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SOME IDENTITIES RELATED TO CHEBYSHEF POLYNOMIALS AND THEIR APPLICATIONS

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Abstract: In this article, we study a kind of sum powers of Chebyshef polynomials. By using elementary method and the properties of the Chebyshef polynomials, several new identities are obtained, which generalizes Melham's conjecture on sums of odd powers of Lucas numbers.

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1 Introduction

In references [1] and [6], Zhang and Ma studied the properties of the Chebyshef polynomials, and obtained some identities involving the Fibonacci numbers and the Lucas numbers. The main results in [1] were the following two identities:

$$\begin{aligned} x^{2n} &= \frac{(2n)!}{4^n (n!)^2} T_0(x) + \frac{2(2n)!}{4^n} \sum_{k=1}^{n} \frac{1}{(n-k)!(n+k)!} T_{2k}(x) \\ &= \frac{(2n)!}{4^n} \sum_{k=0}^{n} \frac{2k+1}{(n-k)!(n+k+1)!} U_{2k}(x) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} x^{2n+1} &= \frac{(2n+1)!}{4^n} \sum_{k=0}^{n} \frac{1}{(n-k)!(n+k+1)!} T_{2k+1}(x) \\ &= \frac{(2n+1)!}{4^n} \sum_{k=0}^{n} \frac{k+1}{(n-k)!(n+k+2)!} U_{2k+1}(x), \end{aligned} \tag{1.2}$$

where $T_n(x)$ is the Chebyshef polynomials of the first kind, and $U_n(x)$ is the Chebyshef polynomials of the second kind, defined respectively by

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right]$$

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and

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}} \left[\left(x + \sqrt{x^2-1} \right)^{n+1} - \left(x - \sqrt{x^2-1} \right)^{n+1} \right].$$

On the other hand, in a private communication with Cooper, Melham suggested that it would be interesting to discover an explicit expansion for

$$L_1 L_2 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$$

as a polynomial in F_{2n+1}, where F_n and L_n denote the Fibonacci number and Lucas number respectively. Melham [5] also proposed following two conjectures:

Conjecture 1.1 Let $m \geq 1$ be a positive integer. Then the sum

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1} \tag{1.3}$$

can be expressed as $(F_{2n+1} - 1)^2 P_{2m-1}(F_{2n+1})$, where $P_{2m-1}(x)$ is a polynomial of degree $2m-1$ with integer coefficients.

Conjecture 1.2 Let $m \geq 0$ be an integer. Then the sum

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1} \tag{1.4}$$

can be expressed as $(L_{2n+1} - 1) Q_{2m}(L_{2n+1})$, where $Q_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients.

Wiemann and Cooper [2] obtained some divisibility properties in the study of some conjectures of Melham related to the sum $\sum_{k=1}^{n} F_{2k}^{2m+1}$. Ozeki [3] proved that

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} \frac{(-1)^j}{L_{2m+1-2j}} \binom{2m+1}{j} \left(F_{(2m+1-2j)(2n+1)} - F_{2m+1-2j} \right).$$

Prodinger [4] studied the more general summation $\sum_{k=0}^{n} F_{2k+\delta}^{2m+1+\epsilon}$, where $\delta,\ \epsilon \in \{0,\ 1\}$, and obtained many interesting identities, two of which are:

$$\sum_{k=0}^{n} F_{2k+1}^{2m+1} = \sum_{l=0}^{m} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l} \frac{5^{l-m}}{L_{2m-2j+1}} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m-2j+1}{2l+1}$$

and

$$\sum_{k=0}^{n} F_{2k}^{2m} = \frac{F_{2n+1}}{5^m} \sum_{r=0}^{m-1} L_{2n+1}^{2r+1} \sum_{j=0}^{m-r-1} \binom{2m}{j} \binom{m-j+r}{m-j-r-1} \frac{(-1)^j}{F_{2m-2j}} + \frac{(-1)^m}{5^m} \binom{2m}{m} \left(n + \frac{1}{2} \right).$$

For the sum of powers of Lucas numbers, Prodinger [4] also obtained similar conclusions:

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{r=0}^{m} L_{2n+1}^{2r+1} \sum_{j=0}^{m-r} \binom{2m+1}{j} \binom{m-j+r}{m-j-r} \frac{2m+1-2j}{2r+1} \frac{1}{L_{2m+1-2j}} + 4^m$$

and

$$\sum_{k=0}^{n} L_{2k+1}^{2m+1} = \sum_{r=0}^{m} L_{2(n+1)}^{2r+1} \sum_{j=0}^{m-r} \binom{2m+1}{j} \binom{m-j+r}{m-j-r} \frac{2m+1-2j}{2r+1} \frac{(-1)^{m-r}}{L_{2m+1-2j}} - \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^{j} \frac{2}{L_{2m+1-2j}}.$$

The main purpose of this paper is using the combination method to solve Conjecture 1.2 completely, and prove the following:

Theorem 1.3 Let $m \geq 0$ be an integer. Then the sum

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1}$$

can be expressed as $(L_{2n+1}-1)\, Q_{2m}(L_{2n+1})$, where $Q_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients.

2 Several Lemmas

In this section, we shall give several simple lemmas which are necessary in the proof of our theorem. First we have the following:

Lemma 2.1 For any nonnegative integers m and n, we have

$$\sum_{k=1}^{m} T_{2k}^{2n}(x) = m\frac{(2n)!}{4^n (n!)^2} + \frac{1}{4^n} \sum_{r=1}^{n} \binom{2n}{n-r} \frac{U_{4rm+2r-1}(x) - U_{2r-1}(x)}{U_{2r-1}(x)}$$

and

$$\sum_{k=0}^{m} T_{2k+1}^{2n}(x) = (m+1)\frac{(2n)!}{4^n (n!)^2} + \frac{1}{4^n} \sum_{r=1}^{n} \binom{2n}{n-r} \frac{U_{4r(m+1)-1}(x)}{U_{2r-1}(x)}.$$

Proof First we give a simple proof for (1.1) and (1.2). In fact, for any positive integer n and real number $x \neq 0$, by using the familiar binomial expansion

$$\left(x + \frac{1}{x}\right)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-2r},$$

we get

$$\left(x + \frac{1}{x}\right)^{2n} = \frac{(2n)!}{(n!)^2} + \sum_{r=1}^{n} \binom{2n}{n-r} \left(x^{2r} + \frac{1}{x^{2r}}\right) \tag{2.1}$$

and

$$\left(x + \frac{1}{x}\right)^{2n+1} = \sum_{r=0}^{n} \binom{2n+1}{n-r} \left(x^{2r+1} + \frac{1}{x^{2r+1}}\right). \tag{2.2}$$

From the properties of the Chebyshef polynomials we know that for any integer n,

$$T_n\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right) = \frac{1}{2}\left(x^n + \frac{1}{x^n}\right). \tag{2.3}$$

Now substituting $\frac{1}{2}\left(x+\frac{1}{x}\right)$ by y in (2.1) and (2.2), and combining (2.3) we may immediately deduce the identities

$$y^{2n}=\frac{(2n)!}{4^n(n!)^2}T_0(y)+\frac{2}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}T_{2r}(y)$$

and

$$y^{2n+1}=\frac{1}{4^n}\sum_{r=0}^{n}\binom{2n+1}{n-r}T_{2r+1}(y).$$

This proves formulas (1.1) and (1.2).

Let $x=\left(y+\sqrt{y^2-1}\right)^k$ be in (2.1) and (2.2), and hence $\frac{1}{x}=\left(y-\sqrt{y^2-1}\right)^k$, $T_n(T_m(y))=T_{mn}(y)$. Then we have

$$T_k^{2n}(y)=\frac{(2n)!}{4^n(n!)^2}+\frac{2}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}T_{2rk}(y) \tag{2.4}$$

and

$$T_k^{2n+1}(y)=\frac{1}{4^n}\sum_{r=0}^{n}\binom{2n+1}{n-r}T_{k(2r+1)}(y). \tag{2.5}$$

Now we prove Lemma 2.1. Let $\alpha=x+\sqrt{x^2-1}$, $\beta=x-\sqrt{x^2-1}$. Then from (2.4) and note that $\alpha\cdot\beta=1$ we may get

$$\begin{aligned}
&\sum_{k=1}^{m}T_{2k}^{2n}(x)=m\frac{(2n)!}{4^n(n!)^2}+\frac{2}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}\sum_{k=1}^{m}T_{4rk}(x)\\
=\;& m\frac{(2n)!}{4^n(n!)^2}+\frac{1}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}\sum_{k=1}^{m}\left(\alpha^{4rk}+\beta^{4rk}\right)\\
=\;& m\frac{(2n)!}{4^n(n!)^2}+\frac{1}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}\left(\frac{\alpha^{4r}(\alpha^{4rm}-1)}{\alpha^{4r}-1}+\frac{\beta^{4r}(\beta^{4rm}-1)}{\beta^{4r}-1}\right)\\
=\;& m\frac{(2n)!}{4^n(n!)^2}+\frac{1}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}\frac{\alpha^{4r}+\beta^{4r}+\alpha^{4rm}-\alpha^{4r(m+1)}+\beta^{4rm}-\beta^{4r(m+1)}-2}{2-\alpha^{4r}-\beta^{4r}}\\
=\;& m\frac{(2n)!}{4^n(n!)^2}+\frac{1}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}\frac{\left(\alpha^{4rm+2r}-\beta^{4rm+2r}\right)\left(\alpha^{2r}-\beta^{2r}\right)-\left(\alpha^{2r}-\beta^{2r}\right)^2}{\left(\alpha^{2r}-\beta^{2r}\right)^2}\\
=\;& m\frac{(2n)!}{4^n(n!)^2}+\frac{1}{4^n}\sum_{r=1}^{n}\binom{2n}{n-r}\frac{U_{4rm+2r-1}(x)-U_{2r-1}(x)}{U_{2r-1}(x)}.
\end{aligned}$$

This proves the first formula of Lemma 2.1. Similarly, we can deduce the second one.

Lemma 2.2 For any nonnegative integers m and n, we have

$$\sum_{k=1}^{m}T_{2k}^{2n+1}(x)=\frac{1}{2}\sum_{r=0}^{n}\frac{\binom{2n+1}{n-r}}{4^n}\frac{U_{4mr+2r+2m}(x)-U_{2r}(x)}{U_{2r}(x)}$$

and

$$\sum_{k=0}^{m} T_{2k+1}^{2n+1}(x) = \frac{1}{2\cdot 4^n}\sum_{r=0}^{n}\binom{2n+1}{n-r}\frac{U_{4mr+4r+2m+1}(x)}{U_{2r}(x)}.$$

Proof From (2.5) we have

$$\begin{aligned}
&\sum_{k=1}^{m} T_{2k}^{2n+1}(x) = \frac{1}{4^n}\sum_{r=0}^{n}\binom{2n+1}{n-r}\sum_{k=1}^{m} T_{2k(2r+1)}(x)\\
= &\sum_{r=0}^{n}\frac{\binom{2n+1}{n-r}}{4^n}\sum_{k=1}^{m}\frac{1}{2}\left(\alpha^{2k(2r+1)}+\beta^{2k(2r+1)}\right)\\
= &\frac{1}{2}\sum_{r=0}^{n}\frac{\binom{2n+1}{n-r}}{4^n}\left(\frac{\alpha^{2(2r+1)}(\alpha^{2m(2r+1)}-1)}{\alpha^{2(2r+1)}-1}+\frac{\beta^{2(2r+1)}(\beta^{2m(2r+1)}-1)}{\beta^{2(2r+1)}-1}\right)\\
= &\frac{1}{2}\sum_{r=0}^{n}\frac{\binom{2n+1}{n-r}}{4^n}\left(\frac{\alpha^{(2m+1)(2r+1)}-\beta^{(2m+1)(2r+1)}-\alpha^{2r+1}+\beta^{2r+1}}{\alpha^{2r+1}-\beta^{2r+1}}\right)\\
= &\frac{1}{2}\sum_{r=0}^{n}\frac{\binom{2n+1}{n-r}}{4^n}\frac{U_{4mr+2r+2m}(x)-U_{2r}(x)}{U_{2r}(x)},
\end{aligned}$$

where we have used the identity $\alpha\cdot\beta = 1$. This proves the first formula of Lemma 2.2. Similarly, we can deduce the second formula of Lemma 2.2.

Lemma 2.3 For any positive integers m and n, we have

$$\sum_{r=1}^{n} L_{2r}^{2m+1} = \sum_{k=0}^{m}\binom{2m+1}{m-k}\frac{L_{(2n+1)(2k+1)}-L_{2k+1}}{L_{2k+1}}.$$

Proof Note that $T_{2k}\left(\frac{\sqrt{5}}{2}\right) = \frac{1}{2}L_{2k}$ and $U_{2r}\left(\frac{\sqrt{5}}{2}\right) = L_{2r+1}$, the Lucas number. From the first formula of Lemma 2.2 we may immediately deduce that

$$\sum_{r=1}^{n} L_{2r}^{2m+1} = \sum_{k=0}^{m}\binom{2m+1}{m-k}\frac{L_{(2n+1)(2k+1)}-L_{2k+1}}{L_{2k+1}}.$$

This proves Lemma 2.3.

3 Proof of Theorem

Now we use Lemma 2.3 to prove Theorem 1.3. For any integer $k\geq 0$, applying mathematical induction we can prove that $L_{2n+1}-1$ divide $L_{(2n+1)(2k+1)}-L_{2k+1}$. In fact note that $L_1 = 1$, so $\left(L_{(2n+1)}-1\right)\mid\left(L_{(2n+1)(2k+1)}-L_{2k+1}\right)$ holds for $k=0$. If $k=1$, then note that $L_{2n+1}^3 = L_{3(2n+1)}-3L_{2n+1}$ (This identity can be deduced directly from the Binet formula) and $L_3 = 4$, we have

$$L_{3(2n+1)}-L_3 = L_{2n+1}^3+3L_{2n+1}-4 = (L_{2n+1}-1)\left(L_{2n+1}^2+L_{2n+1}+4\right).$$

So that $\left(L_{(2n+1)}-1\right)$ divide $\left(L_{3(2n+1)}-L_3\right)$. Suppose that

$$\left(L_{(2n+1)}-1\right) \mid \left(L_{(2n+1)(2k+1)}-L_{2k+1}\right)$$

for all integers $k \le m$. Then for $k=m+1$, note that

$$L_{2(2n+1)}L_{(2n+1)(2m+1)}=L_{(2n+1)(2m+3)}+L_{(2n+1)(2m-1)},$$

we have

$$\begin{aligned} & L_{(2n+1)(2m+3)}-L_{2m+3}=L_{2(2n+1)}L_{(2n+1)(2m+1)}-L_{(2n+1)(2m-1)}-L_{2m+3} \\ = & \left(L_{2n+1}^2+2\right)L_{(2n+1)(2m+1)}-\left(L_{(2n+1)(2m-1)}-L_{2m-1}\right)-3L_{2m}-3L_{2m-1} \\ = & \left(L_{2n+1}^2-1\right)L_{(2n+1)(2m+1)}+3\left(L_{(2n+1)(2m+1)}-L_{2m+1}\right) \\ & -\left(L_{(2n+1)(2m-1)}-L_{2m-1}\right). \end{aligned} \tag{3.1}$$

Now our conclusion follows from (3.1) and the inductive hypothesis.

On the other hand, note that

$$\begin{aligned} L_{(2n+1)(2m+3)} &= L_{2(2n+1)}L_{(2n+1)(2m+1)}-L_{(2n+1)(2m-1)} \\ &= (L_{2n+1}^2+2)L_{(2n+1)(2m+1)}-L_{(2n+1)(2m-1)}, \end{aligned} \tag{3.2}$$

so by mathematical induction we can also prove that

$$L_{(2n+1)(2m+1)}-L_{2m+1}=f_{2m+1}\left(L_{2n+1}\right),$$

where $f_{2m+1}(x)$ is a polynomial of degree $2m+1$ with integer coefficients. In fact if $m=0$, then $L_{(2n+1)(2m+1)}-L_{2m+1}=L_{(2n+1)}-L_1=f_1\left(L_{2n+1}\right)$, where $f_1(x)$ is a polynomial of degree 1 with integer coefficients. Suppose that $L_{(2n+1)(2k+1)}-L_{2k+1}=f_{2k+1}\left(L_{2n+1}\right)$ for all integers $k\le m$. Then for $k=m+1$, note that (3.2), we have

$$\begin{aligned} & L_{(2n+1)(2m+3)}-L_{2m+3}=(L_{2n+1}^2+2)L_{(2n+1)(2m+1)}-L_{2m+3}-L_{(2n+1)(2m-1)} \\ = & (L_{2n+1}^2+2)\left(f_{2m+1}(L_{2n+1})+L_{2m+1}\right)-L_{2m+3}-L_{(2n+1)(2m-1)}=f_{2m+3}(L_{2n+1}), \end{aligned}$$

where $f_{2m+3}(x)$ is a polynomial of degree $2m+3$ with integer coefficients. This proves $L_{(2n+1)(2m+1)}-L_{2m+1}=f_{2m+1}\left(L_{2n+1}\right)$ for all integers $m\ge 0$.

Combining Lemma 2.3, $(L_{2n+1}-1)$ divide $\left(L_{(2n+1)(2k+1)}-L_{2k+1}\right)=f_{2k+1}\left(L_{2n+1}\right)$ and $(L_{2n+1}-1,\ L_{2n+1})=1$, we may immediately deduce that

$$\begin{aligned} & L_1L_3L_5\cdots L_{2m+1}\sum_{k=1}^{n}L_{2k}^{2m+1} \\ = & L_1L_3L_5\cdots L_{2m+1}\left(\sum_{k=0}^{m}\binom{2m+1}{m-k}\frac{L_{(2n+1)(2k+1)}-L_{2k+1}}{L_{2k+1}}\right) \\ = & \left(L_{2n+1}-1\right)Q_{2m}\left(L_{2n+1}\right), \end{aligned}$$

where $Q_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients. This completes the proof of our theorem.

References

[1] Ma Rong, Zhang Wenpeng. Several identities involving the Fibonaccinum bers and Lucas numbers[J]. Fibonacci Quarterly, 2007, 45(2): 164–170.

[2] Wiemann M, Cooper C. Divisibility of an F-L type convolution, applications of Fibonacci numbers[M]. Dordrecht: Kluwer Acad. Publ., 2004.

[3] Ozeki K. On Melham's sum[J]. Fibonacci Quarterly, 2008, 46/47(2): 107–110.

[4] Prodinger H. On a sum of Melham and its variants[J]. Fibonacci Quarterly, 2008/2009, 46/47(2): 107–110.

[5] Melham R S. Some conjectures concerning sums of odd powers of Fibonacci and Lucas numbers[J]. The Fibonacci Quarterly, 2008/2009, 46/47(2):312–315, 386–418.

[6] Zhang Wenpeng. On Chebyshev polynomials and Fibonacci numbers[J]. Fibonacci Quarterly, 2002, 40(2): 424–428.

一些关于Chebyshef多项式以及它们应用的恒等式

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摘要: 本文研究了Chebyshef 多项式的一类幂和问题. 利用初等方法以及Chebyshef 多项式的性质, 获得了一些有趣的恒等式, 推广了Melham关于Lucas数的奇数次幂和的猜想.

关键词: Chebyshef 多项式; Fibonacci 数; Lucas 数; 恒等式; Melham 猜想

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