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LINEAR ADMISSIBLE PREDICTION OF FINITE POPULATION REGRESSION COEFFICIENT UNDER A BALANCED LOSS FUNCTION

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Abstract: In this paper, under a balanced loss function we investigate admissible prediction of finite population regression coefficient in superpopulation models with and without the assumption that the underlying distribution is normal, respectively. By using the statistical decision theory, necessary and sufficient conditions for a homogeneous linear predictor to be admissible in the class of homogeneous linear predictors are obtained in the non-normal case, we also obtain a sufficient and necessary condition for a homogeneous linear predictor to be admissible in the class of all predictors in the normal case, which generalize some relative results under quadratic loss to balanced loss function.

Keywords: admissible prediction; finite population regression coefficient; balanced loss function

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1 Introduction

We open this section with some notations: Given a matrix A, the symbols $\mathscr{M}(A), A'$, $\mathrm{tr}(A)$, $rk(A)$ will stand for the range space, the transpose, the trace, the rank, respectively, of matrix A. The $n\times n$ identity matrix is denoted by I_n. For an $n\times n$ matrix A, $A>0$ means that A is a symmetric positive definite matrix. $A\geq 0$ means that A is a symmetric nonnegative definite matrix, $A\geq B(A\leq B)$ means that $A-B\geq 0(B-A\leq 0)$. $R^{m\times n}$ stands for the set composed of all $m\times n$ real matrices.

Let us consider finite population $\mathscr{P}=\{1,\cdots,N\}$ as the collection of a known number N of identifiable units. Associated with the ith unit of $\mathscr{P}$, there are $p+1$ quantities: $y_i, x_{i1},\cdots,x_{ip}$, where all but y_i are known, $i=1,\cdots,N$. Denote $y=(y_1,\cdots,y_N)'$ and $X=(X_1,\cdots,X_N)'$, where $X_i=(x_{i1},\cdots,x_{ip})'$, $i=1,\cdots,N$. We express the relationships

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among the variables by the linear model

$$y = X\beta + \varepsilon, \tag{1.1}$$

where β is a $p\times 1$ unknown superparameter vector, ε is an $N\times 1$ random vector with mean 0 and covariance matrix $\sigma^2 V$, $V > 0$ is a known matrix, but the parameter $\sigma^2 > 0$ is unknown. If ε is a random vector with multivariate normal distribution, the model (1.1) will be written as

$$y = X\beta + \varepsilon, \varepsilon \sim N(0, \sigma^2 V). \tag{1.2}$$

Denote the finite population regression coefficient as $\beta_N = (X'V^{-1}X)^{-1}X'V^{-1}y$. In the literature, a lot of predictions for finite population regression coefficient have been produced. For example, Bolfarine and Zacks [1, 2] studied Bayes and minimax prediction under square error loss function. Bolfarine et al. [3] obtained the best linear unbiased prediction under the generalized prediction mean squared error. Yu and Xu [4] studied the admissibility of linear predictors under quadratic loss. Xu and Yu [5] obtained the linear admissible predictors under matrix loss. Recently, Bansal and Aggarwal [6–8] considered Bayes prediction of finite population regression coefficient of some superpopulation regression model with normal assumption using Zellner's [9] balanced loss function. However, there is not much literature on the linear admissible prediction of the finite population regression coefficient in the superpopulation models with and without the assumption that the underlying distribution is normal using Zellner's balanced loss function.

In order to predict the finite population regression coefficient β_N, let us select a sample $\mathbf{s}$ of size $n(\leq N)$ from $\mathscr{P}$ according to some specified sampling plan in order to obtain information on β_N. Let $\mathbf{r} = \mathscr{P} - \mathbf{s}$ be the unobserved part of $\mathscr{P}$. After the sample has been selected, we may reorder the elements of y such that we have the corresponding partitions of y, X and V, that is

$$y = \begin{pmatrix} y_s \\ y_r \end{pmatrix}, X = \begin{pmatrix} X_s \\ X_r \end{pmatrix}, V = \begin{pmatrix} V_s & V_{sr} \\ V_{rs} & V_r \end{pmatrix}.$$

Following Bolfarine et al.[3], we can write the finite population regression coefficient β_N as

$$\beta_N = Q_s y_s + Q_r y_r,$$

where

$$\begin{aligned} &Q_s = F^{-1}BC^{-1}, Q_r = F^{-1}DE^{-1},\\ &B = X_s' - X_r'V_r^{-1}V_{rs}, C = V_s - V_{sr}V_r^{-1}V_{rs},\\ &D = X_r' - X_s'V_s^{-1}V_{sr}, E = V_r - V_{rs}V_s^{-1}V_{sr} \end{aligned}$$

and

$$F = BC^{-1}X_s + DE^{-1}X_r.$$

Consider the class of homogeneous linear predictors as $\pounds = \{Ly_s : L \in R^{p\times n}\}$. Let $\delta(y_s)$ be a predictor of β_N, in this article, we use Zellner's balanced loss function

$$L(\delta(y_s), \beta_N) = \theta(y_s - X_s\delta(y_s))'(y_s - X_s\delta(y_s)) + (1-\theta)(\delta(y_s) - \beta_N)'X_s'X_s(\delta(y_s) - \beta_N). \quad (1.3)$$

Here $\theta \in [0, 1]$ is a weight coefficient. Zellner's balanced loss function takes both precision of estimation and goodness of fit of model into account, so it is a more comprehensive and reasonable standard than quadratic loss and residual sum of square. Moreover, we know that the balanced loss function is more sensitive than the quadratic loss function, which means that if a prediction is admissible under the balanced loss function, it is also admissible under the quadratic loss function. Therefore, the study about the admissible prediction under the balanced loss function is significant. Denote the corresponding risk function as $R(\delta(y_s), \beta_N) = E(L(\delta(y_s), \beta_N))$.

In this paper, we discuss the admissibility of linear predictors in the class of homogeneous linear predictors and in the class of all predictors, respectively.

The rest of this paper is organized as follows: Necessary and sufficient conditions for homogeneous linear predictors of β_N to be admissible in $\pounds$ under model (1.1) and loss function (1.3) are placed in Section 2. In Section 3, we give the sufficient conditions for homogeneous linear predictors to be admissible in the class of all predictors under model (1.2) and loss function (1.3). Concluding remarks are given in Section 4.

2 Admissibility of a Homogeneous Linear Predictor in the Class of Linear Predictors

In this section, we give necessary and sufficient conditions for homogeneous linear predictors to be admissible in $\pounds$ under the model (1.1) and the balanced loss function (1.3). First, we give a definition for admissibility.

Definition 2.1 The predictor $\delta_1(y_s)$ is called as good as $\delta_2(y_s)$ if and only if $R(\delta_1(y_s), \beta_N) \leq R(\delta_2(y_s), \beta_N)$ for all $\beta \in R^p$ and $\sigma^2 > 0$, and $\delta_1(y_s)$ is called better than $\delta_2(y_s)$ iff $\delta_1(y_s)$ is as good as $\delta_2(y_s)$ and $R(\delta_1(y_s), \beta_N) \neq R(\delta_2(y_s), \beta_N)$ at some $\beta_0 \in R^p$ and $\sigma_0^2 > 0$. Let $\mathscr{L}$ be a class of predictors, then a predictor $\delta(y_s)$ is said to be admissible for β_N in $\mathscr{L}$ iff $\delta(y_s) \in \mathscr{L}$ and there exists no predictor in $\mathscr{L}$ which is better than $\delta(y_s)$.

Lemma 2.1 (Wu [10]) Consider the following model

$$y_s = X_s\beta + \varepsilon_s, \quad (2.1)$$

where ε_s is a $n \times 1$ unobservable random vector with $E(\varepsilon_s) = 0, \mathrm{Cov}(\varepsilon_s) = \sigma^2 V_s$, X_s and V_s are known $n \times p$ and $n \times n$ matrices, respectively. Whereas $\beta \in R^p$ and $\sigma^2 > 0$ are unknown parameters. If $S\beta$ is a linearly estimable variable under the model (2.1), then under the loss function $(d - S\beta)'(d - S\beta)$, Ly_s is an admissible estimator of $S\beta$ in $\pounds$ if and only if

(1) $L = LX_s(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1}$,

(2) $LX_s(X_s'V_s^{-1}X_s)^{-1}S' - LX_s(X_s'V_s^{-1}X_s)^{-1}X_s'L' \geq 0$.

Theorem 2.1 Under model (1.1), Ly_s is an admissible predictor of β_N in $\pounds$ under the balanced loss function (1.3) if and only if

(1) $L = LX_s(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1} + \theta(X_s'X_s)^{-1}X_s' - \theta(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1}$,

(2) $(L - \theta(X_s'X_s)^{-1}X_s' - (1-\theta)(Q_s + Q_rV_{rs}V_s^{-1}))X_s(X_s'V_s^{-1}X_s)^{-1}(I_p - X_s'L') \geq 0$.

Proof By direct operation, we have

$$\begin{aligned}
& R(Ly_s, \beta_N) \\
= \;& E[\theta(y_s - X_sLy_s)'(y_s - X_sLy_s) + (1-\theta)(Ly_s - Q_sy_s - Q_ry_r)'X_s'X_s(Ly_s - Q_sy_s - Q_ry_r)] \\
= \;& E[(\tilde{L}y_s - \tilde{S}\beta)'(\tilde{L}y_s - \tilde{S}\beta)] \\
& +\sigma^2\mathrm{tr}[\theta V_s + (1-\theta)Q_s'X_s'X_sQ_sV_s + 2(1-\theta)Q_s'X_s'X_sQ_rV_{rs} + (1-\theta)Q_r'X_s'X_sQ_rV_r] \\
& -\sigma^2\mathrm{tr}[H(\theta H^{-1}X_s' + (1-\theta)(Q_s + Q_rV_{rs}V_s^{-1}))V_s(\theta H^{-1}X_s' + (1-\theta)(Q_s + Q_rV_{rs}V_s^{-1}))'],
\end{aligned}$$

where

$$\begin{aligned}
& H = X_s'X_s, \tilde{L} = H^{\frac{1}{2}}(L - \theta H^{-1}X_s' - (1-\theta)(Q_s + Q_rV_{rs}V_s^{-1})), \\
& \tilde{S} = (1-\theta)H^{\frac{1}{2}}(I_p - Q_sX_s - Q_rV_{rs}V_s^{-1}X_s).
\end{aligned}$$

In order to prove that Ly_s is an admissible prediction of β_N in $\pounds$ under balanced loss function (1.3), we need only to show that $\tilde{L}y_s$ is an admissible estimator for $\tilde{S}\beta$ under model (2.1) and loss function $(d - \tilde{S}\beta)'(d - \tilde{S}\beta)$ in $\pounds$. It follows by Lemma 2.1 that Ly_s is an admissible predictor of β_N in $\pounds$ under balanced loss function (1.3) if and only if

(1) $\tilde{L} = \tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1}$,

(2) $\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}\tilde{S}' - \tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}X_s'\tilde{L}' \geq 0$.

On the basis of this, we can obtain the result by direct operation.

It is easy to obtain the following corollary by this theorem.

Corollary 2.1 Under model (1.1) and the loss function (1.3), L_1y_s is an admissible predictor of β_N in $\pounds$, where $L_1 = \theta(X_s'X_s)^{-1}X_s' + (1-\theta)(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1}$.

If $(y_s - X_s\delta(y_s))'(y_s - X_s\delta(y_s))$ is also considered to be a kind of loss function, then $(X_s'X_s)^{-1}X_s'y_s$ is the best linear unbiased prediction in $\pounds$. This corollary illustrates that the linear admissible prediction under loss (1.3) are convex combination between the linear admissible predictions under loss $(y_s - X_s\delta(y_s))'(y_s - X_s\delta(y_s))$ and the linear admissible predictions under loss $(\delta(y_s) - \beta_N)'X_s'X_s(\delta(y_s) - \beta_N)$. Moreover, the weights assigned to the goodness of fit of model and the precision of estimation are consistent to the weights assigned to their corresponding admissible predictions. It is clear to illustrate the use of the balanced loss function (1.3).

3 Admissibility of a Homogeneous Linear Predictor in the Class of All Predictors

In section 2, we have given the necessary and sufficient conditions for a homogeneous linear predictor to be admissible in the class of homogeneous linear predictors. It is interesting to discuss the problem whether the admissible predictor is also admissible in the class of all predictors. In this section, we will answer this problem under model (1.2) and the loss function (1.3). In the following, we first give some lemmas.

Lemma 3.1 (Wu [11]) Consider the following model

$$\tilde{y}_s = \tilde{X}_s\beta + \tilde{\varepsilon}, \tilde{\varepsilon} \sim N(0, \sigma^2 I_n), \tag{3.1}$$

where $\tilde{y_s} \in R^n$, $\tilde{X}_s \in R^{n\times p}$. Let L and S be known $p \times n$ matrices. If L satisfies the following conditions:

(1) $L = L\tilde{X}_s(\tilde{X}_s'\tilde{X}_s)^{-1}\tilde{X}_s'$,

(2) $L\tilde{X}_s(\tilde{X}_s'\tilde{X}_s)^{-1}\tilde{X}_s'L' \le L\tilde{X}_s(\tilde{X}_s'\tilde{X}_s)^{-1}\tilde{X}_s'S'$,

(3) $rk(L\tilde{X}_s(\tilde{X}_s'\tilde{X}_s)^{-1}\tilde{X}_s'(S-L)') \ge rk(L) - 2$.

Then an estimator $L\tilde{y}_s$ of $S\tilde{X}_s\beta$ is admissible in the class of all estimators under loss function $(d - S\tilde{X}_s\beta)'(d - S\tilde{X}_s\beta)$.

Lemma 3.2 (Rao [12]) Let L and S be $m \times n$ matrices. Then LS' is symmetric and $LL' \le LS'$ if and only if there exists an $n \times n$ symmetric matrix $M \ge 0$ such that $L = SM$, $rk(M) = rk(L)$ and the eigenvalues of M are in the closed interval $[0, 1]$.

Lemma 3.3 (Wu [11]) Let L and S be $m \times n$ matrices. Then the following two statements are equivalent.

(1) LS' is symmetric, $LL' \le LS'$ and $rk(LS' - LL') \ge rk(L) - 2$,

(2) There exists an $n \times n$ symmetric matrix $M \ge 0$ such that $L = SM$, $rk(M) = rk(L)$, the eigenvalues of M are in $[0, 1]$ and at most two of them are equal to one.

Lemma 3.4 (Rao [12]) Let $h(y)$ be an admissible estimator of $g(\gamma)$ under $(d-g(\gamma))'(d-g(\gamma))$. Then for every constant matrix K, $Kh(y)$ is an admissible estimator of $Kg(\gamma)$ under $(d_1 - Kg(\gamma))'(d_1 - Kg(\gamma))$.

Theorem 3.1 Under the model (1.2) and the loss function (1.3), a predictor Ly_s of β_N is admissible in the class of all predictors if L satisfied the following conditions:

(1) $\tilde{L} = \tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1}$,

(2) $\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}X_s'\tilde{L}' \le \tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}\tilde{S}'$,

(3) $rk(\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}(\tilde{S} - \tilde{L}X_s)') \ge, rk(\tilde{L}) - 2$,

where

$$\tilde{L} = H^{\frac{1}{2}}(L - \theta H^{-1}X_s' - (1-\theta)(Q_s + Q_rV_{rs}V_s^{-1})),$$
$$\tilde{S} = (1-\theta)H^{\frac{1}{2}}(I_p - Q_sX_s - Q_rV_{rs}V_s^{-1}X_s).$$

Proof According to the proof of Theorem 2.1, we have

$$\begin{aligned} & R(Ly_s, \beta_N) \\ = \ & E[(\tilde{L}y_s - \tilde{S}\beta)'(\tilde{L}y_s - \tilde{S}\beta)] + C^2 \\ = \ & E[(\tilde{L}_1\tilde{y}_s - \tilde{S}_1\tilde{X}_s\beta)'(\tilde{L}_1\tilde{y}_s - \tilde{S}_1\tilde{X}_s\beta)] + C^2, \end{aligned}$$

where

$$\tilde{L}_1 = \tilde{L}V_s^{\frac{1}{2}}, \tilde{y}_s = V_s^{-\frac{1}{2}}y_s, \tilde{X}_s = V_s^{-\frac{1}{2}}X_s,$$
$$\tilde{S}_1 = (1-\theta)H^{\frac{1}{2}}((X_s'X_s)^{-1}X_s' - Q_s - Q_rV_{rs}V_s^{-1})V_s^{\frac{1}{2}}$$

and

$$\begin{aligned} C^2 &= \sigma^2\mathrm{tr}[\theta V_s+(1-\theta)Q_s'X_s'X_sQ_sV_s+2(1-\theta)Q_s'X_s'X_sQ_rV_{rs}+(1-\theta)Q_r'X_s'X_sQ_rV_r \\ &\quad -H(\theta H^{-1}X_s'+(1-\theta)(Q_s+Q_rV_{rs}V_s^{-1}))V_s(\theta H^{-1}X_s'+(1-\theta)(Q_s+Q_rV_{rs}V_s^{-1}))']. \end{aligned}$$

Therefore, to prove that Ly_s is an admissible predictor of β_N in the class of all predictors, we need only to show that $\tilde{L}_1\tilde{y}_s$ is an admissible estimator of $\tilde{S}_1\tilde{X}_s\beta$ in the class of all estimators under model (3.1) and the loss function $(d-\tilde{S}_1\tilde{X}_s\beta)'(d-\tilde{S}_1\tilde{X}_s\beta)$. By Lemma 3.1, we obtain the result.

Corollary 3.1 Under model (1.2) and the loss function (1.3), L_1y_s is admissible in the class of all predictors, where

$$L_1=\theta(X_s'X_s)^{-1}X_s'+(1-\theta)(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1}.$$

The proof of this corollary is omitted here since it is easy to verify that L_1y_s satisfies the conditions of Theorem 3.1.

Theorem 3.2 Let $\mathscr{M}(X_s'\tilde{L}')\subset\mathscr{M}(\tilde{S}')$ and Ly_s be an admissible predictor of β_N in the class of all predictors under the model (1.2) and the loss function (1.3). Then L satisfies the following conditions:

(1) $\tilde{L}=\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}X_s'V_s^{-1}$,

(2) $\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}X_s'\tilde{L}'\leq\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}\tilde{S}'$,

(3) $rk(\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}(\tilde{S}-\tilde{L}X_s)')\geq rk(\tilde{L})-2$,

where

$$\begin{aligned} \tilde{L}&=H^{\frac{1}{2}}(L-\theta H^{-1}X_s'-(1-\theta)(Q_s+Q_rV_{rs}V_s^{-1})), \\ \tilde{S}&=(1-\theta)H^{\frac{1}{2}}(I_p-Q_sX_s-Q_rV_{rs}V_s^{-1}X_s). \end{aligned}$$

Proof Take an $n\times p$ matrix $\tilde{P}$ such that $\mathscr{M}(\tilde{P})=\mathscr{M}(\tilde{X}_s)$ and $\tilde{P}'\tilde{P}=I_p$, where $p=rk(\tilde{X}_s)$. Then $\tilde{X}_s(\tilde{X}_s'\tilde{X}_s)^{-1}\tilde{X}_s'=\tilde{P}\tilde{P}'$. If $Ly_s\in\pounds$ is an admissible predictor of β_N in the class of all predictors under the model (1.2) and the loss function (1.3), then $\tilde{L}_1\tilde{y}_s$ is an admissible estimator of $\tilde{S}_1\tilde{X}_s\beta$ in the class of all estimators under the model (3.1) and the loss function $(d-\tilde{S}_1\tilde{X}_s\beta)'(d-\tilde{S}_1\tilde{X}_s\beta)$ according to the proof of Theorem 3.1. This shows that conditions (1) and (2) of this theorem hold by Lemma 2.1. Therefore, we will show that (3) holds using (1) and (2). Suppose, to the contrary, that (3) does not hold, i.e., $rk(\tilde{L}X_s(X_s'V_s^{-1}X_s)^{-1}(\tilde{S}-\tilde{L}X_s)')<rk(\tilde{L})-2$, which is equivalent to

$$rk(\tilde{L}_1\tilde{X}_s(\tilde{X}_s'\tilde{X}_s)^{-1}\tilde{X}_s'(\tilde{S}_1-\tilde{L}_1)')<rk(\tilde{L}_1)-2.$$

By equation $rk(\tilde{L}_1)=rk(\tilde{L}_1\tilde{X}_s)=rk(\tilde{L}_1\tilde{P})$, Lemmas 3.2, 3.3 and condition $\tilde{L}_1\tilde{P}\tilde{P}'\tilde{L}\leq\tilde{L}_1'\tilde{P}\tilde{P}'\tilde{S}_1'$, there exists a $p\times p$ symmetric matrix $M\geq0$ such that $\tilde{L}_1\tilde{P}=\tilde{S}_1\tilde{P}M$, $rk(M)=rk(\tilde{L}_1)$ and the eigenvalues of M are in $[0,1]$ and at least three eigenvalues are equal to 1. By the spectral decomposition of $\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}$, we write $\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}=\Gamma\tilde{\Delta}\Gamma'$. Here Γ is an orthogonal matrix of order p, $\tilde{\Delta}=\mathrm{diag}(\tau_1,\cdots,\tau_q,0\cdots,0)$ and $\Delta=\mathrm{diag}(\tau_1,\cdots,\tau_q)>0$ with $q=$

$rk(\tilde{S}_1\tilde{P}) = rk(\tilde{S}_1\tilde{X}_s)$. Since $\mathscr{M}(\tilde{X}_s'\tilde{L}_1') \subset \mathscr{M}(\tilde{X}_s'\tilde{S}_1')$ if and only if $\mathscr{M}(\tilde{P}'\tilde{L}_1') \subset \mathscr{M}(\tilde{P}'\tilde{S}_1')$, it follows from the definition of M and $\mathscr{M}(X_s'\tilde{L}') \subset \mathscr{M}(\tilde{S}')$ that

$$\mathscr{M}(M) = \mathscr{M}(\tilde{P}'\tilde{L}_1') \subset \mathscr{M}(\tilde{P}'\tilde{S}_1') = \mathscr{M}(\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}).$$

Moreover, $\mathscr{M}(\Gamma'M\Gamma) \subset \mathscr{M}(\tilde{\Delta})$. This derives that $\Gamma'M\Gamma = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $M_1 \geq 0$ is a symmetric matrix of order q and has the same nonzero eigenvalues as those of M. Take a $q \times q$ orthogonal matrix N_1 such that $N_1'M_1N_1 = \begin{pmatrix} I_t & 0 \\ 0 & U \end{pmatrix}$, where $t \geq 3$ and $U = diag(\omega_1, \cdots, \omega_{q-t}) \geq 0$. Then $N = \begin{pmatrix} N_1 & 0 \\ 0 & I_{p-q} \end{pmatrix}$ is an orthogonal matrix of order p and

$$N'\Gamma'M\Gamma N = \begin{pmatrix} I_t & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{pmatrix} \triangleq G.$$

We here write

$$z = N'\Gamma'\tilde{P}'\tilde{y}_s, \gamma = N'\Gamma'\tilde{P}'\tilde{X}_s\beta \tag{3.2}$$

and $(n-p)\hat{\sigma}^2 = \tilde{y}_s'(I_n - \tilde{X}_s(\tilde{X}_s'\tilde{X}_s)^{-1}\tilde{X}_s')\tilde{y}_s$. Then eq. (3.2) implies that

$$z \sim N_p(\gamma, \sigma^2 I_p), (n-p)\sigma^{-2}\hat{\sigma}^2 \sim \chi^2_{n-p}, \tag{3.3}$$

and $z, \hat{\sigma}^2$ are mutually independent. We also have

$$\tilde{L}_1\tilde{y}_s = \tilde{S}_1\tilde{P}\Gamma NGz, \tilde{S}_1\tilde{X}_s\beta = \tilde{S}_1\tilde{P}\Gamma N\gamma. \tag{3.4}$$

By $\tilde{S}_1\tilde{P}\Gamma(\Gamma'\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}\Gamma)^-\Gamma'\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}\Gamma = \tilde{S}_1\tilde{P}\Gamma$, Lemma 3.4 and eq. (3.4), $\tilde{L}_1\tilde{y}_s$ is an admissible estimator of $\tilde{S}_1\tilde{X}_s\beta$ under loss $(d-\tilde{S}_1\tilde{X}_s\beta)'(d-\tilde{S}_1\tilde{X}_s\beta)$ if and only if $\Gamma'\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}\Gamma NGz$ is an admissible estimator of $g(\gamma)$ under the loss function $(d-g(\gamma))'(d-g(\gamma))$, where

$$g(\gamma) = \Gamma'\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}\Gamma N\gamma.$$

Since

$$N'\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & I_{p-q} \end{pmatrix}\Gamma'\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}\Gamma NGz = Gz \text{ and } N'\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & I_{p-q} \end{pmatrix}\Gamma'\tilde{P}'\tilde{S}_1'\tilde{S}_1\tilde{P}\Gamma N\gamma = \tilde{I}_q\gamma,$$

where $\tilde{I}_q = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}$, $\tilde{L}_1\tilde{y}_s$ is an admissible estimator of $\tilde{S}_1\tilde{X}_s\beta$ under loss $(d-\tilde{S}_1\tilde{X}_s\beta)'(d-\tilde{S}_1\tilde{X}_s\beta)$ if and only if Gz is an admissible estimator of $\tilde{I}_q\gamma$ under the loss function $(d-\tilde{I}_q\gamma)'(d-\tilde{I}_q\gamma)$ by Lemma 3.4. Partition z and γ as

$$z = \begin{pmatrix} z_{(1)} \\ z_{(2)} \\ z_{(3)} \end{pmatrix} \begin{matrix} t \times 1 \\ (q-t) \times 1 \\ (p-q) \times 1 \end{matrix}, \quad \gamma = \begin{pmatrix} \gamma_{(1)} \\ \gamma_{(2)} \\ \gamma_{(3)} \end{pmatrix} \begin{matrix} t \times 1 \\ (q-t) \times 1 \\ (p-q) \times 1 \end{matrix}.$$

Then the loss of Gz is expressed as

$$\begin{aligned} & (Gz - \tilde{I}_q\gamma)'(Gz - \tilde{I}_q\gamma) \\ = \; & (z_{(1)} - \gamma_{(1)})'(z_{(1)} - \gamma_{(1)}) + (Uz_{(2)} - \gamma_{(2)})'(Uz_{(2)} - \gamma_{(2)}). \end{aligned}$$

Hence, to verify this theorem, we need only to show that $z_{(1)}$ is an inadmissible estimator of $\gamma_{(1)}$ under the loss function $(d - \gamma_{(1)})'(d - \gamma_{(1)})$. By eq. (3.3), $z_{(1)} \sim N_t(\gamma_{(1)}, \sigma^2 I_t)$ with $t \geq 3$. Take

$$w = [1 - 2c(n-p)(n-p+2)^{-1}\hat{\sigma}^2(z'_{(1)}z_{(1)})^{-1}]z_{(1)}$$

with a constant c. Using integration by parts, we have

$$\begin{aligned} & E(z_{(1)} - \gamma_{(1)})'(z_{(1)} - \gamma_{(1)}) - E(w - \gamma_{(1)})'(w - \gamma_{(1)}) \\ = \; & \frac{4c(n-p)}{(n-p+2)}E[\frac{(z_{(1)} - \gamma_{(1)})'z_{(1)}\hat{\sigma}^2}{z'_{(1)}z_{(1)}} - \frac{c(n-p)\hat{\sigma}^4}{(n-p+2)z'_{(1)}z_{(1)}}] \\ = \; & \frac{4c(n-p)(t-2-c)\sigma^4}{n-p+2}E(\frac{1}{z'_{(1)}z_{(1)}}) > 0, \end{aligned}$$

if c is specified as one satisfying $0 < c < t-2$. This proves that $z_{(1)}$ is inadmissible.

4 Concluding Remarks

In this paper, necessary and sufficient conditions are given for homogeneous linear predictors to be admissible in the class of homogeneous linear predictors under the linear model (1.1). Sufficient conditions are also given for homogeneous linear predictors to be admissible in the class of all predictors under the linear model (1.2). They are proved to be necessary under additional conditions. However, it is also interesting to study the minimaxity of homogeneous linear predictors of the finite population regression coefficient under a balanced loss function. This will be studied in the other paper.

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平衡损失下有限总体回归系数的线性可容许预测

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摘 要: 本文研究了平衡损失函数下正态总体和非正态总体中有限回归系数的可容许预测. 利用统计决策理论，获得了非正态总体中齐次线性预测为可容许预测的充分必要条件和在正态总体中齐次线性预测在一切预测类中可容许性的充要条件, 推广了二次损失下的若干相关结果.

关键词: 可容许预测; 有限总体回归系数; 平衡损失函数

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