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HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN A HYPERBOLIC SPACE

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Abstract: In the paper, we study an n-dimensional complete connected and oriented hypersurface M^N in $H^{n+1}(-1)$ with constant mean curvature and two distinct principal curvatures, one of which is simple. By using the moving frames, we obtain that if the squared norm of second fundamental form of M^n satisfies a rigidity condition (1.3), the M^n is isometric to hyperbolic cylinder.

Keywords: hypersurfaces; hyperbolic space form; constant mean curvature

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1 Introduction

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian space form with constant sectional curvature c. According to $c>0$, $c=0$ or $c<0$, it is called sphere space, Euclidean space or hyperbolic space, respectively, and it denoted by $S^{n+1}(c)$, R^{n+1} or $H^{n+1}(c)$. Let M^n be an $n-$dimensional hypersurface in $M^{n+1}(c)$. As it is well known there many rigidity results with constant mean curvature, constant scalar curvature or constant k-th mean curvature in $M^{n+1}(c)$, for example, see [1–6] in $S^{n+1}(c)$ or R^{n+1} and [7–9] in hyperbolic space $H^{n+1}(c)$.

In [9], Wu proved the following theorem.

Theorem 1.1 Let M^n $(n\geq 3)$ be a complete hypersurface in $H^{n+1}(-1)$ with constant mean curvature H ($|H|>1$) and two distinct principal curvatures with multiplicities $n-1$, 1. Set

$$S_{\pm}=-n+\frac{n^3H^2}{2(n-1)}\mp\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4-4(n-1)H^2}. \tag{1.1}$$

If the square length of the second fundamental form satisfies $S\leq S_+$ or $S\geq S_-$, then $S=S_+$ or $S=S_-$, and M^n is isometric to hyperbolic cylinder $S^{n-1}(\lambda_+^2-1)\times H^1(\frac{1}{\lambda_+^2}-1)$

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or $H^{n-1}(\lambda_-^2-1)\times S^1(\frac{1}{\lambda_-^2}-1)$, here

$$\lambda_\pm=\frac{n|H|\pm\sqrt{n^2H^2-4(n-1)}}{2(n-1)}. \tag{1.2}$$

In this note, we shall also investigate n-dimensional hypersurfaces with constant curvature H ($|H|>1$) in $H^{n+1}(-1)$ and obtain the following result:

Theorem 1.2 Let M^n ($n\geq 3$) be a complete hypersurface in $H^{n+1}(-1)$ with constant mean curvature H ($|H|>1$) and two distinct principal curvatures with multiplicities $n-1$, 1. If the square length of the second fundamental form satisfies

$$S_+\leq S\leq S_-, \tag{1.3}$$

then $S=S_+$ or $S=S_-$, and M^n is isometric to hyperbolic cylinder $S^{n-1}(\lambda_+^2-1)\times H^1(\frac{1}{\lambda_+^2}-1)$ or $H^{n-1}(\lambda_-^2-1)\times S^1(\frac{1}{\lambda_-^2}-1)$, here

$$\lambda_\pm=\frac{n|H|\pm\sqrt{n^2H^2-4(n-1)}}{2(n-1)}. \tag{1.4}$$

2 Preliminaries

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian space form with constant sectional curvature c. Let M^n be an n-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)$. We choose a local orthonormal frame $e_1,\cdots,e_n,e_{n+1}$ in $M^{n+1}(c)$ such that $e_1,\cdots,e_n$ are tangent to M^n. Let $\omega_1,\cdots,\omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1\leq A,B,\cdots\leq n+1;\quad 1\leq i,j,\cdots\leq n.$$

The structure equations of $M^{n+1}(c)$ are given by

$$d\omega_A=\sum_B\omega_{AB}\wedge\omega_B,\quad \omega_{AB}+\omega_{BA}=0, \tag{2.1}$$

$$d\omega_{AB}=\sum_C\omega_{AC}\wedge\omega_{CB}+\Omega_{AB}, \tag{2.2}$$

$$\Omega_{AB}=-\frac{1}{2}\sum_{CD}K_{ABCD}\omega_C\wedge\omega_D, \tag{2.3}$$

$$K_{ABCD}=c(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}). \tag{2.4}$$

Restricting to M^n such that

$$\omega_{n+1}=0,\quad \omega_{n+1i}=\sum_j h_{ij}\omega_j,\quad h_{ij}=h_{ji}. \tag{2.5}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.8}$$

$$n(n-1)(r-c) = n^2H^2 - S, \tag{2.9}$$

where $n(n-1)r$ is the scalar curvature, H is the mean curvature and S is the squared of the second fundamental form of M^n.

Let M^n be an n $(n \geq 3)$ dimensional complete connected and oriented hypersurface in $M^{n+1}(c)$ with constant mean curvature and with two distinct principal curvatures, one of which is simple. Without loss of generality , we may assume

$$\lambda_1 = \cdots = \lambda_{n_1} = \lambda, \quad \lambda_n = \mu, \tag{2.10}$$

where λ_i for $i = 1, \cdots, n$ are the principal curvatures of M^n. We have

$$(n-1)\lambda + \mu = nH, \quad S = (n-1)\lambda^2 + \mu^2. \tag{2.11}$$

From (2.9) and (2.11), we have, for $c = -1$, that

$$\lambda\mu = (n-1)(r+1) - (n-2)H^2 + (n-2)\sqrt{H^4 - (r+1)H^2}, \tag{2.12}$$

on M^n, or

$$\lambda\mu = (n-1)(r+1) - (n-2)H^2 - (n-2)\sqrt{H^4 - (r+1)H^2}, \tag{2.13}$$

on M^n.

3 Proof of Theorem

Let M^n be a connected hypersurface in $H^{n+1}(-1)$ with constant mean curvature and two distinct principal curvatures λ, μ with multiplicities $n-1, 1$. Since the multiplicites are constant, it is easy to know that their eigenspaces are completely integrable. Let s be the parameter of arc length of the goedesics corresponding to μ, and we may put $\omega_n = ds$. Then λ and μ are locally functions of s. Let $\omega = |\lambda - H|^{-\frac{1}{n}}$. In [9], B.Y. Wu got the following equations:

$$\frac{d^2\omega}{ds^2} + \omega(-1 + H^2 + (2-n)H\omega^{-n} + (1-n)\omega^{-2n}) = 0 \tag{3.1}$$

for $\lambda > H$ or

$$\frac{d^2\omega}{ds^2} + \omega(-1 + H^2 + (n-2)H\omega^{-n} + (1-n)\omega^{-2n}) = 0 \tag{3.2}$$

for $\lambda < H$. Integrating (3.1) and (3.2), we get

$$(\frac{d\omega}{ds})^2 + (-1+H^2)\omega^2 + 2H\omega^{2-n} + \omega^{2-2n} = C \tag{3.3}$$

for $\lambda > H$ or

$$(\frac{d\omega}{ds})^2 + (-1+H^2)\omega^2 - 2H\omega^{2-n} + \omega^{2-2n} = C \tag{3.4}$$

for $\lambda < H$.

We first obtain the following propositions:

Proposition 3.1 Let M^n be an n $(n \geq 3)$-dimensional complete connected hypersurface in $H^{n+1}(-1)$ with constant mean curvature H $(|H| > 1)$ and two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. If $\lambda\mu - 1 \geq 0$, then M^n is isometric to hyperbolic cylinder $S^{n-1}(\lambda_+^2 - 1) \times H^1(\frac{1}{\lambda_+^2} - 1)$ or $H^{n-1}(\lambda_-^2 - 1) \times S^1(\frac{1}{\lambda_-^2} - 1)$.

Proof Let λ and μ be the two distinct principal curvatures of M^n with multiplicities $(n-1)$ and 1 respectively. Then, from $nH = (n-1)\lambda + \mu$ and $\omega = |\lambda - H|^{-\frac{1}{n}}$, we have the following :

$$\lambda\mu - 1 = -1 + H^2 + (2-n)H\omega^{-n} + (1-n)\omega^{-2n} \tag{3.5}$$

for $\lambda > H$ or

$$\lambda\mu - 1 = -1 + H^2 + (n-2)H\omega^{-n} + (1-n)\omega^{-2n} \tag{3.6}$$

for $\lambda < H$.

Then, if $\lambda\mu - 1 \geq 0$, we obtain

$$-1 + H^2 + (2-n)H\omega^{-n} + (1-n)\omega^{-2n} \geq 0 \tag{3.7}$$

for $\lambda > H$ or

$$-1 + H^2 + (n-2)H\omega^{-n} + (1-n)\omega^{-2n} \geq 0 \tag{3.8}$$

for $\lambda < H$. From (3.1) and (3.2), we have $\frac{d^2\omega}{ds^2} \leq 0$. Thus $\frac{d\omega}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, $\omega(s)$ must monotonic when s tends to infinity. From (3.3) and (3.4), we know that the positive function $\omega(s)$ is bounded from above. Since $\omega(s)$ is bounded and is monotonic when s tends infinity, we find that both $\lim_{s\to+\infty}\omega(s)$ and $\lim_{s\to-\infty}\omega(s)$ exist and then we have

$$\lim_{s\to+\infty}\frac{d\omega(s)}{ds} = \lim_{s\to-\infty}\frac{d\omega(s)}{ds} = 0. \tag{3.9}$$

By the monotonicity of $\frac{d\omega(s)}{ds}$, we see that $\frac{d\omega(s)}{ds} \equiv 0$ and $\omega(s)$ is constant. Then we have λ and μ are constant, that is, M^n is isoparametric. According to Cartan [10], we know that M^n is isometric to the hyperbolic cylinder $S^{n-1}(\lambda_+^2 - 1) \times H^1(\frac{1}{\lambda_+^2} - 1)$ or $H^{n-1}(\lambda_-^2 - 1) \times S^1(\frac{1}{\lambda_-^2} - 1)$.

By using the same method in Proposition 3.1, we can obtain the following proposition:

Proposition 3.2 Let M^n be an n $(n \geq 3)$-dimensional complete connected hypersurface in $H^{n+1}(-1)$ with constant mean curvature H $(|H| > 1)$ and two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. If $\lambda\mu - 1 \leq 0$, then M^n is isometric to hyperbolic cylinder $S^{n-1}(\lambda_+^2 - 1) \times H^1(\frac{1}{\lambda_+^2} - 1)$ or $H^{n-1}(\lambda_-^2 - 1) \times S^1(\frac{1}{\lambda_-^2} - 1)$.

Proof of Theorem 1.2 Since M^n is oriented and the mean curvature H is constant, we can choose an orientation for M^n such that $H > 0$. From (2.9), we know that the inequality $S_+ \leq S \leq S_-$ is equivalent to

$$\begin{aligned}&\frac{1}{2(n-1)^2}[n^2H^2 - n\sqrt{n^2H^4 - 4(n-1)H^2} - 2(n-1)] \\ \leq\ & \frac{n(r+1)-2}{n-2} \leq \frac{1}{2(n-1)^2}[n^2H^2 + n\sqrt{n^2H^4 - 4(n-1)H^2} - 2(n-1)],\end{aligned} \tag{3.10}$$

where $n(n-1)r$ is the scalar curvature of M^n.

We define the function

$$f(x) = (n-1)^2x^2 - [n^2H^2 - 2(n-1)]x + 1. \tag{3.11}$$

Since $f(0) = 1$ and $|H| = H > 1$, we know that function (3.11) has two positive real roots.

$$x_{1,2} = \frac{1}{2(n-1)^2}[n^2H^2 \pm n\sqrt{n^2H^4 - 4(n-1)H^2} - 2(n-1)]. \tag{3.12}$$

It can be easily checked that $x_1 \leq x_2$ and if $x_1 \leq x \leq x_2$, then $f(x) \leq 0$.

Now we set $x = \frac{n(r+1)-2}{n-2}$, from (3.10), we have

$$f(\frac{n(r+1)-2}{n-2}) \leq 0. \tag{3.13}$$

If there exists a point p on M^n such that (2.12) and (2.13) hold at p, we have $H^2 = r+1$ at p, from (2.9), we have $S = nH^2$ at p, that is, p is a umbilical point on M^n, that is contradiction to M^n has no umbilical points. Therefore, we only consider two cases:

Case I If (2.12) holds on M^n, we shall prove that $-1 + \lambda\mu \geq 0$ on M^n. We consider three subcases:

(a) If $-1 + (n-1)(r+1) - (n-2)H^2 \geq 0$, then from (2.12), we have $-1 + \lambda\mu \geq 0$ on M^n.

(b) If $-1 + (n-1)(r+1) - (n-2)H^2 < 0$, suppose $-1 + \lambda\mu < 0$ on M^n, from (2.12), we have

$$(n-2)\sqrt{H^4 - (r+1)H^2} < -[-1 + (n-1)(r+1) - (n-2)H^2]. \tag{3.14}$$

Therefore, we have

$$\frac{(n-2)^2}{n^2}\{(n-1)^2[\frac{n(r+1)-2}{n-2}]^2 - [n^2H^2 - 2(n-1)]\frac{n(r+1)-2}{n-2} + 1\} > 0, \tag{3.15}$$

that is, $f(\frac{n(r+1)-2}{n-2})>0$. This is a contradiction to (3.13), we deduce that $-1+\lambda\mu\geq 0$ on M^n.

(c) If $-1+(n-1)(r+1)-(n-2)H^2\geq 0$ at a point p of M^n and $-1+(n-1)(r+1)-(n-2)H^2<0$ at other points of M^n, in this case, from (a) and (b), we have $-1+\lambda\mu\geq 0$ on M^n.

Therefore, we know that if (2.12) holds on M^n, then $-1+\lambda\mu\geq 0$ on M^n. By Proposition 3.1, we obtain that M^n is isometric to hyperbolic cylinder $S^{n-1}(\lambda_+^2-1)\times H^1(\frac{1}{\lambda_+^2}-1)$ or $H^{n-1}(\lambda_-^2-1)\times S^1(\frac{1}{\lambda_-^2}-1)$, and $\lambda_\pm=\frac{n|H|\pm\sqrt{n^2H^2-4(n-1)}}{2(n-1)}$.

Case II If (2.13) holds on M^n, we consider three subcases:

(d) If $-1+(n-1)(r+1)-(n-2)H^2\leq 0$, then from (2.13), we have $-1+\lambda\mu\leq 0$ on M^n.

(e) If $-1+(n-1)(r+1)-(n-2)H^2>0$ on M^n, suppose $-1+\lambda\mu>0$ on M^n, from (2.13), we have

$$(n-2)\sqrt{H^4-(r+1)H^2}<-1+(n-1)(r+1)-(n-2)H^2. \tag{3.16}$$

Therefore, we have

$$\frac{(n-2)^2}{n^2}\{(n-1)^2[\frac{n(r+1)-2}{n-2}]^2-[n^2H^2-2(n-1)]\frac{n(r+1)-2}{n-2}+1\}>0, \tag{3.17}$$

that is, $f(\frac{n(r+1)-2}{n-2})>0$. This is a contradiction to (3.13), we deduce that $-1+\lambda\mu\leq 0$ on M^n.

(f) If $-1+(n-1)(r+1)-(n-2)H^2\leq 0$ at a point p of M^n and $-1+(n-1)(r+1)-(n-2)H^2>0$ at other points of M^n, in this case, from (d) and (e), we have $-1+\lambda\mu\leq 0$ on M^n.

Therefore, we know that if (2.13) holds on M^n, then $-1+\lambda\mu\leq 0$ on M^n. By Proposition 3.2, we obtain that M^n is isometric to hyperbolic cylinder $S^{n-1}(\lambda_+^2-1)\times H^1(\frac{1}{\lambda_+^2}-1)$ or $H^{n-1}(\lambda_-^2-1)\times S^1(\frac{1}{\lambda_-^2}-1)$, and $\lambda_\pm=\frac{n|H|\pm\sqrt{n^2H^2-4(n-1)}}{2(n-1)}$.

This proves Theorem 1.2.

Remark Wu in Theorem 1.1 (Theorem 5.2 in [9]) considered the complete hypersurfaces in $H^{n+1}(-1)$ which satisfied the condition: $S\geq S_-$ or $S\leq S_+$. He obtained the existence of the global solutions of (3.3) or (3.4) under some conditions for C. From the existence of global solutions, he proved the Theorem 1.1. On the other hand, we in Theorem 1.2 consider the complete hypersurfaces in $H^{n+1}(-1)$ which satisfy the condition: $S_+\leq S\leq S_-$. We obtain that the sectional curvature $\lambda\mu-1$ of M^n satisfies that $\lambda\mu-1\geq 0$ or $\lambda\mu-1\leq 0$. From Proposition 3.1 or Proposition 3.2, we can prove Theorem 1.2.

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References

[1] Cheng Q M. Complete hypersurfaces in a Euclidean space R^{n+1} with constant scalar curvature[J]. Indiana Univ. Math. J., 2002, 51: 53–68.

[2] Cheng Q M. Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature[J]. J. of London Math. Society, 2001, 64: 755–768.

[3] Hu Z J. Complete hypersurfaces with constant mean curvature and nonnegative sectional curvature[J]. Pro. American Math. Soci., 1995, 123: 2835–2840.

[4] Shu S C, Liu S Y. Hypersurfaces with constant mean curvature in a real space form[J]. Turk. J. Math., 2010, 34: 1–10.

[5] Wei G X. Complete hypersurfaces with constant mean curvature in a unit sphere[J]. Monatsh. Math., 2006, 149: 251–258.

[6] Wei G X. Complete hypersurfaces in a Euclidean space R^{n+1} with constant mth mean curvature[J]. Diff. Geom. Appl., 2008, 26: 298–306.

[7] Hu Z J, Scherfner M, Zhai S. Hypersurfaces of the hyperbolic space with constant curvature[J]. Diff. Geo. Appli., 2007, 25: 594–611.

[8] Shu S C, Han A Y. Hypersurfaces in a hypervolic space with constant k-th mean curvarture[J]. Bull. Math. Soc. Sci. Math. Romumanie, 2009, 52: 65–78.

[9] Wu B Y. On complete hypersurfaces with two distinct principal curvatures in a hyperbolic space[J]. Balkan J. Geometry Appli., 2010, 15(2): 126–137.

[10] Cartan E. Sur des families remarquables d'hypersurfaces isoparamétriques dans les espaces shpériques[J]. Math. Z., 1939, 45: 335–367.

双曲空间形式中具有常平均曲率的超曲面

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摘要: 本文研究了双曲空间形式$H^{n+1}(-1)$中具有常平均曲率及两个离散主曲率(其中一个主曲率是1 -重)的完备连通可定向的n -维超曲面M^n. 利用活动标架, 得到如果M^n的基本形式的模长满足刚性条件(1.3), 那么M^n同构双曲柱面.

关键词: 超曲面; 双曲空间形式; 常平均曲率

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