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INTERCHANGE BETWEEN WEAK ORLICE-HARDY SPACES WITH CONCAVE FUNCTIONS THROUGH MARTINGALE TRANSFORMS

GUO Hong-ping[1], YU Lin[2], JIANG Qin[3]

(*1.Department of Mathematics and Finance, Hanjiang Normal University, Shiyan 442000, China*)
(*2.School of Science, China Three Gorges University, Yichang 443002, China*)
(*3.Department of Computer Science, Hanjiang Normal University, Shiyan 442000, China*)

Abstract: In this paper, we consider the interchanging relation between two weak Orlicz-Hardy spaces associated concave functions of martingales. By the means of martingale transform, we prove the result that the elements in weak Orlicz-Hardy space $\mathrm{w}\mathcal{H}_{\Phi_1}$ are none other than the martingale transforms of those in $\mathrm{w}\mathcal{H}_{\Phi_2}$, where Φ_1 is a concave Young function, Φ_2 is a concave or a convex Young function and $\Phi_1 \preceq \Phi_2$ in some sense. It extends the corresponding results in the literature from strong-type spaces to the setting of weak-type spaces, from norm inequalities to quasi-norm inequalities as well.

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1 Introduction

In this paper, we extend some classical results of martingale transforms from the strong-type spaces (normed space) to the setting of weak-type spaces (quasi-normed space). More precisely, we are interested in the characterization about the interchanging between weak Orlicz-Hardy space $\mathrm{w}\mathcal{H}_{\Phi_1}$ and $\mathrm{w}\mathcal{H}_{\Phi_2}$ in terms of Burkholder's martingale transforms.

The first motivation in this paper comes from the classical results of Chao and Long [2], as well as the similar results of Garsia [3] and Weisz [10]. The concept of martingale transforms was first introduced by Burkholder [1]. It is shown that the martingale transforms are especially useful to study the relations between the "predictable" Hardy spaces of martingales, such as $\mathcal{H}_p$, which is associated with the conditional quadratic variation of martingales. The "characterization" of such spaces via martingale transforms were provided in [2]: the elements in the space $\mathcal{H}_{p_1}$ are none other than the martingale transforms of those

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Biography: Guo Hongping (1987–), female, born at Xiantao, Hubei, lecturer, major in martingale theory and functional analysis.

in $\mathcal{H}_{p_2}$ for $0 < p_1 < p_2 < \infty$. All of those results can be found also in the monographs of Long [7] and Weisz [11].

Generally, the similar conclusions were obtained also in the case of Orlicz-Hardy spaces for martingales by Ishak and Mogyoródi [4], Meng and Yu [8] and Yu [14–15], according to different situations, respectively.

On the other hand, we also note that in recent years, the weak spaces, including their applications to harmonic analysis and martingale theory, have been got more and more attention. See for example Jiao [5], Nakai [9], Weisz [12–13]. Particularly, Liu, Hou and Wang [6] firstly introduced the weak Orlicz-Hardy spaces of martingales and discussed its basic properties and some martingale inequalities. Jiao [5] investigated the embedding relations between weak Orlicz martingale spaces.

This article will focus its attention on the relationship between the weak Orlicz-Hardy spaces $\mathrm{w}\mathcal{H}_{\Phi_1}$ and $\mathrm{w}\mathcal{H}_{\Phi_2}$, where Φ_1 and Φ_2 are two generalized Young functions (not need to be convex) and $\Phi_1 \preceq \Phi_2$ in some sense (see Definition 2.1). It will be shown that the elements in weak Orlicz-Hardy space $\mathrm{w}\mathcal{H}_{\Phi_1}$ are none other than the martingale transforms of those in $\mathrm{w}\mathcal{H}_{\Phi_2}$, which extend the corresponding results in Chao and Long [2] from strong-type spaces to the setting of weak-type spaces. In this paper, we are interested in the case Φ_1 is not convex.

2 Notations and Lemmas

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure space, let $(\mathcal{F}_n, n \in \mathbb{N})$ be a sequence of nondecreasing sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \bigvee \mathcal{F}_n$, and let $f = (f_n, n \in \mathbb{N})$ be a martingale adapted to $(\mathcal{F}_n, n \in \mathbb{N})$. Denote by $df = (df_n, n \in \mathbb{N})$ the sequence of martingale differences with $df_n = f_n - f_{n-1}$, $n \geq 1$, and set $f_0 \equiv 0$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. The conditional quadratic variation of a martingale f is defined by

$$s_n(f) := \left(\sum_{i=1}^{n} E(|df_i|^2|\mathcal{F}_{i-1}) \right)^{\frac{1}{2}}, \quad s(f) := \left(\sum_{i=1}^{\infty} E(|df_i|^2|\mathcal{F}_{i-1}) \right)^{\frac{1}{2}},$$

Then for $0 < p \leq \infty$, we define martingale Hardy space as below

$$\mathcal{H}_p := \{f = (f_n, n \in \mathbb{N}) : s(f) \in L_p \text{ and } \|f\|_{\mathcal{H}_p} := \|s(f)\|_p < \infty\}.$$

A non-decreasing function $\Phi(x)$ is called a generalized Young function (convex or concave), if $\Phi(x) = \int_0^x \varphi(t)\mathrm{d}t, x \geq 0$, where $\varphi(x)$ is a left-continuous, non-negative function on $[0, +\infty)$. When $\Phi(x)$ is a convex Young function, we can define the inverse of $\varphi(t)$ by $\psi(s) := \inf\{t : \varphi(t) \geq s\}$. It is well known that its integral $\Psi(x) = \int_0^x \psi(t)\mathrm{d}t$ is a convex function and $\Psi(x)$ is called the Young's complementary function of Φ. The upper index and lower index are defined by

$$p_\Phi = \sup_{0<x<\infty} \frac{x\varphi(x)}{\Phi(x)}, \quad q_\Phi = \inf_{0<x<\infty} \frac{x\varphi(x)}{\Phi(x)}$$

If $p_\Phi < +\infty$, then the inverse function Φ^{-1} of Φ exists and has the form

$$\Phi^{-1}(x) = \int_0^x m_\Phi(t)\mathrm{d}t.$$

If Φ is convex then $m_\Phi(t)$ is a decreasing function and we can easily see that (see Ishak and Mogyoródi [4])

$$m_\Phi(t) = \frac{1}{\varphi(\Phi^{-1}(t))}, \ \ t > 0.$$

A function $\Phi(x)$ is said to satisfy the Δ_2 condition (denote $\Phi \in \Delta_2$) if there is a constant C such that $\Phi(2t) \le C\Phi(t)$ for all $t > 0$. It is well known that if $\Phi(x)$ is a convex function with $p_\Phi < +\infty$ then $\Phi \in \Delta_2$ and if $\Phi(x)$ is a concave function with $q_\Phi > 0$ then $\Phi \in \Delta_2$.

Let $\Phi(x)$ be a generalized Young function. We say that the random variable f belongs to the weak Orlicz space $\mathrm{w}L_\Phi = \mathrm{w}L_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ if there exists an $c > 0$ such that $\Phi(\frac{t}{c})\mathbb{P}(|f| > t) < +\infty$ for all $t > 0$. In this case we put

$$\|f\|_{\mathrm{w}L_\Phi} := \inf\Big\{c > 0 : \ \Phi\Big(\frac{t}{c}\Big)\mathbb{P}(|f| > t) \le 1, \ \forall t > 0\Big\}.$$

The class $\mathrm{w}L_\Phi$ is said to be a weak Orlicz space. Some basic facts on weak Orlicz spaces were discussed in Liu, Hou and Wang [6]. For example, $\|\cdot\|_{\mathrm{w}L_\Phi}$ is a quasi-norm, $\mathrm{w}L_\Phi$ is a quasi-Banach space, and $L_\Phi \hookrightarrow \mathrm{w}L_\Phi$. If $\|f\|_{\mathrm{w}L_\Phi} < +\infty$, then

$$\sup_{t>0} \Phi\Big(\frac{t}{\|f\|_{\mathrm{w}L_\Phi}}\Big)\mathbb{P}(|f| > t) \le 1.$$

We define the weak Orlicz-Hardy spaces of martingales as below

$$\mathrm{w}\mathcal{H}_\Phi := \{f = (f_n, n \in \mathbb{N}) : s(f) \in \mathrm{w}L_\Phi \text{ and } \|f\|_{\mathrm{w}\mathcal{H}_\Phi} := \|s(f)\|_{\mathrm{w}L_\Phi} < \infty\}.$$

A new type of partial ordering between pairs of Young functions was introduced by [14–15] as below.

Definition 2.1 [14–15] Let Φ_1, Φ_2 be two generalized Young functions. We call that Φ_2 is more convex than Φ_1, $\Phi_2 \succeq \Phi_1$ or $\Phi_1 \preceq \Phi_2$ in symbols, if the composition $\Phi_1^{-1} \circ \Phi_2$ is a convex function.

Lemma 2.1 (see [16]) Let $\Phi_1 \preccurlyeq \Phi_2$ be two generalized Young functions having lower index $q_{\Phi_1} > 0$ and upper index $p_{\Phi_2} < \infty$. Then $q_{\Phi_{1,2}} > 0$ and $p_{\Phi_{1,2}} < \infty$. More exactly, we have that

(i) $\frac{q_{\Phi_2}}{p_{\Phi_1}} \leqslant q_{\Phi_{1,2}} \leqslant \frac{q_{\Phi_2}}{q_{\Phi_1}}$;

(ii) $\frac{p_{\Phi_2}}{p_{\Phi_1}} \leqslant p_{\Phi_{1,2}} \leqslant \frac{p_{\Phi_2}}{q_{\Phi_1}}$.

Remark 2.1 Since $\Phi_{1,2}(x)$ is a convex Young function, we denote by $\varphi_{1,2}(x)$ and $\psi_{1,2}(x)$ the density functions such that $\Phi_{1,2}(x) = \displaystyle\int_0^x \varphi_{1,2}(t)\mathrm{d}t$ and its Young's complementary function $\Psi_{1,2}(x) = \displaystyle\int_0^x \psi_{1,2}(t)\mathrm{d}t$, respectively.

Remark 2.2 It is shown in Lemma 2.1 that $\Phi_{1,2}(x) = \Phi_1^{-1} \circ \Phi_2(x)$ has finite upper index, then the inverse function $\Phi_{1,2}^{-1}(x) = \Phi_2^{-1} \circ \Phi_1(x)$ of $\Phi_{1,2}(x)$ exists and it has the form

$$\Phi_{1,2}^{-1}(x) = \int_0^x m_{\Phi_{1,2}}(t)\mathrm{d}t, \ \ x \geq 0.$$

Since $\Phi_{1,2}(x)$ is convex, then its inverse function $\Phi_{1,2}^{-1}(x)$ is concave, therefore $m_{\Phi_{1,2}}(x)$ is a decreasing function and we also have that

$$m_{\Phi_{1,2}}(x) = \frac{1}{\varphi_{1,2} \circ \Phi_{1,2}^{-1}(x)}.$$

Lemma 2.2 (see [6]) Let $\Phi \in \Delta_2$, then there exists a constant $K_\Phi \geq 1$ depending only on Φ, such that

$$\|f+g\|_{\mathrm{w}L_\Phi} \leq K_\Phi(\|f\|_{\mathrm{w}L_\Phi} + \|g\|_{\mathrm{w}L_\Phi}), \ \ \forall f, g \in \mathrm{w}L_\Phi.$$

Let $v = (v_n, n \in \mathbb{N})$ be a process adapted to $(\mathcal{F}_n, n \in \mathbb{N})$, the martingale transform T_v for a given martingale f is defined by $T_v f = (T_v f_n, n \in \mathbb{N})$ where $T_v f_n := \sum\limits_{i=1}^{n} v_{i-1} \cdot df_i$. It can easily be seen that $T_v f$ is still a martingale.

The Lemma below is well known and can be found in Long [7] and Weisz [11].

Lemma 2.3 (see [7, 13]) Let $f = (f_n, n \in \mathbb{N})$ be a martingale. Then f_n converges a.s. on the set of $\{\omega : \ s(f) < \infty\}$.

3 Main Results and Their Proofs

At first, we prove a necessary lemma, which can be seen as a weak version of the generalized Hölder's inequality and has an independent existence value.

Lemma 3.1 Let Φ_1 be a concave Young function with $q_{\Phi_1} > 0$, Φ_2 a concave Young function with $q_{\Phi_2} > 0$ or a convex Young function with $p_{\Phi_2} < +\infty$, and let $\Phi_1 \preceq \Phi_2$, $\Phi_{1,2}(x) = \Phi_1^{-1} \circ \Phi_2(x)$ with Young's complementary function $\Psi_{1,2}(x)$. If $f \in \mathrm{w}L_{\Phi_2}$, $g \in \mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}$, then $f \cdot g \in \mathrm{w}L_{\Phi_1}$ and we have

$$\|f \cdot g\|_{\mathrm{w}L_{\Phi_1}} \leq 2K_{\Phi_1} \|f\|_{\mathrm{w}L_{\Phi_2}} \cdot \|g\|_{\mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}}. \tag{3.1}$$

Proof For any $f \in \mathrm{w}L_{\Phi_2}$ and $g \in \mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}$, if $\|f\|_{\mathrm{w}L_{\Phi_2}} \cdot \|g\|_{\mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}} = 0$, then (3.1) is obvious. Now we assume that $\|f\|_{\mathrm{w}L_{\Phi_2}} \cdot \|g\|_{\mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}} > 0$. For the sake of convenience, denote $\|f\|_{\mathrm{w}L_{\Phi_2}} = A$ and $\|g\|_{\mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}} = B$. Because $(\Phi_{1,2}, \Psi_{1,2})$ is a pair of conjugate Young functions, by Young's inequality, we have that

$$\frac{|f \cdot g|}{A \cdot B} \leq \Phi_1^{-1} \circ \Phi_2\Big(\frac{|f|}{A}\Big) + \Psi_{1,2}\Big(\frac{|g|}{B}\Big).$$

Since $q_{\Phi_1} > 0$ and $0 < q_{\Phi_2} \leq p_{\Phi_2} < +\infty$, $\Phi_1, \Phi_2 \in \Delta_2$. Applying Lemma 2.2, we obtain

$$\frac{\|f \cdot g\|_{\mathrm{w}L_{\Phi_1}}}{A \cdot B} \leq K_{\Phi_1}\bigg(\Big\|\Phi_1^{-1} \circ \Phi_2\Big(\frac{|f|}{A}\Big)\Big\|_{\mathrm{w}L_{\Phi_1}} + \Big\|\Psi_{1,2}\Big(\frac{|g|}{B}\Big)\Big\|_{\mathrm{w}L_{\Phi_1}}\bigg). \tag{3.2}$$

Because $0 < A = \|f\|_{\mathrm{w}L_{\Phi_2}} < +\infty$, so $\Phi_2\left(\frac{t}{A}\right)\mathbb{P}(|f| > t) \le 1$ for all $t > 0$. Since both Φ_1 and Φ_2 are continuous and bijective from $[0, +\infty)$ to itself, then for any $s > 0$, there exists a $t > 0$ such that $\Phi_1(s) = \Phi_2(t/A)$. Moreover, for any $s > 0$, we have

$$\begin{aligned}
& \Phi_1(s)\mathbb{P}(\Phi_1^{-1} \circ \Phi_2(|f|/A) > s) = \Phi_1(s)\mathbb{P}(\Phi_2(|f|/A) > \Phi_1(s)) \\
= \quad & \Phi_1(s)\mathbb{P}(\Phi_2(|f|/A) > \Phi_2(t/A)) = \Phi_2(t/A)\mathbb{P}(|f| > t) \le 1.
\end{aligned}$$

This implies that $\left\|\Phi_1^{-1}\circ\Phi_2\left(\frac{|f|}{A}\right)\right\|_{\mathrm{w}L_{\Phi_1}} \le 1$. Similarly, we can prove that $\left\|\Psi_{1,2}\left(\frac{|g|}{B}\right)\right\|_{\mathrm{w}L_{\Phi_1}} \le 1$. Substituting these to (3.2), then (3.1) is proved.

Theorem 3.1 Let Φ_1 be a concave Young function with $q_{\Phi_1} > 0$, Φ_2 a concave Young function with $q_{\Phi_2} > 0$ or a convex Young function with $p_{\Phi_2} < +\infty$, and $\Phi_1 \preceq \Phi_2$. Let $f = (f_n, n \in \mathbb{N}) \in \mathrm{w}\mathcal{H}_{\Phi_1}$, and define the martingale transform $T(f)$ by

$$Tf_0 = 0,\ a.s.,\ \ Tf_n = \sum_{i=1}^{n} m_{\Phi_{1,2}}(s_i(f)) \cdot df_i,\ \ n \ge 1.$$

Then the martingale $T(f) = (Tf_n, n \in \mathbb{N})$ belongs to $\mathrm{w}\mathcal{H}_{\Phi_2}$ and

$$\|T(f)\|_{\mathrm{w}\mathcal{H}_{\Phi_2}} \le \|\Phi_2^{-1} \circ \Phi_1(s(f))\|_{\mathrm{w}L_{\Phi_2}} \le \|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}}. \tag{3.3}$$

Additionally, $\{Tf_n\}_{n\ge1}$ converges a.s. to a limit Tf_∞.

Proof Setting $s_0(f) = 0$, for all $i \ge 1$, we have $E(|df_i|^2|\mathcal{F}_{i-1}) = s_i^2(f) - s_{i-1}^2(f)$, and

$$E(|d(Tf_i)|^2|\mathcal{F}_{i-1}) = E(m_{\Phi_{1,2}}^2(s_i(f))|df_i|^2|\mathcal{F}_{i-1}) = m_{\Phi_{1,2}}^2(s_i(f)) \cdot E(|df_i|^2|\mathcal{F}_{i-1}).$$

Then for all $n \ge 1$, we have

$$s_n^2(T(f)) = \sum_{i=1}^{n} E(|d(Tf_i)|^2|\mathcal{F}_{i-1}) = \sum_{i=1}^{n} m_{\Phi_{1,2}}^2(s_i(f))(s_i^2(f) - s_{i-1}^2(f)).$$

The sequence $\{s_n(f)\}_{n\ge1}$ is non-negative and non-decreasing, the function $m_{\Phi_{1,2}}(x)$ is non-negative and decreasing, so for all $i \ge 1$, we have

$$\begin{aligned}
& m_{\Phi_{1,2}}^2(s_i(f))(s_i^2(f) - s_{i-1}^2(f)) \\
= \quad & \left[m_{\Phi_{1,2}}(s_i(f))(s_i(f) - s_{i-1}(f))\right] \cdot \left[m_{\Phi_{1,2}}(s_i(f))(s_i(f) + s_{i-1}(f))\right] \\
\le \quad & \left[m_{\Phi_{1,2}}(s_i(f))(s_i(f) - s_{i-1}(f))\right] \cdot \left[m_{\Phi_{1,2}}(s_i(f))s_i(f) + m_{\Phi_{1,2}}(s_{i-1}(f))s_{i-1}(f)\right] \\
\le \quad & \int_{s_{i-1}(f)}^{s_i(f)} m_{\Phi_{1,2}}(t)\mathrm{d}t \cdot \left(\int_0^{s_i(f)} m_{\Phi_{1,2}}(t)\mathrm{d}t + \int_0^{s_{i-1}(f)} m_{\Phi_{1,2}}(t)\mathrm{d}t\right) \\
= \quad & \left[\Phi_{1,2}^{-1}(s_i(f)) - \Phi_{1,2}^{-1}(s_{i-1}(f))\right] \cdot \left[\Phi_{1,2}^{-1}(s_i(f)) + \Phi_{1,2}^{-1}(s_{i-1}(f))\right] \\
= \quad & \left[\Phi_{1,2}^{-1}(s_i(f))\right]^2 - \left[\Phi_{1,2}^{-1}(s_{i-1}(f))\right]^2.
\end{aligned}$$

Consequently, for any $n \ge 1$, we get

$$s_n^2(T(f)) \le \sum_{i=1}^{n} \left(\left[\Phi_{1,2}^{-1}(s_i(f))\right]^2 - \left[\Phi_{1,2}^{-1}(s_{i-1}(f))\right]^2\right) = \left[\Phi_{1,2}^{-1}(s_n(f))\right]^2.$$

In other words, we have that $s(T(f)) \leq \Phi_{1,2}^{-1}(s(f))$ a.s.. Given $f \in \mathrm{w}\mathcal{H}_{\Phi_1}$, then $\|s(f)\|_{\mathrm{w}L_{\Phi_1}} = \|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}} < +\infty$. By the homogeneity of quasi-norm, we may assume that $\|s(f)\|_{\mathrm{w}L_{\Phi_1}} = 1$ for simplicity. Then

$$\sup_{t>0} \Phi_1(t)\mathbb{P}(s(f) > t) = \sup_{t>0} \Phi_1\left(\frac{t}{\|s(f)\|_{\mathrm{w}L_{\Phi_1}}}\right)\mathbb{P}(s(f) > t) \leq 1.$$

Since both Φ_1 and Φ_2 are bijective from $[0, +\infty)$ to itself, for any $s \in (0, +\infty)$, there exists a $t \in (0, +\infty)$, such that $\Phi_1(t) = \Phi_2(s)$. For any $s > 0$, we have that

$$\begin{aligned} \Phi_2(s)\mathbb{P}(\Phi_{1,2}^{-1}(s(f)) > s) &= \Phi_2(s)\mathbb{P}(\Phi_2^{-1} \circ \Phi_1(s(f)) > s) = \Phi_2(s)\mathbb{P}(\Phi_1(s(f)) > \Phi_2(s)) \\ &= \Phi_1(t)\mathbb{P}(\Phi_1(s(f)) > \Phi_1(t)) = \Phi_1(t)\mathbb{P}(s(f) > t) \leq 1. \end{aligned}$$

This means that $\Phi_{1,2}^{-1}(s(f)) \in \mathrm{w}L_{\Phi_2}$ and $\|\Phi_{1,2}^{-1}(s(f))\|_{\mathrm{w}L_{\Phi_2}} \leq \|s(f)\|_{\mathrm{w}L_{\Phi_1}}$. Since

$$\Phi_2\left(\frac{t}{\|\Phi_{1,2}^{-1}(s(f))\|_{\mathrm{w}L_{\Phi_2}}}\right) \cdot \mathbb{P}(s(T(f)) > t) \leq \Phi_2\left(\frac{t}{\|\Phi_{1,2}^{-1}(s(f))\|_{\mathrm{w}L_{\Phi_2}}}\right) \cdot \mathbb{P}(\Phi_{1,2}^{-1}(s(f)) > t) \leq 1,$$

then $\|s(T(f))\|_{\mathrm{w}L_{\Phi_2}} \leq \|\Phi_{1,2}^{-1}(s(f))\|_{\mathrm{w}L_{\Phi_2}} \leq \|s(f)\|_{\mathrm{w}L_{\Phi_1}}$. This means that $T(f) \in \mathrm{w}\mathcal{H}_{\Phi_2}$ and

$$\|T(f)\|_{\mathrm{w}\mathcal{H}_{\Phi_2}} \leq \|\Phi_2^{-1} \circ \Phi_1(s(f))\|_{\mathrm{w}L_{\Phi_2}} \leq \|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}}.$$

The inequality (3.3) is proved.

Moreover, if we denote $\|\Phi_2^{-1} \circ \Phi_1(s(f))\|_{\mathrm{w}L_{\Phi_2}} = A$, then

$$\mathbb{P}\big(\Phi_2^{-1} \circ \Phi_1(s(f)) > t\big) \leq \frac{1}{\Phi_2(t/A)}, \ \ \forall t > 0.$$

Note that $\lim\limits_{t\to+\infty} \Phi_2(t/A) = +\infty$, so

$$\begin{aligned} \mathbb{P}(\Phi_2^{-1} \circ \Phi_1(s(f)) = +\infty) &= \lim_{n\to\infty} \mathbb{P}\Big(\bigcap_{k=1}^{n}\{\Phi_2^{-1} \circ \Phi_1(s(f)) > k\}\Big) \\ &\leq \lim_{n\to\infty} \mathbb{P}(\Phi_2^{-1} \circ \Phi_1(s(f)) > n) \leq \lim_{n\to\infty} \frac{1}{\Phi_2(n/A)} = 0. \end{aligned}$$

On the other hand, since $s(T(f)) \leq \Phi_2^{-1} \circ \Phi_1(s(f))$, then $\{s(T(f)) < +\infty\} \supset \{\Phi_2^{-1} \circ \Phi_1(s(f)) < +\infty\}$. Hence, we have that

$$\begin{aligned} 1 \geq \mathbb{P}(s(T(f)) < +\infty) &\geq \mathbb{P}(\Phi_2^{-1} \circ \Phi_1(s(f)) < +\infty) \\ &= 1 - \mathbb{P}(\Phi_2^{-1} \circ \Phi_1(s(f)) = +\infty) = 1. \end{aligned}$$

This means that $s(T(f)) < +\infty$ a.s.. Consequently, by Lemma 2.3, $\{Tf_n\}_{n\geq 1}$ converges a.s. to a limit Tf_∞. The proof is completed.

Theorem 3.2 Let the generalized Young functions Φ_1 and Φ_2, the martingales f and $T(f)$ be as in Theorem 3.1. Then

$$\|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}} \leq 2\sqrt{2}K_{\Phi_1}\|\varphi_{1,2} \circ \Phi_{1,2}^{-1}(s(f))\|_{\mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}} \cdot \|T(f)\|_{\mathrm{w}\mathcal{H}_{\Phi_2}}.$$

Proof With $s_0(T(f))=0$, we have

$$E(|dTf_i|^2|\mathcal{F}_{i-1})=s_i^2(T(f))-s_{i-1}^2(T(f))$$

for all $i\geq 1$. From the representation of Tf_n figuring in the statement of Theorem 3.1, we have

$$|df_i|=\frac{|dTf_i|}{m_{\Phi_{1,2}}(s_i(f))},\ \ i\geq 1$$

(if $m_{\Phi_{1,2}}(s_i(f))=0$, then we can add an $\varepsilon>0$ to each $s_i(f)$ and at the end let $\varepsilon\to 0$). Therefore, by Abel's rearrangement, we have

$$\begin{aligned} s_n^2(f) &= \sum_{i=1}^n E(|dTf_i|^2|\mathcal{F}_{i-1})=\sum_{i=1}^n E\left[\Big(\frac{|dTf_i|}{m_{\Phi_{1,2}}(s_i(f))}\Big)^2\Big|\mathcal{F}_{i-1}\right] \\ &= \sum_{i=1}^n \frac{s_i^2(T(f))-s_{i-1}^2(T(f))}{m_{\Phi_{1,2}}^2(s_i(f))} \\ &= \sum_{i=1}^n \left[s_i^2(T(f))-s_{i-1}^2(T(f))\right]\cdot\varphi_{1,2}^2(\Phi_{1,2}^{-1}(s_i(f))) \\ &= s_n^2(T(f))\cdot\varphi_{1,2}^2(\Phi_{1,2}^{-1}(s_n(f))) \\ &\quad -\sum_{i=1}^{n-1} s_i^2(T(f))\Big[\varphi_{1,2}^2(\Phi_{1,2}^{-1}(s_{i+1}(f)))-\varphi_{1,2}^2(\Phi_{1,2}^{-1}(s_i(f)))\Big]. \end{aligned}$$

Noticing that both the sequences $\{s_n(T(f))\}_{n\geq 0}$ and $\{\varphi_{1,2}\circ\Phi_{1,2}^{-1}(s_n(f))\}_{n\geq 0}$ are nonnegative and nondecreasing, then we get that

$$s_n^2(f)\leq 2s_n^2(T(f))\cdot\varphi_{1,2}^2(\Phi_{1,2}^{-1}(s_n(f))),\quad n\geq 0.$$

Therefore

$$s(f)\leq\sqrt{2}s(T(f))\cdot\varphi_{1,2}(\Phi_{1,2}^{-1}(s(f))).$$

Thus applying Lemma 3.1, we have that

$$\begin{aligned} \|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}} &\leq \sqrt{2}\|s(T(f))\cdot\varphi_{1,2}(\Phi_{1,2}^{-1}(s(f)))\|_{\mathrm{w}L_{\Phi_1}} \\ &\leq 2\sqrt{2}K_{\Phi_1}\|s(T(f))\|_{\mathrm{w}L_{\Phi_2}}\cdot\|\varphi_{1,2}(\Phi_{1,2}^{-1}(s(f)))\|_{\mathrm{w}L_{\Phi_1\circ\Psi_{1,2}}} \\ &= 2\sqrt{2}K_{\Phi_1}\|T(f)\|_{\mathrm{w}\mathcal{H}_{\Phi_2}}\cdot\|\varphi_{1,2}(\Phi_{1,2}^{-1}(s(f)))\|_{\mathrm{w}L_{\Phi_1\circ\Psi_{1,2}}}. \end{aligned}$$

This proves the assertion.

Now, combining Theorem 3.1 and 3.2, we obtain the following corollary, one of the main results of the present article.

Corollary 3.1 Let Φ_1 be a concave Young function with $q_{\Phi_1}>0$, Φ_2 a concave Young function with $q_{\Phi_2}>0$ or a convex Young function with $p_{\Phi_2}<+\infty$, and $\Phi_1\preceq\Phi_2$. Then for any martingale $f=(f_n,n\in\mathbb{N})\in\mathrm{w}\mathcal{H}_{\Phi_1}$, there exists a martingale $g=(g_n,n\in\mathbb{N})\in\mathrm{w}\mathcal{H}_{\Phi_2}$, such that f is the martingale transform of g. Namely, we have

$$f_0=0,\text{a.s.},\ \ f_n=\sum_{i=1}^n v_{i-1}\cdot dg_i,\ \ n\geq 1,$$

where $v_i=\varphi_{1,2}\circ\Phi_{1,2}^{-1}(s_i(f))$ $(i=0,1,2,3,\cdots)$. We have

$$\|v_\infty\|_{\mathrm{w}L_{\Phi_1\circ\Psi_{1,2}}}\le\max\{1,(p_{\Phi_{1,2}}-1)\|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}}\} \tag{3.4}$$

and

$$\|g\|_{\mathrm{w}\mathcal{H}_{\Phi_2}}\le\|\Phi_2^{-1}\circ\Phi_1(s(f))\|_{\mathrm{w}L_{\Phi_2}}\le\|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}}.$$

Proof From Theorem 3.1 and 3.2, only the inequality (3.4) needs to be proved. In fact, since $(\Phi_{1,2},\Psi_{1,2})$ is a pair of conjugate Young functions, so

$$u\varphi_{1,2}(u)=\Phi_{1,2}(u)+\Psi_{1,2}(\varphi_{1,2}(u)),\ \ \forall\, u>0. \tag{3.5}$$

Because $p_{\Phi_{1,2}}=\sup\limits_{u>0}\frac{u\varphi_{1,2}(u)}{\Phi_{1,2}(u)}$, then

$$p_{\Phi_{1,2}}\Phi_{1,2}(u)\ge u\varphi_{1,2}(u),\ \ \forall\, u>0. \tag{3.6}$$

By (3.5) and (3.6), we get

$$p_{\Phi_{1,2}}\Phi_{1,2}(u)\ge\Phi_{1,2}(u)+\Psi_{1,2}(\varphi_{1,2}(u)),\ \ \forall\, u>0,$$

and then

$$\Psi_{1,2}(\varphi_{1,2}(u))\le(p_{\Phi_{1,2}}-1)\Phi_{1,2}(u),\ \ \forall\, u>0. \tag{3.7}$$

Substituted u in (3.7) by $\Phi_{1,2}^{-1}(s(f))$, we have

$$\Psi_{1,2}(\varphi_{1,2}\circ\Phi_{1,2}^{-1}(s(f)))\le(p_{\Phi_{1,2}}-1)\Phi_{1,2}(\Phi_{1,2}^{-1}(s(f)))\le(p_{\Phi_{1,2}}-1)\cdot s(f). \tag{3.8}$$

Employing (3.8), on the one hand, by the convexity of $\Psi_{1,2}$, for all $t>0$, we have

$$\begin{aligned}\Phi_1\circ\Psi_{1,2}\left(\frac{t}{\max\{1,(p_{\Phi_{1,2}}-1)\|s(f)\|_{\mathrm{w}L_{\Phi_1}}\}}\right)&\le\Phi_1\left(\frac{\Psi_{1,2}(t)}{\max\{1,(p_{\Phi_{1,2}}-1)\|s(f)\|_{\mathrm{w}L_{\Phi_1}}\}}\right)\\&\le\Phi_1\left(\frac{\Psi_{1,2}(t)}{(p_{\Phi_{1,2}}-1)\|s(f)\|_{\mathrm{w}L_{\Phi_1}}}\right).\end{aligned} \tag{3.9}$$

On the other hand, for any $t>0$, we have

$$\begin{aligned}\mathbb{P}\big(\varphi_{1,2}\circ\Phi_{1,2}^{-1}(s(f))>t\big)&=\mathbb{P}\big(\Psi_{1,2}(\varphi_{1,2}\circ\Phi_{1,2}^{-1}(s(f)))>\Psi_{1,2}(t)\big)\\&\le\mathbb{P}\big((p_{\Phi_{1,2}}-1)s(f)>\Psi_{1,2}(t)\big).\end{aligned} \tag{3.10}$$

Since $f\in\mathrm{w}\mathcal{H}_{\Phi_1}$, we have $s(f)\in\mathrm{w}L_{\Phi_1}$, furthermore, we have $(p_{\Phi_{1,2}}-1)s(f)\in\mathrm{w}L_{\Phi_1}$ too, and $\|(p_{\Phi_{1,2}}-1)s(f)\|_{\mathrm{w}L_{\Phi_1}}=(p_{\Phi_{1,2}}-1)\|s(f)\|_{\mathrm{w}L_{\Phi_1}}=(p_{\Phi_{1,2}}-1)\|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}}$. Therefore for any $u>0$, we have

$$\Phi_1\left(\frac{u}{\|(p_{\Phi_{1,2}}-1)s(f)\|_{\mathrm{w}L_{\Phi_1}}}\right)\mathbb{P}\big((p_{\Phi_{1,2}}-1)s(f)>u\big)\le1. \tag{3.11}$$

From (3.9), (3.10) and (3.11), for any $t > 0$, we have that

$$\begin{aligned}
& \Phi_1 \circ \Psi_{1,2}\left(\frac{t}{\max\{1, (p_{\Phi_{1,2}} - 1)\|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}}\}}\right)\mathbb{P}\big(\varphi_{1,2} \circ \Phi_{1,2}^{-1}(s(f)) > t\big) \\
= \ & \Phi_1 \circ \Psi_{1,2}\left(\frac{t}{\max\{1, (p_{\Phi_{1,2}} - 1)\|s(f)\|_{\mathrm{w}L_{\Phi_1}}\}}\right)\mathbb{P}\big(\varphi_{1,2} \circ \Phi_{1,2}^{-1}(s(f)) > t\big) \\
\leq \ & \Phi_1\left(\frac{\Psi_{1,2}(t)}{\|(p_{\Phi_{1,2}} - 1)s(f)\|_{\mathrm{w}L_{\Phi_1}}}\right)\mathbb{P}\big((p_{\Phi_{1,2}} - 1)s(f) > \Psi_{1,2}(t)\big) \leq 1.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|v_\infty\|_{\mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}} &= \|\varphi_{1,2} \circ \Phi_{1,2}^{-1}(s(f))\|_{\mathrm{w}L_{\Phi_1 \circ \Psi_{1,2}}} \\
&\leq \max\{1, (p_{\Phi_{1,2}} - 1)\|f\|_{\mathrm{w}\mathcal{H}_{\Phi_1}}\}.
\end{aligned}$$

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凹函数定义的弱Orlicz-Hardy空间之间的鞅变换

郭红萍[1], 于 林[2], 姜 琴[3]

(1.汉江师范学院数学与财经系, 湖北 十堰 442000)

(2.三峡大学理学院, 湖北 宜昌 443002)

(3.汉江师范学院计算机科学系, 湖北 十堰 442000)

摘要: 本文研究了两个弱Orlicz-Hardy鞅空间中元素之间相互转换关系的问题. 利用鞅变换的方法, 证明了: 设Φ_1是凹Young函数, Φ_2是凹或者凸Young函数, 且$q_{\Phi_1} > 0$, $0 < q_{\Phi_2} \le p_{\Phi_2} < +\infty$, 则当$\Phi_1 \preceq \Phi_2$时, $\mathrm{w}\mathcal{H}_{\Phi_1}$中的元素是$\mathrm{w}\mathcal{H}_{\Phi_2}$中元素的鞅变换的结果, 所得结果将已有的相关结论由强型空间(赋范空间) 推广到弱型空间(赋拟范空间).

关键词: 鞅变换; 弱Orlicz-Hardy空间; 凹函数

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