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COMPLETE MOMENT CONVERGENCE OF WEIGHTED SUMS FOR SEQUENCES OF φ-MIXING RANDOM VARIABLES

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Abstract: The complete moment convergence of weighted sums for φ-mixing sequences is investigated. By using moment inequality and truncation method, the sufficient conditions for complete moment convergence of weighted sums for φ-mixing sequences are obtained, which generalize the corresponding results of Ahmed et al.(2002) and Chen and Wang (2010).

Keywords: φ-mixing; weighted sums; complete moment convergence; complete convergence

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1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables defined on probability space $(\Omega, \mathscr{F}, P)$. Write $\mathscr{F}_j^k = \sigma\{X_i; j \leq i \leq k\}$, $1 \leq j \leq k \leq \infty$,

$$\varphi(m) = \sup_{k \geq 1}\{|P(B|A) - P(B)|;\ \ A \in \mathscr{F}_1^k,\ \ B \in \mathscr{F}_{k+m}^{\infty},\ \ P(A) > 0\},\ m \geq 1.$$

We call $\{X_n,\ n \geq 1\}$ a φ-mixing sequence if $\lim\limits_{m \to \infty} \varphi(m) = 0$.

It is obvious that $\varphi(m) = 0$ for any $m \geq 1$ for independent sequences. So independent sequences are the special case of φ-mixing sequences. φ-mixing is a wide range of dependent sequence and has valuable applications. Many authors studied the convergence properties for sequences of φ-mixing random variables. We refer the reader to Shao [1] for moment inequality, Wang et al. [2] for strong law of large numbers and growth rate, Kim and Ko [3], Chen and Wang [4], Guo and He [5] for complete moment convergence.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$, $\sum\limits_{n=1}^{\infty} P(|X_n - a| > \varepsilon) < \infty$. This notion was given firstly by Hsu and Robbins [6].

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In view of the Borel-Cantelli lemma, the above result implies that $X_n \to a$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables. The converse theorem was proved by Erdös [7]. This results has been generalized and extended in several directions, see Baum and Katz [8], Gut [9], Taylor et al. [10] and Cai and Xu [11]. In particular, Ahmed et al. [12] obtained the following result in Banach space.

Theorem A Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of rowwise independent random elements in a separable real Banach space $(B, \|\cdot\|)$. Let $P(\| X_{ni} \| > x) \leq CP(|X| > x)$ for some random variable X, constant C and all n, i and $x > 0$. Suppose that $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants such that $\sup\limits_{i\geq 1} |a_{ni}| = O(n^{-r})$ for some $r > 0$, and $\sum\limits_{i=1}^{\infty} |a_{ni}| = O(n^{\alpha})$ for some $\alpha \in [0, r)$. Let β be such that $\alpha + \beta \neq -1$ and fix $\delta > 0$ such that $1 + \alpha/r < \delta \leq 2$. Denote $s = \max(1 + (\alpha + \beta + 1)/r,\ \delta)$. If $E|X|^s < \infty$ and $S_n = \sum\limits_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then $\sum\limits_{n=1}^{\infty} n^{\beta} P(\| S_n \| > \epsilon) < \infty$ for all $\epsilon > 0$.

The concept of complete moment convergence was given firstly by Chow [13]. Wang and Su [14] extended and generalized Chow's result to a Rademacher type $p(1 < p < 2)$ Banach space. Recently, Chen and Wang [4] obtained the following complete qth moment convergence result for φ-mixing sequences.

Theorem B Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed φ-mixing random variables and denote $S_n = \sum\limits_{i=1}^{n} X_i, n \geq 1$. Suppose that $r > 1, 0 < p < 2, q > 0$. Then the following statements are equivalent:

$$\begin{cases} E|X|^q < \infty, & \text{if } q > rp, \\ E|X|^{rp}\log(1 + |X|) < \infty, & \text{if } q = rp, \\ E|X|^{rp} < \infty, & \text{if } 0 < q < rp, \end{cases} \tag{1.1}$$

$$\sum_{n=1}^{\infty} n^{r-2-q/p} E(\max_{1\leq k\leq n} |S_k - kb| - \epsilon n^{1/p})_+^q < \infty, \qquad \forall \epsilon > 0, \tag{1.2}$$

where and in the following $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ if $x < 0$, and x_+^q means $(x_+)^q$, $b = EX$ if $rp \geq 1$ and $b = 0$ if $0 < rp < 1$.

The main purpose of this paper is to discuss again the above results for weighted sums of φ-mixing sequences. The result of Ahmed et al. [12] is extended to φ-mixing case. The result of Chen and Wang [4] is extended to the case of weighted sums.

For the proofs of the main results, we need to restate a few definitions and lemmas for easy reference. Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance, $I(A)$ denotes the indicator function of A. $[x]$ denotes the maximum integer not larger than x. For a finite set B, the symbol $\sharp B$ denotes the number of elements in the set B. Let $a_n \ll b_n$ denote that there exists a constant $C > 0$ such that $a_n \le Cb_n$ for sufficiently large n.

Definition 1.1 A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if $\lim\limits_{x\to\infty} \dfrac{l(x\lambda)}{l(x)} = 1$ for each $\lambda > 0$.

By the properties of slowly varying function and Fubini's theorem, we can easily prove the following lemma. Here we omit the details of the proof.

Lemma 1.1 Let X be a random variable and $l(x) > 0$ be a slowly varying function. Then

(i) $\sum\limits_{n=1}^{\infty} n^{-1}E|X|^{\alpha}I(|X| > n^{\gamma}) \le CE|X|^{\alpha}\log(1 + |X|)$ for any $\alpha \ge 0, \gamma > 0$;

(ii) $\sum\limits_{n=1}^{\infty} n^{\beta}l(n)E|X|^{\alpha}I(|X| > n^{\gamma}) \le CE|X|^{\alpha+(\beta+1)/\gamma}l(|X|^{1/\gamma})$ for any $\beta > -1, \alpha \ge 0, \gamma > 0$;

(iii) $\sum\limits_{n=1}^{\infty} n^{\beta}l(n)E|X|^{\alpha}I(|X| \le n^{\gamma}) \le CE|X|^{\alpha+(\beta+1)/\gamma}l(|X|^{1/\gamma})$ for any $\beta < -1, \alpha \ge 0, \gamma > 0$.

The following two lemmas will play an important role in the proof of our main results. The proof is due to Shao [1].

Lemma 1.2 Let $\{X_i, i \ge 1\}$ be a φ-mixing sequence with mean zero and $E|X_i|^2 < \infty$ for all $i \ge 1$. Then for all $n \ge 1$ and $k \ge 0$, we have

$$E(\sum_{i=k+1}^{k+n} X_i)^2 \le 8000n\exp\{6\sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i)\} \max_{k<i\le k+n} EX_i^2.$$

Lemma 1.3 Let $\{X_i, i \ge 1\}$ be a φ-mixing sequence. Suppose that there exists an array $\{C_{nk},\ k \ge 0, n \ge 1\}$ of positive numbers such that $E(\sum_{i=k+1}^{k+m} X_i)^2 \le C_{nk}$ for any $k \ge 0, n \ge 1, m \le n$. Then for any $q \ge 2$, there exists $C = C(q, \varphi(\cdot))$ such that

$$E\max_{1\le j\le n} |\sum_{i=k+1}^{k+j} X_i|^q \le C[C_{nk}^{q/2} + E(\max_{k<i\le k+n} |X_i|^q)].$$

Lemma 1.4 (see [15]) Let $\{X_i, i \ge 1\}$ be a φ-mixing sequence with $EX_i = 0,\ EX_i^2 < \infty$ and $\sum\limits_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Then $E|\sum\limits_{i=1}^{n} X_i|^2 \le C\sum\limits_{i=1}^{n} EX_i^2$.

By Lemma 1.3 and Lemma 1.4, we deduce the following lemma.

Lemma 1.5 Under the conditions of Lemma 1.4, then for any $q \ge 2$, there exists $C = C(q, \varphi(\cdot))$ such that $E\sup\limits_{1\le j\le n} |\sum\limits_{i=1}^{j} X_i|^q \le C\{(\sum\limits_{i=1}^{n} EX_i^2)^{q/2} + \sum\limits_{i=1}^{n} E|X_i|^q\}$.

By monotone convergence theorem and Lemma 1.5, we can obtain the following lemma.

Lemma 1.6 Under the conditions of Lemma 1.4, then for any $q \ge 2$, there exists $C = C(q, \varphi(\cdot))$ such that $E\sup\limits_{j\ge1} |\sum\limits_{i=1}^{j} X_i|^q \le C\{(\sum\limits_{i=1}^{\infty} EX_i^2)^{q/2} + \sum\limits_{i=1}^{\infty} E|X_i|^q\}$.

2 Main Results and Proofs

Theorem 2.1 Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed φ-mixing random variables with $EX = 0$ and $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants such that

$$\max_{1 \leq i \leq n} |a_{ni}| \ll n^{-r} \text{ for some } r > 0 \tag{2.1}$$

and

$$\sum_{i=1}^{n} |a_{ni}| \ll n^{\alpha} \text{ for some } \alpha \in [0, r). \tag{2.2}$$

Let $\beta > -1$ and $s = 1 + (\alpha + \beta + 1)/r$. If

$$\begin{cases} E|X|^q < \infty, & \text{if } q > s, \\ E|X|^s \log(1 + |X|) < \infty, & \text{if } q = s, \\ E|X|^s < \infty, & \text{if } 1 \leq q < s, \end{cases} \tag{2.3}$$

then

$$\sum_{n=1}^{\infty} n^{\beta} E(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} a_{ni} X_i| - \epsilon)_{+}^{q} < \infty \text{ for all } \epsilon > 0. \tag{2.4}$$

Proof Without loss of generality, from (2.1) and (2.2), we can assume

$$\max_{1 \leq i \leq n} |a_{ni}| \leq n^{-r}, \quad \sum_{i=1}^{n} |a_{ni}| \leq n^{\alpha}. \tag{2.5}$$

It is obvious that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\beta} E(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} a_{ni} X_i| - \epsilon)_{+}^{q} = \sum_{n=1}^{\infty} n^{\beta} \int_0^{\infty} P(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} a_{ni} X_i| > \epsilon + x^{1/q}) \mathrm{d}x \\ \leq \ & \sum_{n=1}^{\infty} n^{\beta} P(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} a_{ni} X_i| > \epsilon) + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} a_{ni} X_i| > x^{1/q}) \mathrm{d}x =: I_1 + I_2. \end{aligned}$$

Thus, it suffices to show that $I_1 < \infty$ and $I_2 < \infty$. We prove only $I_2 < \infty$, the proof of $I_1 < \infty$ is analogous. Set for all $n \geq 1$ and $1 \leq i \leq n$, $X_{ni} = a_{ni} X_i I(|a_{ni} X_i| \leq x^{1/q})$. Note that

$$\begin{aligned} I_2 \ \leq \ & \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} \sum_{i=1}^{n} P(|a_{ni} X_i| > x^{1/q}) \mathrm{d}x + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_{ni}| > x^{1/q}) \mathrm{d}x \\ =: \ & I_3 + I_4. \end{aligned}$$

For any $q \geq 1$, by (2.5), we have

$$\sum_{i=1}^{n} |a_{ni}|^q = \sum_{i=1}^{n} |a_{ni}| |a_{ni}|^{q-1} \leq n^{-r(q-1)} \sum_{i=1}^{n} |a_{ni}| \leq n^{\alpha - r(q-1)}. \tag{2.6}$$

For I_3, noting that $\int_1^\infty P\left(|a_{ni}X_i| > x^{1/q}\right) \mathrm{d}x \le E|a_{ni}X_i|^q I(|a_{ni}X_i| > 1)$, by Lemma 1.1, (2.3) and (2.6), we have

$$\begin{aligned}
I_3 &\le \sum_{n=1}^\infty n^\beta \sum_{i=1}^n E|a_{ni}X_i|^q I(|a_{ni}X_i| > 1) \le \sum_{n=1}^\infty n^\beta \sum_{i=1}^n |a_{ni}|^q E|X|^q I(|X| > n^r) \\
&\le \sum_{n=1}^\infty n^{\beta+\alpha-r(q-1)} E|X|^q I(|X| > n^r) \\
&\ll \begin{cases} \sum_{n=1}^\infty n^{\beta+\alpha-r(q-1)} E|X|^q, & \text{if } q > s, \\ \sum_{n=1}^\infty n^{-1} E|X|^s I(|X| > n^r), & \text{if } q = s, \\ \sum_{n=1}^\infty n^{\beta+\alpha-r(q-1)} E|X|^q I(|X| > n^r), & \text{if } 1 \le q < s, \end{cases} \\
&\ll \begin{cases} \sum_{n=1}^\infty n^{\beta+\alpha-r(q-1)}, & \text{if } q > s, \\ E|X|^s \log(1+|X|), & \text{if } q = s, \\ E|X|^s, & \text{if } 1 \le q < s, \end{cases} \\
&< \infty.
\end{aligned} \tag{2.7}$$

Next we deal with I_4. We first verify that

$$\sup_{x\ge 1} x^{-1/q} \max_{1\le k\le n} |\sum_{i=1}^k EX_{ni}| \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Since (2.3) implies $E|X|^{1+\alpha/r} < \infty$, we have by $EX = 0$ that

$$\begin{aligned}
&\sup_{x\ge 1} x^{-1/q} \max_{1\le k\le n} |\sum_{i=1}^k EX_{ni}| = \sup_{x\ge 1} x^{-1/q} \max_{1\le k\le n} |\sum_{i=1}^k Ea_{ni}X_i I(|a_{ni}X_i| > x^{1/q})| \\
\le\ & \sum_{i=1}^n E|a_{ni}X_i| I(|a_{ni}X_i| > 1) \le n^\alpha E|X| I(|X| > n^r) \\
\le\ & E|X|^{1+\alpha/r} I(|X| > n^r) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, to prove $I_4 < \infty$, we need only to show that

$$I_5 =: \sum_{n=1}^\infty n^\beta \int_1^\infty P(\max_{1\le k\le n} |\sum_{i=1}^k (X_{ni} - EX_{ni})| > x^{1/q}) \mathrm{d}x < \infty.$$

By Lemma 1.5, Markov's inequality and C_r inequality, for any $t \ge 2$, we have

$$I_5 \le \sum_{n=1}^\infty n^\beta \int_1^\infty x^{-t/q} E \max_{1\le k\le n} |\sum_{i=1}^k (X_{ni} - EX_{ni})|^t \mathrm{d}x$$

$$\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} x^{-t/q} (\sum_{i=1}^{n} E a_{ni}^2 X_i^2 I(|a_{ni} X_i| \le x^{1/q}))^{t/2} \mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} x^{-t/q} \sum_{i=1}^{n} E|a_{ni} X_i|^t I(|a_{ni} X_i| \le x^{1/q}) \mathrm{d}x \\
&=: I_6 + I_7.
\end{aligned}$$

For I_6, since $\beta > -1$, we can choose $s' < 2$ such that $1 + \alpha/r < s' < s$. Taking sufficiently large t such that $-s't/(2q) < -1$ and $\beta + (\alpha - r(s'-1))t/2 < -1$, by (2.3), (2.5) and (2.6) we obtain

$$\begin{aligned}
I_6 &= \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} x^{-t/q} (\sum_{i=1}^{n} E|a_{ni} X_i|^{s'} |a_{ni} X_i|^{2-s'} I(|a_{ni} X_i| \le x^{1/q}))^{t/2} \mathrm{d}x \\
&\le \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} x^{-s't/(2q)} \left(\sum_{i=1}^{n} |a_{ni}|^{s'} E|X|^{s'} \right)^{t/2} \mathrm{d}x \ll \sum_{n=1}^{\infty} n^{\beta + (\alpha - r(s'-1))t/2} < \infty.
\end{aligned}$$

Finally, we deal with I_7. Set

$$I_{nj} = \{ i \ge 1 \mid (n(j+1))^{-r} < |a_{ni}| \le (nj)^{-r} \}, \ \ j = 1, 2, \cdots .$$

Then $\cup_{j \ge 1} I_{nj} = N$, where N is the set of positive integers. Note also that for all $k \ge 1,\ n \ge 1,\ t \ge 1$,

$$\begin{aligned}
n^{\alpha} &\ge \sum_{i=1}^{\infty} |a_{ni}| = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}| \ge \sum_{j=1}^{\infty} (\sharp I_{nj}) (n(j+1))^{-r} \\
&\ge n^{-r} \sum_{j=k}^{\infty} (\sharp I_{nj}) (j+1)^{-rt} (k+1)^{rt-r}.
\end{aligned}$$

Hence, we have

$$\sum_{j=k}^{\infty} (\sharp I_{nj}) j^{-rt} \le C n^{\alpha + r} k^{r - rt}. \tag{2.9}$$

Note that

$$\begin{aligned}
I_7 &= \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} x^{-t/q} \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} E|a_{ni} X|^t I(|a_{ni} X| \le x^{1/q}) \mathrm{d}x \\
&\le \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\sharp I_{nj}) (nj)^{-rt} \int_1^{\infty} x^{-t/q} E|X|^t I(|X| \le x^{1/q} n^r (j+1)^r) \mathrm{d}x \\
&= q \sum_{n=1}^{\infty} n^{\beta - rq} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rt} \sum_{k=n}^{\infty} \int_{k^r}^{(k+1)^r} y^{-t+q-1} E|X|^t I(|X| \le y(j+1)^r) \mathrm{d}y \\
&\ll \sum_{n=1}^{\infty} n^{\beta - rq} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rt} \sum_{k=n}^{\infty} k^{-rt + rq - 1} E|X|^t I(|X| \le (k+1)^r (j+1)^r)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\beta-rq} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rt} \sum_{k=n}^{\infty} k^{-rt+rq-1} E|X|^t I(|X| \le 2^r (k+1)^r) \\
&\quad + \sum_{n=1}^{\infty} n^{\beta-rq} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rt} \sum_{k=n}^{\infty} k^{-rt+rq-1} E|X|^t I(2^r (k+1)^r < |X| \le (k+1)^r (j+1)^r) \\
&=: I_8 + I_9. \qquad (2.10)
\end{aligned}
$$

For I_8, we choose sufficiently large t such that $-rt+rq-1<-1,\ \alpha+\beta-rt+r<-1$, by Lemma 1.1 and (2.9) we have

$$
\begin{aligned}
I_8 &= \sum_{n=1}^{\infty} n^{\beta-rq} \sum_{k=n}^{\infty} k^{-rt+rq-1} E|X|^t I(|X| \le 2^r (k+1)^r) \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rt} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha+\beta-rq+r} \sum_{k=n}^{\infty} k^{-rt+rq-1} E|X|^t I(|X| \le 2^r (k+1)^r) \\
&= \sum_{k=1}^{\infty} k^{-rt+rq-1} E|X|^t I(|X| \le 2^r (k+1)^r) \sum_{n=1}^{k} n^{\alpha+\beta-rq+r} \\
&\ll \begin{cases} \sum_{k=1}^{\infty} k^{-rt+rq-1} E|X|^t I(|X| \le 2^r (k+1)^r), & \text{if } q > s, \\ \sum_{k=1}^{\infty} k^{-rt+rq-1} (\log k) E|X|^t I(|X| \le 2^r (k+1)^r), & \text{if } q = s, \\ \sum_{k=1}^{\infty} k^{\alpha+\beta-rt+r} E|X|^t I(|X| \le 2^r (k+1)^r), & \text{if } 1 \le q < s, \end{cases} \\
&\ll \begin{cases} E|X|^q, & \text{if } q > s, \\ E|X|^s \log(1+|X|), & \text{if } q = s, \\ E|X|^s, & \text{if } 1 \le q < s, \end{cases} \\
&< \infty. \qquad (2.11)
\end{aligned}
$$

For I_9, noting that $rq-r-1>-1$ for any $q \ge s > 1$ and $\alpha+\beta>-1$, by Lemma 1.1 and (2.9) we have

$$
\begin{aligned}
I_9 &\le \sum_{n=1}^{\infty} n^{\beta-rq} \sum_{k=n}^{\infty} k^{-rt+rq-1} \sum_{j=1}^{\infty} (\sharp I_{nj}) j^{-rt} \sum_{i=2(k+1)}^{(j+1)(k+1)} E|X|^t I(i^r < |X| \le (i+1)^r) \\
&\le \sum_{n=1}^{\infty} n^{\beta-rq} \sum_{k=n}^{\infty} k^{-rt+rq-1} \sum_{i=2(k+1)}^{\infty} E|X|^t I(i^r < |X| \le (i+1)^r) \sum_{j=[i(k+1)^{-1}]-1}^{\infty} (\sharp I_{nj}) j^{-rt} \\
&\ll \sum_{n=1}^{\infty} n^{\beta-rq} \sum_{k=n}^{\infty} k^{-rt+rq-1} \sum_{i=2(k+1)}^{\infty} n^{r+\alpha} i^{r(1-t)} k^{-r(1-t)} E|X|^t I(i^r < |X| \le (i+1)^r) \\
&= \sum_{k=1}^{\infty} k^{rq-r-1} \sum_{i=2(k+1)}^{\infty} i^{r(1-t)} E|X|^t I(i^r < |X| \le (i+1)^r) \sum_{n=1}^{k} n^{\alpha+\beta-rq+r}
\end{aligned}
$$

$$
\begin{aligned}
&\ll \begin{cases} \sum_{k=1}^{\infty} k^{rq-r-1} \sum_{i=2(k+1)}^{\infty} i^{r(1-t)} E|X|^t I(i^r < |X| \le (i+1)^r), & \text{if } q > s, \\ \sum_{k=1}^{\infty} k^{rq-r-1} (\log k) \sum_{i=2(k+1)}^{\infty} i^{r(1-t)} E|X|^t I(i^r < |X| \le (i+1)^r), & \text{if } q = s, \\ \sum_{k=1}^{\infty} k^{\alpha+\beta} \sum_{i=2(k+1)}^{\infty} i^{r(1-t)} E|X|^t I(i^r < |X| \le (i+1)^r), & \text{if } 1 \le q < s, \end{cases} \\
&\ll \begin{cases} E|X|^q, & \text{if } q > s, \\ E|X|^s \log(1+|X|), & \text{if } q = s, \\ E|X|^s, & \text{if } 1 \le q < s, \end{cases} \\
&< \infty.
\end{aligned} \tag{2.12}
$$

The proof of (2.4) is completed.

Remark 2.1 As in Remark 2.1 of Guo and Zhu [16], (2.4) implies

$$\sum_{n=1}^{\infty} n^{\beta} P(\max_{1\le k\le n} |\sum_{i=1}^{k} a_{ni} X_i| > \epsilon) < \infty \text{ for all } \epsilon > 0.$$

Without necessarily imposing any extra conditions, we not only promote and improve the result of Ahmed et al. [12] from i.i.d. to φ-mixing setting but also obtain the complete moment convergence of maximum weighted sums for φ-mixing sequences.

Theorem 2.2 Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed φ-mixing random variables with $EX = 0$ and $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (2.1) and (2.2). If $\alpha > 0$ and

$$\begin{cases} E|X|^q < \infty, & \text{if } 1+\alpha/r < q < 2, \\ E|X|^{1+\alpha/r} \log(1+|X|) < \infty, & \text{if } q = 1+\alpha/r, \\ E|X|^{1+\alpha/r} < \infty, & \text{if } 1 \le q < 1+\alpha/r, \end{cases} \tag{2.13}$$

then

$$\sum_{n=1}^{\infty} n^{-1} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_{ni} X_i| - \epsilon)_+^q < \infty \text{ for all } \epsilon > 0. \tag{2.14}$$

Proof Applying the same notation and method of Theorem 2.1, denoting $\beta = -1$, we need only to give the different parts. Note that (2.13) implies that $I_3 < \infty$. It is obvious that (2.13) implies $E|X|^{1+\alpha/r} < \infty$. Therefore, we get that (2.8) holds. Thus, to complete the proof of (2.14), it suffices to show that

$$I_5 =: \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} P(\max_{1\le k\le n} |\sum_{i=1}^{k} (X_{ni} - EX_{ni})| > x^{1/q}) \mathrm{d}x < \infty.$$

In fact, noting that $\alpha+\beta+1>0, -2r+rq-1<-1, \alpha-1-r<-1$ and (2.13), by taking t=2 in the proof of (2.10), (2.11) and (2.12), we deduce that

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{-1}\int_1^{\infty} x^{-2/q}\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i|\le x^{1/q})\mathrm{d}x\\ \ll&\begin{cases}E|X|^q<\infty, & \text{if } 1+\alpha/r<q<2,\\ E|X|^{1+\alpha/r}\log(1+|X|)<\infty, & \text{if } q=1+\alpha/r,\\ E|X|^{1+\alpha/r}<\infty, & \text{if } 1\le q<1+\alpha/r,\end{cases}\end{aligned} \tag{2.15}$$

Then, by Markov's inequality, (2.15) and Lemma 1.5, we have

$$I_5 \ll \sum_{n=1}^{\infty} n^{-1}\int_1^{\infty} x^{-2/q}\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i|\le x^{1/q})\mathrm{d}x<\infty.$$

Corollary 2.1 Let $\{X, X_n, n\ge 1\}$ be a sequence of identically distributed $\varphi-$mixing random variables with $EX=0$ and $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$. Suppose that $r>1, 1\le p<2, q\ge 1$. Then (1.1) implies

$$\sum_{n=1}^{\infty} n^{r-2-q/p}E(\max_{1\le k\le n}|\sum_{i=1}^{k}X_i|-\epsilon n^{1/p})_+^q<\infty, \qquad \forall\epsilon>0.$$

Proof Take $\beta=r-2, a_{ni}=n^{-1/p}$ for $1\le i\le n,\ n\ge 1$, and $\alpha=1-1/p$ in Theorem 2.1. It is obvious that a_{ni} satisfies (2.1) and (2.2). Thus, by (1.1) and Theorem 2.1, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{r-2-q/p}E(\max_{1\le k\le n}|\sum_{i=1}^{k}X_i|-\epsilon n^{1/p})_+^q\\ =&\sum_{n=1}^{\infty} n^{r-2}E(\max_{1\le k\le n}|\sum_{i=1}^{k}a_{ni}X_i|-\epsilon)_+^q<\infty.\end{aligned} \tag{2.16}$$

Corollary 2.2 Let $\{X, X_n, n\ge 1\}$ be a sequence of identically distributed φ-mixing random variables with $EX=0$ and $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$. Suppose that $1\le p<2, 1\le q<2$. Then

$$\begin{cases}E|X|^q<\infty, & \text{if } p<q<2,\\ E|X|^p\log(1+|X|)<\infty, & \text{if } q=p,\\ E|X|^p<\infty, & \text{if } 1\le q<p\end{cases} \tag{2.17}$$

implies $\sum_{n=1}^{\infty} n^{-1-q/p}E(\max_{1\le k\le n}|\sum_{i=1}^{k}X_i|-\epsilon n^{1/p})_+^q<\infty, \forall\epsilon>0.$

Proof Take $\beta=-1, a_{ni}=n^{-1/p}$ for $1\le i\le n,\ n\ge 1$, and $r=1/p,\ \alpha=1-1/p$ in Theorem 2.2. It is obvious that a_{ni} satisfies (2.1) and (2.2). Thus, by (2.17) and Theorem 2.2, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{-1-q/p}E(\max_{1\le k\le n}|\sum_{i=1}^{k}X_i|-\epsilon n^{1/p})_+^q=\sum_{n=1}^{\infty} n^{-1}E(\max_{1\le k\le n}|\sum_{i=1}^{k}a_{ni}X_i|-\epsilon)_+^q\\ <&\ \infty. \qquad (2.18)\end{aligned}$$

Remark 2.2 When $1\le p<2$, by Theorem 2.1 and Theorem 2.2, we establish the results of Chen and Wang [4]. Theorem 2.1 and Theorem 2.2 deal with more general weights, and generalize and extend those of Chen and Wang [4].

Theorem 2.3 Let $\{X, X_n, n\ge 1\}$ be a sequence of identically distributed φ-mixing random variables with $EX=0$ and $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$. Let $\{a_{ni}, i\ge 1, n\ge 1\}$ be an array of constants such that

$$\max_{i\ge 1}|a_{ni}|\ll n^{-r}\ \text{ for some } r>0 \qquad (2.19)$$

and

$$\sum_{i=1}^{\infty}|a_{ni}|\ll n^{\alpha}\ \text{ for some } \alpha\in[0,r). \qquad (2.20)$$

Let $\beta>-1$ and $s=1+(\alpha+\beta+1)/r$. Then (2.3) implies

$$\sum_{n=1}^{\infty} n^{\beta}E(\sup_{k\ge 1}|\sum_{i=1}^{k}a_{ni}X_i|-\epsilon)_+^q<\infty\ \text{ for all } \epsilon>0.$$

Proof The proof is the same as that of Theorem 2.1 except that we use Lemma 1.6 instead of Lemma 1.5 and it is omitted.

Theorem 2.4 Let $\{X, X_n, n\ge 1\}$ be a sequence of identically distributed φ-mixing random variables with $EX=0$ and $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$. Let $\{a_{ni}, i\ge 1, n\ge 1\}$ be an array of constants satisfying (2.19) and (2.20). If $\alpha>0$, then (2.13) implies

$$\sum_{n=1}^{\infty} n^{-1}E(\sup_{k\ge 1}|\sum_{i=1}^{k}a_{ni}X_i|-\epsilon)_+^q<\infty\ \text{ for all } \epsilon>0.$$

Proof The proof is the same as that of Theorem 2.2 except that we use Lemma 1.6 instead of Lemma 1.5 and it is omitted.

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φ -混合序列加权和的矩完全收敛性

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摘 要: 本文研究了φ -混合序列加权和的矩完全收敛性. 利用矩不等式和截尾的方法, 获得了φ -混合序列加权和的矩完全收敛性的充分条件. 所得结果推广了Ahmed等(2002)及陈平炎和王定成(2010)的结论.

关 键 词: φ -混合; 加权和; 矩完全收敛性; 完全收敛性

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