left[\frac{a_{13}m_{13}x_3^*}{(m_{13}x_3^*+x_1^*)^2}+\frac{a_{13}}{m_{13}x_3^*+x_1^*}\right]x_1^*,\\
q_1 &= \left[\frac{a_{21}}{(m_{12}x_2^*+x_1^*)}-\frac{a_{21}x_1^*}{(m_{12}x_2^*+x_1^*)^2}\right]x_2^*, q_2=-\left[\frac{a_{21}m_{12}x_1^*}{(m_{12}x_2^*+x_1^*)^2}\right]x_2^*,\\
q_3 &= \left[\frac{a_{21}m_{12}^2x_1^*x_2^*}{(m_{12}x_2^*+x_1^*)^3}-\frac{a_{21}m12x_1^*}{(m_{12}x_2^*+x_1^*)^2}\right],\\
q_4 &= \left[\frac{2a_{21}m_{12}x_1^*}{(m_{12}x_2^*+x_1^*)^3}-\frac{a_{21}}{(m_{12}x_3^*+x_1^*)^2}\right]\left[\frac{a_{21}}{(m_{12}x_2^*+x_1^*)}-\frac{a_{21}x_1^*}{(m_{12}x_2^*+x_1^*)^2}-a_{11}\right]x_2^*,\\
q_5 &= \left[\frac{2a_{21}m_{12}x_1^*}{(m_{12}x_2^*+x_1^*)^3}-\frac{a_{21}m_{12}}{(m_{12}^2x_2^*+x_1^*)^2}\right]\left[\frac{a_{21}}{m_{12}x_2^*+x_1^*}-\frac{a_{21}x_1^*}{(m_{12}x_2^*+x_1^*)^2}\right]x_2^*,\\
q_6 &= \left[\frac{a_{21}}{(m_{12}x_2^*+x_1^*)^3}\right]x_2^*, q_7=\left[\frac{a_{21}m_{12}^3x_1^*}{(m_{12}x_2^*+x_1^*)^4}\right]x_2^*,\\
q_8 &= \left[\frac{3a_{21}m_{12}x_1^*}{(m_{12}x_2^*+x_1^*)^4}+\frac{2a_{21}m_{12}x_1^*}{(m_{12}^2x_2^*+x_1^*)^3}\right]\left[\frac{a_{21}}{(m_{12}x_2^*+x_1^*)}-\frac{a_{21}x_1^*}{(m_{12}x_2^*+x_1^*)^2}-a_{11}\right]x_2^*,\\
q_9 &= \left[\frac{3a_{21}m_{12}^2x_1^*}{(m_{12}x_2^*+x_1^*)^4}+\frac{a_{21}m_{12}^2x_1^*}{(m_{12}^2x_2^*+x_1^*)^3}\right]\left[\frac{2a_{21}m_{12}x_1^*}{(m_{12}x_2^*+x_1^*)^3}-\frac{a_{21}}{(m_{12}x_3^*+x_1^*)^2}\right]x_2^*,\\
r_1 &= \left[\frac{a_{31}}{(m_{13}x_3^*+x_1^*)}-\frac{a_{31}x_1^*}{(m_{13}x_3^*+x_1^*)^2}\right]x_2^*, r_2=-\left[\frac{a_{31}m_{13}x_1^*}{(m_{13}x_3^*+x_1^*)^2}\right]x_3^*,\\
r_3 &= \left[\frac{a_{31}m_{13}^2x_1^*x_3^*}{(m_{13}x_3^*+x_1^*)^3}-\frac{a_{31}m13x_1^*}{(m_{13}x_3^*+x_1^*)^2}\right],\\
r_4 &= \left[\frac{2a_{31}m_{13}x_1^*}{(m_{13}x_3^*+x_1^*)^3}-\frac{a_{31}}{(m_{13}x_3^*+x_1^*)^2}\right]\left[\frac{a_{21}}{(m_{13}x_3^*+x_1^*)}-\frac{a_{31}x_1^*}{(m_{12}x_3^*+x_1^*)^2}\right]x_2^*,\\
r_5 &= \left[\frac{2a_{31}m_{13}x_1^*}{(m_{13}x_3^*+x_1^*)^3}-\frac{a_{31}m_{13}}{(m_{13}^2x_2^*+x_1^*)^2}\right]\left[\frac{a_{31}}{m_{13}x_3^*+x_1^*}-\frac{a_{31}x_1^*}{(m_{13}x_2^*+x_1^*)^2}\right]x_2^*,\\
r_6 &= \left[\frac{a_{31}}{(m_{13}x_2^*+x_1^*)^3}\right]x_3^*, r_7=\left[\frac{a_{31}m_{13}^3x_1^*}{(m_{13}x_3^*+x_1^*)^4}\right]x_2^*,\\
r_8 &= \left[\frac{3a_{31}m_{13}x_1^*}{(m_{13}x_3^*+x_1^*)^4}+\frac{2a_{31}m_{13}x_1^*}{(m_{13}^2x_3^*+x_1^*)^3}\right]\left[\frac{a_{31}}{(m_{13}x_3^*+x_1^*)}-\frac{a_{31}x_1^*}{(m_{13}x_3^*+x_1^*)^2}\right]x_3^*,\\
r_9 &= \left[\frac{3a_{31}m_{13}^2x_1^*}{(m_{13}x_2^*+x_1^*)^4}+\frac{a_{31}m_{13}^2x_1^*}{(m_{13}^2x_3^*+x_1^*)^3}\right]\left[\frac{2a_{31}m_{13}x_1^*}{(m_{13}x_2^*+x_1^*)^3}-\frac{a_{31}}{(m_{13}x_3^*+x_1^*)^2}\right]x_3^*.
\end{aligned}
$$

附录 B

$$
\begin{aligned}
k_0 &= s_0^4, k_1=2m_1s_0^3,\\
k_2 &= [n_1s_0-s_0n_0(m_1+s_1)]^2-2s_0^2n_0[s_0(m_2+n_1s_1+n_2s_0)\\
&\quad -m_2s_0n_0-n_1s_1]-2s_0^2(m-2s_0-s_1m_1+s_1^2),\\
k_3 &= -2(m_1n_1+s_0n_0)[n_1s_0-s_0n_0(m_1+s_1)]+2m_1s_0(m-2s_0-s_1m_1+s_1^2),
\end{aligned}
$$

$$\begin{aligned}
k_4 &= 2(m_1+s_1)(m_2+n_1s_1+n_2s_0)[n_1s_0-s_0n_0(m_1+s_1)]\\
&\quad -2s_0^2n_0m_2(m_2+n_1s_1+n_2s_0)+2m_1s_0(s_0-s_1-m_1m_2),\\
k_5 &= -2(m_1+s_1)(m_2+n_1s_1+n_2s_0)(m_1n_1+s_0n_0)-2s_0^2m_2\\
&\quad -2(s_0-s_1-m_1m_2)(m_2s_0-s_1m_1+s_1^2),\\
k_6 &= s_0(m_2+n_1s_1+n_2s_0)-m_2s_0n_0-n_1s_1\\
&\quad -2(n_1-m_2-n_1s_1-n_2s_0)(m_1n_1+s_0n_0)-(s_0-s_1-m_1m_2),\\
k_7 &= 2m_2(m_2s_0-m_1s_1+s_1^2)-2(n_1-m_2-n_1s_1-n_2s_0)\\
&\quad \times(m_1+s_1)(m_2+n_1s_1+n_2s_0),\\
k_8 &= 2m_2(s_0-s_1-m_1m_2)-m_2(m_2+n_1s_1+n-2s_0)^2\\
&\quad -(n_1-m_2-n_1s_1-n_2s_0)^2.
\end{aligned}$$