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ADJACENT VERTEX DISTINGUISHING TOTAL COLORINGS OF GRAPHS WITH SMALLER DEGREES

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Abstract: In this paper, we study the adjacent vertex distinguishing total colorings of graphs with maximum degree three and no adjacent Δ-vertices. By the technique of splitting edges, graphs with more special situations are constructed. And then we obtain the upper bound of adjacent vertex distinguishing total chromatic numbers of these graphs. Up to present graph having maximum degree three such that its adjacent vertex distinguishing total chromatic number is six has been reported in current literature, our conclusion answers this problem partially.

Keywords: total colorings; adjacent vertex distinguishing total colorings

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1 Introduction

The graphs considered in this paper are connected, limited, undirected, simple graphs. The shorthand $[s,t]$ stands for a set $\{s, s+1, s+2, \dots, t\}$, where s and t are integers with $0 \le s < t$. Δ-vertices of a graph are the vertices of maximum degree. $N_G(u)$ is the set of neighbors of a vertex u in graph G.

Definition 1.1 Let $f : V(G) \cup E(G) \to C = [1,k]$ be a proper total coloring of a graph G. Let $C(u,f)$ be the color set of colors assigned to the vertex u and the edges incident to u such that $|C(u,f)| = d_G(u)+1$. If $C(u,f) \ne C(v,f)$ for every edge $uv \in E(G)$, then we called f an adjacent vertex distinguishing total k-coloring (k-AVDTC) of G. The minimum number of k colors required for which G admits a k-AVDTC is denoted as $\chi''_{as}(G)$, called the adjacent vertex distinguishing total chromatic number (AVDTCN).

$C = [1,k]$ in Definition 1.1 is called a color set of G. We write the color set $C = [1,k]$ as $f(V(G) \cup E(G)) = \{f(x) \mid x \in V(G) \cup E(G)\}$, sometimes.

Zhang et al. presented a conjecture on adjacent vertex distinguishing total coloring in [4]. It seems quite difficult to settle down this conjecture, since most research show it

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only by several special classes of graphs up to now. Meanwhile, no counterexamples to the conjecture were found. For simple graphs having maximum degree three, Wang [5], Hulgan [6] and Chen [7], independently, confirmed positively the conjecture by showing the adjacent vertex distinguishing total chromatic numbers of simple graphs having maximum three is at most 6. Wang and Wang [8, 9] showed that

(1) Let G be an outerplanar graph with $\Delta(G) \geq 3$. Then $\Delta(G)+1 \leq \chi''_{as}(G) \leq \Delta(G)+2$, and furthermore $\chi''_{as}(G) = \Delta(G)+2$ if and only if G has adjacent Δ-vertices.

(2) Let G be a planar graph. If the girth $g(G) \geq 8$ and $\Delta(G) = 3$, then $\chi''_{as}(G) \leq 5$. However, Hulgan [6] pointed out: although complete graphs of odd order show the conjectured bound is sharp for even maximum degree, many maximum degree three graphs, including the K_4, $K_{3,3}$ and Petersen graphs, have an AVDTC with only 5. He proposed the following problem:

Problem 1.2 For a graph G with $\Delta(G) = 3$, is the bound $\chi''_{as}(G) \leq 6$ sharp?

Conjecture 1.3 [4] Let G be a connected graph of order $n \geq 2$, then $\chi''_{as}(G) \leq \Delta(G)+3$.

Lemma 1.4 [4] Let T be a tree on order $n \geq 2$. Then

$$\chi''_{as}(T) = \begin{cases} \Delta(T)+1, & \text{if } T \text{ contians no adjacent } \Delta-\text{vertices;} \\ \Delta(T)+2, & \text{if } T \text{ contians adjacent } \Delta-\text{vertices.} \end{cases}$$

2 Main Results

Lemma 2.1 Let G be a simple graph with maximum degree 3 and minimum degree 2. If G has no adjacent Δ-vertices, and every vertex of degree 2 is adjacent only to two vertices of degree 3, then $\chi''_{as}(G) = 4$.

Proof By induction on orders of graphs.

Basis Step There are just two graphs that have two and four Δ-vertices shown in Figure 1, since no graphs G having five Δ-vertices when $\Delta(G) = 3$ and $\delta(G) \geq 2$. These graphs hold $\chi''_{as}(G) = 4$.

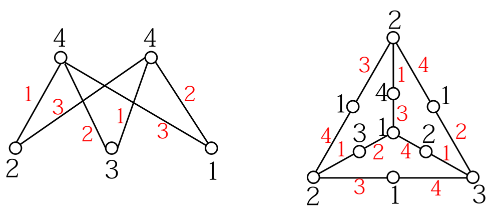

Figure 1 Two graphs with smaller orders.

Induction Step Suppose that the lemma holds for graphs of smaller orders. We have a graph $G' = G - \{u, w, v\} + \{a, b\}$, a is the vertex such that x adhere with y, b is the vertex such that s adhere with t, where G is Figure 2(a) and G' is Figure 2(b). From $|G| > |G'|$ and

G' satisfies the the lemma's hypothesis, we have $\chi''_{as}(G') = 4$ by induction hypothesis. Our goal is to show $\chi''_{as}(G) = 4$. Let f be an AVDTC of G' with $f(V(G') \cup E(G') = C = [1, 4]$.

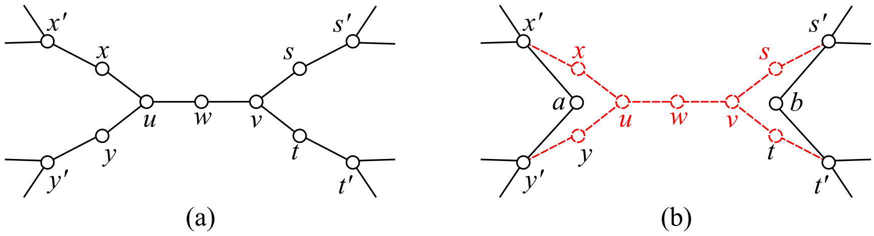

Figure 2 A diagram for illustrating Induction Step in the proof of Lemma 1.2.

We define a total coloring g of the graph G as: $g(x) = g(y) = f(a)$, $g(xx') = f(ax')$, $g(ux) = f(ay')$, $g(yy') = f(ay')$, $g(uy) = f(ax')$, $g(s) = g(t) = f(b)$, $g(ss') = f(bs')$, $g(vs) = f(bt')$, $g(tt') = f(bt')$, $g(vt) = f(bs')$. Therefore, $C(x, f) = C(y, f)$, $C(s, f) = C(t, f)$. Take $g(z) = f(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{u, w, v, x, y, s, t, ux, uy, vs, vt, uw, wv, xx', yy', ss', tt'\}$. By analysis, if we accord to color G in the mapping of g, then some situations don't satisfy a proper total coloring with four colors. Now we will use the following cases to solve it. Let h be a new total coloring of the graph G.

Case 1 $f(a) = f(b)$.

Without loss of generality, suppose $f(a) = f(b) = 1$, $f(ax') = 2$, $f(ay') = 3$, then $g(u) = 4$, $g(uw) = 1$.

Case 1.1 $C(x, g) = C(y, g) = C(s, g) = C(t, g) = \{1, 2, 3\}$.

Case 1.1.1 $f(bs') = 2$, $f(bt') = 3$.

The possible case is $g(t') = 2$ or 4. If $g(t') = 2$, we set $h(t) = 4$, $h(vt) = 1$, thus $h(v) = 2$ and $h(wv) = 4$, $h(w) = 3$. If $g(t') = 4$, then we set $h(t) = 2$ and $h(vt) = 1$. Consequently, $h(v) = 4$, $h(wv) = 2$ and $h(w) = 3$. Next, we define $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vt, wv\}$. Therefore, h is an AVDTC of G.

Case 1.1.2 $f(bs') = 3$, $f(bt') = 2$.

In this case, $g(t') = 3$ or 4. If $g(t') = 3$, then set $h(t) = 4$, $h(vt) = 1$. Hence, $h(v) = 3$, $h(wv) = 4$, $h(w) = 2$. If $g(t') = 4$, then set $h(t) = 3$, $h(vt) = 1$, which induce $h(v) = 4$, $h(wv) = 3$ and $h(w) = 2$. Next, we let $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vt, wv\}$. So, h is an AVDTC of G.

Case 1.2 $C(x, g) = C(y, g) = \{1, 2, 3\}$, $C(s, g) = C(t, g) = \{1, 2, 4\}$.

Case 1.2.1 $f(bs') = 2$, $f(bt') = 4$.

We know $g(t') = 2$ or 3 in this case. If $g(t') = 2$, then set $h(t) = 3$, $h(vt) = 1$. Clearly, $h(v) = 2$, $h(wv) = 4$, $h(vs) = 3$, $h(w) = 3$. If $g(t') = 3$, then set $h(t) = 2$, $h(vt) = 1$. We can define $h(v) = 4$, $h(wv) = 2$, $h(vs) = 3$, $h(w) = 3$, and set $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vs, vt, wv\}$. We can say that h is an AVDTC of G.

Case 1.2.2 $f(bs') = 4$, $f(bt') = 2$.

Clearly, $g(t') = 3$ or 4. If $g(t') = 3$, then set $h(t) = 4$, $h(vt) = 1$, which induce $h(v) = 3$,

$h(wv) = 4$ and $h(w) = 2$. If $g(t') = 4$, then set $h(t) = 3$, $h(vt) = 1$. Consequently, we set $h(v) = 4$, $h(wv) = 3$, $h(w) = 2$. Hence, we can define $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vt, wv\}$. Thereby, h is an AVDTC of G.

Case 1.3 $C(x, g) = C(y, g) = \{1, 2, 3\}$, $C(s, g) = C(t, g) = \{1, 3, 4\}$.

Case 1.3.1 $f(bs') = 3$, $f(bt') = 4$.

It is not hard to see that $g(t') = 2$ or 3. If $g(t') = 2$, then set $h(t) = 3$, $h(vt) = 1$, which induce $h(v) = 4$, $h(wv) = 3$, $h(vs) = 2$ and $h(w) = 2$. If $g(t') = 3$, then set $h(t) = 2$, $h(vt) = 1$. We will get $h(v) = 3$, $h(wv) = 4$, $h(vs) = 2$, $h(w) = 2$. We define $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vs, vt, wv\}$. This total coloring of h is an AVDTC of G.

Case 1.3.2 $f(bs') = 4$, $f(bt') = 3$.

Here, $g(t') = 2$ or 4. If $g(t') = 2$, then set $h(t) = 4$, $h(vt) = 1$, which induce $h(v) = 2$, $h(wv) = 4$ and $h(w) = 3$. If $g(t') = 4$, then set $h(t) = 2$, $h(vt) = 1$. We define $h(v) = 4$, $h(wv) = 2$, $h(w) = 3$ and $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vt, wv\}$. So, h is an AVDTC of G.

Case 2 $f(a) \neq f(b)$.

Without loss of generality, we assume that $f(a) = 1$, $f(b) = 2$, $f(ax') = 2$ and $f(ay') = 3$. Then $g(u) = 4$, $g(uw) = 1$.

Case 2.1 $C(x, g) = C(y, g) = C(s, g) = C(t, g) = \{1, 2, 3\}$.

Clearly, we can set $h(wv) = 2$, $h(w) = 3$, and $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, wv\}$. Hence, h is an AVDTC of G.

Case 2.2 $C(x, g) = C(y, g) = \{1, 2, 3\}$, $C(s, g) = C(t, g) = \{1, 2, 4\}$.

Case 2.2.1 $f(bs') = 1$, $f(bt') = 4$.

We know $g(t') = 1$ or 3. If $g(t') = 1$, we set $h(t) = 3$, $h(vt) = 2$, and furthermore $h(v) = 1$, $h(wv) = 3$, $h(w) = 2$; if $g(t') = 3$, then set $h(t) = 1$, $h(vt) = 2$. We can set $h(v) = 3$, $h(wv) = 1$, $h(w) = 2$, and $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vt, wv\}$. It means that h is an AVDTC of G.

Case 2.2.2 $f(bs') = 4$, $f(bt') = 1$.

It is not hard to see $g(t') = 3$ or 4. If $g(t') = 3$, we set $h(t) = 4$, $h(vt) = 2$, and define $h(v) = 3$, $h(wv) = 4$, $h(w) = 2$; if $g(t') = 4$, we let $h(t) = 3$, $h(vt) = 2$. For other vertices and edges, we set $h(v) = 4$, $h(wv) = 3$, $h(w) = 2$, and $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, v, t, vt, wv\}$. It is obvious that h is an AVDTC of G.

Case 2.3 $C(x, g) = C(y, g) = \{1, 2, 3\}$, $C(s, g) = C(t, g) = \{2, 3, 4\}$.

Thereby, set $h(wv) = 2$, $h(w) = 3$. We define $h(z) = g(z)$ for $z \in \{V(G) \cup E(G)\} \setminus \{w, wv\}$. Eventually, h is an AVDTC of G.

The lemma follows from the principle of induction.

Lemma 2.2 Let G be a simple graph with maximum degree 3 and minimum degree 2. Suppose that G has no adjacent Δ-vertices and every vertices of degree 2 is adjacent only to vertices of degree 3. Replacing an edge uv with a path uwv, where $uv \in E(G)$ and $w \notin V(G)$, produces a new graph H such that $\chi''_{as}(H) \leq 5$.

Proof Let f be a k-AVDTC of a simple graph G of maximum degree 3 with $k \geq$

$\chi''_{as}(G)$, and $C = [1, k]$ be the color set of G under the coloring f. We show the lemma in the following cases. Let $f(uv) = \alpha$ for an edge $uv \in E(G)$, where $d_G(u) = 2$, $d_G(v) = \Delta(G) = 3$, $N_G(u) = \{x, v\}$, $N_G(v) = \{u, w_1, w_2\}$. An edge uv in G is replaced by a path uwv, where $w \notin V(G)$. We define a total coloring h of H as following. First, set $h(wv) = \alpha$.

Case 1 $f(x) \neq \alpha$.

Let $\{a_1\} \subseteq C \setminus C(u, f)$, then $\alpha \neq a_1$, $h(ux) \neq a_1$. We define $h(u) = \alpha$, $h(uw) = a_1$, $h(w) = f(u)$, $h(z) = f(z)$ for $z \in (S_1 \setminus \{uv\}) \subseteq (S_2 \setminus \{uw, w, wv, u\})$, where $S_1 = V(G) \cup E(G)$, $S_2 = V(H) \cup E(H)$. Notice that the color set of colors assigned to every vertex $z \in V(H) \setminus \{u, w\}$ is as the same as one in G. Since $C(u, h) = \{h(ux), \alpha, a_1\}$, $C(w, h) = \{f(u), \alpha, a_1\}$, $h(ux) \in C(u, h)$ and $h(ux) \notin C(w, h)$, therefore, $C(u, h) \neq C(w, h)$. Based on $d_G(u) \neq d_G(x)$, $d_G(w) \neq d_G(v)$, we have $C(u, h) \neq C(x, h)$, $C(w, h) \neq C(v, h)$. We conclude that h is an AVDTC of H, and $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Case 2 $f(x) = \alpha$.

Case 2.1 $f(v) \neq f(ux)$.

Let $h(u) = f(v)$, $h(uw) = f(u)$, $h(w) \in C \setminus \{\alpha, h(u), h(v)\}$. And $h(z) = f(z)$ for $z \in (S_1 \setminus \{uv\}) \subseteq (S_2 \setminus \{uw, w, wv, u\})$, where $S_1 = V(G) \cup E(G)$ and $S_2 = V(H) \cup E(H)$. We can see that the color set of each vertex $z \in V(H) \setminus \{u, w\}$ is as the same as one in G. Notice that $C(u, h) = \{h(ux), h(u), h(uw)\}$, $C(w, h) = \{\alpha, h(uw), h(w)\}$; for $\alpha \in C(w, h)$, $\alpha \notin C(u, h)$, Thereby, $C(u, h) \neq C(w, h)$. Since $d_G(w) \neq d_G(v)$ for $d_G(u) \neq d_G(x)$, we obtain $C(u, h) \neq C(x, h)$ and $C(w, h) \neq C(v, h)$. The above facts show that h is an AVDTC of H, and $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Case 2.2 $f(v) = f(ux)$.

In this case, we cannot give graph H an AVDTC with four colors. Then we prove that graph H satisfies an AVDTC with five colors. Let $h(u) = f(vw_1)$, $h(uw) = f(vw_2)$, $h(w) \in [1, 5] \setminus \{\alpha, f(v), f(vw_1), f(vw_2)\}$. And $h(z) = f(z)$ for $z \in (S_1 \setminus \{uv\}) \subseteq (S_2 \setminus \{uw, w, wv, u\})$, where $S_1 = V(G) \cup E(G)$ and $S_2 = V(H) \cup E(H)$. We can see that the color set of each vertex $z \in V(H) \setminus \{u, w\}$ is as the same as one in G. Notice that $C(u, h) = \{f(v), f(vw_1), f(vw_2)\}$, $C(w, h) = \{\alpha, f(vw_2), h(w)\}$; for $h(w) \in C(w, h)$, $h(w) \notin C(u, h)$, Thereby, $C(u, h) \neq C(w, h)$. Since $d_G(w) \neq d_G(v)$ for $d_G(u) \neq d_G(x)$, we obtain $C(u, h) \neq C(x, h)$ and $C(w, h) \neq C(v, h)$. The above facts show that h is an 5-AVDTC of H, which implies $\chi''_{as}(H) \leq 5$.

The lemma is covered.

Lemma 2.3 By Lemma 2.2, we can construct a graph G with maximum degree 3 such that G has a path $P = ux_1x_2x_3v$, where $d_G(u) = d_G(v) = 3$, $d_G(x_i) = 2$ for $i = 1, 2, 3$. Adding a new vertex w to G, and then joining w and x_2 by an edge produces a new graph H. Then $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Proof Let f be an AVDTC of a simple graph G, and C is the color set under the coloring f. We define a total coloring τ of H in the following: $\tau(wx_2) \in C \setminus C(x_2, f)$, $\tau(w) = f(x_1x_2)$ and $\tau(z) = f(z)$, $z \in V(H) \cup E(H) \setminus \{wx_2, w\}$. Clearly, τ is an AVDTC of H, that is, $\chi''_{as}(H) \leq \chi''_{as}(G)$, as desired.

By Lemma 2.3, we can construct a graph G with maximum degree 3 such that G contains an edge uv that $d_G(u) = 1$, $d_G(v) = \Delta(G) = 3$.

Lemma 2.4 Let G be a simple graph with maximum degree 3. If $d_G(u) = 1$, $d_G(v) = \Delta(G) = 3$ for an edge $uv \in E(G)$, and then replacing edge uv by a path uw_0v ($w_0 \notin V(G)$) produces a new graph H holding $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Proof Let f be a k-AVDTC of a simple graph G of maximum degree 3 with $k \geq \chi''_{as}(G)$. Let $f(uv) = \alpha$ for an edge $uv \in E(G)$. Let $N_G(v) = \{u, w_1, w_2\}$. We need only consider the colors assigned to the vertex w_0 and the edges uw_0, w_0v in H. Define a total coloring g of G as following: $g(w_0v) = \alpha$, $g(u) = \alpha$, $g(uw_0) = f(u)$, $g(w_0) \in C \setminus \{\alpha, g(v), g(uw_0)\}$; $g(z) = f(z)$ for $z \in (S_1 \setminus \{uv\}) \subseteq (S_2 \setminus \{uw_0, w_0, w_0v, u\})$, where $S_1 = V(G) \cup E(G)$, $S_2 = V(H) \cup E(H)$. Observe that the color set of each vertex $z \in V(H) \setminus \{w\}$ keeps no change G. Obviously, $C(u, g) \neq C(w_0, g)$ and $C(w_0, g) \neq C(v, g)$ since $d_H(w_0) = 2$. Hence, g is an AVDTC of G, which it leads to $\chi''_{as}(H) \leq \chi''_{as}(G)$, we are done.

Lemma 2.5 Let G be a simple graph with maximum degree 3. Suppose that G has no pair of vertices with the same degree and every vertices of degree 1 is adjacent only to vertex of degree 3. Joining a pair of vertices of degree 1 with an edge produces a new graph H. Then $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Proof Let f be an adjacent total coloring of G. Let f^* be a total coloring of H.

Graph H is constructed from G, see in Figure 3(a).

Case 1 $f(v_1) = f(v_2)$. Set $f^*(v_2) = f(vv_1)$, $f^*(v_1v_2) = f(v)$, $f^*(x) = f(x), x \in V(G) \cup E(G) \setminus \{v_2\}$. Then we have $C(u_1, f^*) \neq C(v_1, f^*)$ for $f^*(v_1) \notin C(v_2, f^*)$, $f^*(v_1) \in C(v_1, f^*)$. $C(v, f^*) \neq C(v_1, f^*)$, $C(v, f^*) \neq C(v_2, f^*)$ for $|C(v, f^*)| = 4$, $|C(v_1, f^*)| = |C(v_2, f^*)| = 3$. Therefore, f^* is an adjacent total coloring of H, and $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Case 2 $f(v_1) \neq f(v_2)$. Set $f^*(v_1v_2) = f(v)$, $f^*(x) = f(x), x \in V(G) \cup E(G)$. Then we have $C(u_1, f^*) \neq C(v_1, f^*)$ for $f^*(vv_2) \notin C(v_1, f^*)$, $f^*(vv_2) \in C(v_2, f^*)$. $C(v, f^*) \neq C(v_1, f^*)$, $C(v, f^*) \neq C(v_2, f^*)$ for $|C(v, f^*)| = 4$, $|C(v_1, f^*)| = |C(v_2, f^*)| = 3$. Therefore, f^* is an adjacent total coloring of H, and $\chi''_{as}(H) \leq \chi''_{as}(G)$.

Graph H is constructed from G, see in Figure 3(b). Set

$$\begin{aligned}
&f^*(v_1) = f(u), f^*(vv_1) = f(u_1), f^*(wv) = f(u),\\
&f^*(v) = C(v, f) \setminus \{f^*(v_1), f^*(w), f^*(wv), f^*(vv_1)\},\\
&f^*(vv_2) = C(v, f) \setminus \{f^*(v), f^*(wv), f^*(vv_1)\},\\
&f^*(u_1v_1) = C(v, f) \setminus \{f^*(u_1), f^*(v_1), f^*(uu_1), f^*(vv_1)\},\\
&f^*(v_2) = C(v, f) \setminus \{f^*(v), f^*(vv_2)\},\\
&f^*(x) = f(x), x \in V(G) \cup E(G) \setminus \{v, v_1, v_2, vv_1, vv_2, wv\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&C(u_1, f^*) \neq C(v_1, f^*) \text{ for } f^*(u) \notin C(u_1, f^*), f^*(u) \in C(v_1, f^*);\\
&C(v, f^*) \neq C(v_1, f^*), C(v, f^*) \neq C(v_2, f^*) \text{ for } |C(u, f^*)| = |C(v, f^*)| = 4,\\
&|C(v_1, f^*)| = |C(v_2, f^*)| = 3.
\end{aligned}$$

Therefore, f^* is an adjacent total coloring of H, and $\chi''_{as}(H) \leq \chi''_{as}(G)$.

The lemma is completed.

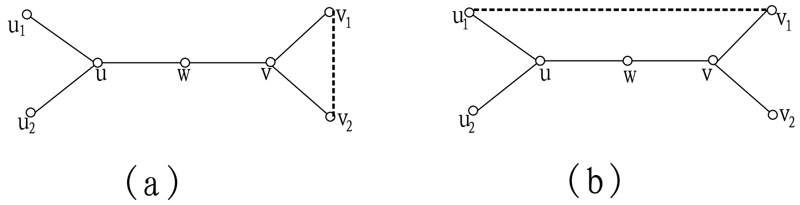

Figure 3 Situations of Lemma 2.5.

Theorem 2.6 Let H be a simple graph with maximum degree 3. Then $\chi''_{as}(H) \leq 5$ if H has no adjacent Δ-vertices.

Proof The theorem is proved in three cases as following:

Case 1 Let H be a tree with $\Delta(H) = 3$, by Lemma 1.4, $\chi''_{as}(H) = 4$ if H contains no adjacent Δ-vertices.

Case 2 If every vertex of degree 2 is adjacent only to vertices of degree 3 and minimum degree is 2 in the graph H of the theorem, we have $\chi''_{as}(H) = 4$ by Lemma 2.1.

Case 3 If H contains 2-degree vertices such that they are adjacent to some 2-degree vertices or 1-degree vertices, or 3-degree vertices are adjacent to some 1-degree vertices. In this case, we can use the methods in the proofs of Lemmas 2.2, 2.3, 2.4 and 2.5 to construct H from a simple graph G such that

(1) G is a simple graph with $\Delta(G) = 3$, $\delta(G) = 2$ and every vertex of degree 2 is adjacent only to two vertices of degree 3;

(2) G is a simple graph with $\Delta(G) = 3$ and no pair of vertices with the same degree, except some cycles with order 3 or 5 have adjacent vertices of degree 2. Lemmas 2.2, 2.3, 2.4 and 2.5 enable us to obtain $\chi''_{as}(H) \leq 5$.

The Theorem is completed.

3 Problem

Remark For a simple graph H with $\Delta(H) = 3$ and H has no adjacent Δ-vertices, is the bound $\chi''_{as}(H) \leq 5$ sharp?

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小度数图的邻点可区别全染色

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摘要: 本文研究了最大度为3 且没有相邻最大度的图的邻点可区别全染色. 利用边剖分的方法, 构造了此类图更为一般的情形, 得到了它们的邻点可区别全色数的上界. 目前, 未找到最大度为3 的图且它的邻点可区别全色数是6. 本文的结果部分地回答了这个问题.

关键词: 全染色; 邻点可区别全染色

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