Vol. 35 (2015)
No. 5

数 学 杂 志
J. of Math. (PRC)

TWO DIMENSIONAL LINEAR ELLIPTIC PDES IN A SEMI-DISK

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Abstract: In this paper, two dimensional linear linear ellliptic in a semi-disk are considered. By using the effective approach by Fokas to solve the linear elliptic PDEs in convex polygonal domain, we improve this method to study the boundary value problems for Laplace, Helmholtz and modified Helmholtz equations in a semi-disk domain. The integral representations for the solutions of these elliptic PDEs are derived. The generalized Dirichlet to Neumann map for the Helmholtz equation is investigated.

Keywords: boundary value problem; Fokas transform method; Riemann-Hilbert technique; generalized Dirichlet to Neumann map

2010 MR Subject Classification: 35C15; 35J05; 35J25

Document code: A **Article ID:** 0255-7797(2015)05-1148-11

1 Introduction

In 1997 Fokas [1] introduced a novel flexible approach to solve initial or boundary value problems for various two dimensional linear and integrable nonlinear PDE's and then to deal with multidimensional problems [2, 3]. In particular, this method was applied in [4] to treat the Laplace equation in a convex polygon and in [5] to other linear two-dimensional PDEs, including the modified Helmholtz and Helmholtz equations. The readers are also referred to see [6–8] for a systematic exposition of the method, its various applications and more references therein. The fundamental problems for the elliptic PDEs in nonpolygonal convex regions remain open [6], which is our main motivation of the present investigation.

We observe that the implementation of Fokas transform method has two fundamental steps. The first step is that Riemann-Hilbert technique is used to construct integral representation of the solution in terms of spectral functions, which is base on the Lax pair or the differential form of equation. The second one is known as a generalized Dirichlet to Neumann map, which determines unknown functions including in the spectral functions. This step is accomplished through so-called the global relation and some invariant properties. In some particular cases, these unknown functions can be reduced to seek a solution of a

* **Received date:** 2013-08-21 **Accepted date:** 2013-10-30

Foundation item: Supported by National Natural Science Foundation of China (11161004) and Natural Science Foundation of Guangxi (2013GXNSFAA019015).

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system of algebraic equations or one of a Riemann-Hilbert problem. However, it is difficult to determine these unknown functions in many cases.

The aim of this paper is to improve Fokas transform approach to study boundary value problems for the following basic elliptic PDEs in a semi-disk domain Ω

$$\Delta q(z,\overline{z}) + 4\alpha q(z,\overline{z}) = 0, \quad z \in \Omega, \tag{1.1}$$

where

$$\Omega = \{(x,y) : x^2 + y^2 \leq r^2, -r \leq x \leq r, y \geq 0\}$$

and set

$$\begin{aligned} L_1 &= \{(x,y) : y = 0, -r < x < r\}, \\ L_2 &= \{(x,y) : x^2 + y^2 = r^2, -r \leq x \leq r, y \geq 0\}, \\ \partial\Omega &= L_1 \cup L_2 \end{aligned}$$

(see Figure 1).

Without loss of generality, we may assume $r = 1$. When $\alpha = 0$, equation (1.1) becomes a Laplace equation; when $\alpha = -\beta^2$, equation (1.1) is said to be a modified Helmholtz equation; when $\alpha = \beta^2$, equation (1.1) is called a Helmholtz equation.

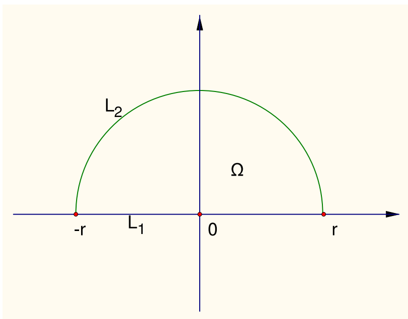

Figure 1 A domain Ω and its boundary $\partial\Omega = L_1 \cup L_2$

The rest of the paper is arranged as follows. In Section 2 we derive the integral representations for the solutions of the elliptic PDEs above-metioned in terms of the spectral functions using the differential forms of equations and the Riemann-Hilbert technique. In Section 3 we discuss generalized Dirichlet to Neumann maps. As an illustration, the Dirichlet boundary value problem for Helmholtz equation in a semi-disk domain is investigated. Using the global relation and some symmetric properties, the unknown functions are determined by the solution of a Fredholm's integral equation of the first kind. A short summary of this work is given in Section 4.

2 The Integral Representations of Solutions

In this section, based on the differential form of equation and Riemann-Hilbert technique we will derive the integral representations for the solutions of Laplace, modified Helmholtz and Helmholtz equation in terms of spectral functions. That is, we have

Theorem 1 Let Ω be a semi-disk domain in the complex plane $\mathbb{C}$ (see Figure 1). Assume that the modified Helmholtz equation has a solution $q(z,\overline{z})$ in Ω such that it is sufficiently smooth to $\partial\Omega$. Then $q(z,\overline{z})$ can be expressed by

$$q(x,y)=\frac{1}{4\pi i}\sum_{j=1}^{4}\int_{l_j} e^{i\beta(kz-\frac{\bar{z}}{k})}\rho_j(x,y,k)\frac{dk}{k}, \tag{2.1}$$

where

$$z=x+iy,\bar{z}=x-iy,k\in\mathbb{C},$$

$$\rho_1(x,y,k)=\int_{-1}^{1} e^{-i\beta(k-\frac{1}{k})x}[-iq_y(x,0)+i\beta(k+\frac{1}{k})q(x,0)]dx, \tag{2.2}$$

$$\rho_2(x,y,k)=\int_{\pi}^{\theta_2} e^{-i\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta)-\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta, \tag{2.3}$$

$$\rho_3(x,y,k)=\int_{\theta_1}^{\theta_2} e^{-i\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta)-\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta, \tag{2.4}$$

$$\rho_4(x,y,k)=\int_{\theta_1}^{0} e^{-i\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta)-\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta, \tag{2.5}$$

$$\theta_1=\arg[\frac{(x+1)^2-y^2}{(x+1)^2+y^2}+i\frac{2y(x+1)}{(x+1)^2+y^2}],\theta_2=\arg[\frac{y^2-(x-1)^2}{(x-1)^2+y^2}+i\frac{2y(1-x)}{(x-1)^2+y^2}],$$

$$0\le\theta_1,\theta_2\le\pi$$

and the rays $l_j=\{k\in\mathbb{C}:\arg k=\frac{(j-1)\pi}{2}\}$ for $j=1,2,3,4$, and l_1 and l_3 are oriented from zero to infinity, while l_2 and l_4 are oriented from infinity to zero; see Figure 2.

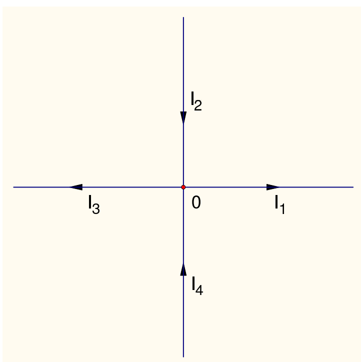

Figure 2 The integral curves $l_j(j=1,2,3,4)$ for Laplace and modified Helmholtz equation in the k-plane

Furthermore, the following global relation is valid:

$$
\begin{aligned}
&\int_{-1}^{1} e^{-i\beta(k-\frac{1}{k})x}[-iq_y(x,0)+i\beta(k+\frac{1}{k})q(x,0)]dx \\
=&\int_{\pi}^{0} e^{-i\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta)-\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta. \qquad (2.6)
\end{aligned}
$$

Proof First, it sees from [6] that function $q(z,\overline{z})$ satisfies the modified Helmholtz equation if and only if the following differential form is closed:

$$dW = e^{-i\beta(kz-\frac{\overline{z}}{k})}[(q_z+ik\beta q)dz-(q_{\overline{z}}+\frac{\beta}{ik}q)d\overline{z}], \quad k\in\mathbb{C}. \tag{2.7}$$

This implies the associated global relation

$$\int_{L_1\cup L_2} e^{-i\beta(kz-\frac{\overline{z}}{k})}[(q_z+ik\beta q)dz-(q_{\overline{z}}+\frac{\beta}{ik}q)d\overline{z}]=0, \tag{2.8}$$

where L_1 and L_2 are shown as in Figure 1. Note that if $z=x+iy,\overline{z}=x-iy$, then

$$\partial_z=\frac{1}{2}(\partial_x-i\partial_y),\partial_{\overline{z}}=\frac{1}{2}(\partial_x+i\partial_y);$$

if $z=re^{i\theta},\overline{z}=re^{-i\theta}$, then

$$\partial_z=\frac{1}{2}e^{-i\theta}(\partial_r+\frac{1}{ir}\partial_\theta),\partial_{\overline{z}}=\frac{1}{2}e^{i\theta}(\partial_r-\frac{1}{ir}\partial_\theta).$$

Hence we deduce from (2.8) that (2.6) holds.

Next, we will derive the integral representation (2.1). We perform the spectral analysis of the differential form

$$d[e^{-i\beta(kz-\frac{\overline{z}}{k})}\mu(z,\overline{z},k)]=dW(z,\overline{z},k). \tag{2.9}$$

It can be derived directly from (2.7) and (2.9) that the modified Helmholtz equation has the following Lax pair equations

$$\mu_z-ik\beta\mu=q_z+ik\beta q, \tag{2.10}$$

$$\mu_{\overline{z}}+\frac{i\beta}{k}\mu=-(q_{\overline{z}}+\frac{\beta}{ik}q). \tag{2.11}$$

Integrating (2.9), we find that for all $z\in\Omega$,

$$\mu_j(z,\overline{z},k)=\int_{z_j}^{z} e^{i\beta[k(z-\zeta)-\frac{1}{k}(\overline{z-\zeta})]}[(q_\zeta+ik\beta q)d\zeta-(q_{\overline{\zeta}}+\frac{\beta}{ik}q)d\overline{\zeta}],$$

here $\mu_j(z,\overline{z},k)$ depend only on point z_j and are independent of the paths of integrations. Meanwhile, $\mu_j(z,\overline{z},k)$ is also the particular solution of the Lax pair equations (2.10) and (2.11).

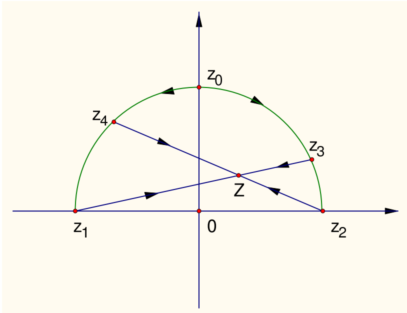

Figure 3 The points $z_j (j = 0, 1, 2, 3, 4)$ and the paths of integrations $\mu_j(z, \overline{z}, k)$

We now choose the point z_j and a suitable path such that we can define a piecewise analytic function $\mu_j(z, \overline{z}, k)$ in the k-plane. Set

$$z_0 = 1 + i, z_1 = -1, z_2 = 1,$$
$$z_3 = \frac{(x+1)^2 - y^2}{(x+1)^2 + y^2} + i\frac{2y(x+1)}{(x+1)^2 + y^2}, z_4 = \frac{y^2 - (x-1)^2}{(x-1)^2 + y^2} + i\frac{2y(1-x)}{(x-1)^2 + y^2},$$

where z_3 is a intersection point of straight line through z_1, z with the semicircle L_2 and z_4 is one of straight line through the point z_2, z with L_2, which are uniquely determined by $z \in \Omega$; see Figure 3. We define $\mu_j(z, \overline{z}, k)(j = 1, 2, 3, 4)$ by

$$\mu_j(z, \overline{z}, k) = \int_{z_j}^{z} e^{i\beta[k(z-\zeta) - \frac{1}{k}\overline{(z-\zeta)}]}[(q_\zeta + ik\beta q)d\zeta - (q_{\overline{\zeta}} + \frac{\beta}{ik}q)d\overline{\zeta}] \ (j = 1, 2),$$
$$\mu_j(z, \overline{z}, k) = \int_{\widehat{z_0 z_j} z} e^{i\beta[k(z-\zeta) - \frac{1}{k}\overline{(z-\zeta)}]}[(q_\zeta + ik\beta q)d\zeta - (q_{\overline{\zeta}} + \frac{\beta}{ik}q)d\overline{\zeta}] \ (j = 3, 4),$$

where the paths of integrations $\mu_j(z, \overline{z}, k)(j = 1, 2)$ are taken as the straight-line segment and the paths of integration $\mu_j(z, \overline{z}, k)(j = 3, 4)$ is chosen as the sum of the circular arc $\widehat{z_0 z_j}$ and the straight-line segment $\overline{z_j z}$, as shown in Figure 3. Let

$$\mu_j^0(z, \overline{z}, k) = \int_{\widehat{z_0 z_j}} e^{i\beta[k(z-\zeta) - \frac{1}{k}\overline{(z-\zeta)}]}[(q_\zeta + ik\beta q)d\zeta - (q_{\overline{\zeta}} + \frac{\beta}{ik}q)d\overline{\zeta}],$$
$$\mu_j^*(z, \overline{z}, k) = \int_{\overline{z_j z}} e^{i\beta[k(z-\zeta) - \frac{1}{k}\overline{(z-\zeta)}]}[(q_\zeta + ik\beta q)d\zeta - (q_{\overline{\zeta}} + \frac{\beta}{ik}q)d\overline{\zeta}]$$

for $j = 3, 4$. Then we have

$$\mu_j(z, \overline{z}, k) = \mu_j^0(z, \overline{z}, k) + \mu_j^*(z, \overline{z}, k) \ (j = 3, 4).$$

It is easy to see the boundedness and analytic domains of the functions $\mu_1, \mu_2, \mu_3^*, \mu_4^*$ in the k-plane are $D_j (j = 1, 2, 3, 4)$, respectively, which are defined by

$$D_1 = \{k \in \mathbb{C} : 0 \leq \arg k \leq \frac{\pi}{2}\}, \quad D_2 = \{k \in \mathbb{C} : \frac{3\pi}{2} \leq \arg k \leq 2\pi\},$$
$$D_3 = \{k \in \mathbb{C} : \pi \leq \arg k \leq \frac{3\pi}{2}\}, \quad D_4 = \{k \in \mathbb{C} : \frac{\pi}{2} \leq \arg k \leq \pi\};$$

see Figure 4. Indeed, put $k=|k|e^{i\varphi}, \zeta=\xi+i\eta$, then we find

$$\mathrm{Re}[ik(z-\zeta)-\frac{i}{k}\overline{(z-\zeta)}]=-(|k|+\frac{1}{|k|})[(x-\xi)\sin\varphi+(y-\eta)\cos\varphi].$$

Consider first the function $\mu_1(z,\overline{z},k)$. Notice that $x-\xi\geq 0, y-\eta\geq 0$, so the exponential associated with $\mu_1(z,\overline{z},k)$ is bounded if and only if $\sin\varphi\geq 0$ and $\cos\varphi\geq 0$. This gives the domain of $\mu_1(z,\overline{z},k)$ is equal to D_1. Similarly, we obtain that the domains of μ_2,μ_3^*,μ_4^* are equal to $D_j(j=2,3,4)$, respectively.

The Lax pair equation (2.10) gives

$$\mu=-q+O(\frac{1}{k}),\quad k\to\infty, \tag{2.12}$$

which can be verified directly for each of the functions $\mu_1,\mu_2,\mu_3^*,\mu_4^*$. Thus we are able to formulate a Riemann-Hilbert problem for the sectionally analytic function μ which is defined by

$$\mu=\begin{cases}\mu_1(z,\overline{z},k), & k\in D_1,\\ \mu_2(z,\overline{z},k), & k\in D_2,\\ \mu_3^*(z,\overline{z},k), & k\in D_3,\\ \mu_4^*(z,\overline{z},k), & k\in D_4\end{cases}$$

for $z\in\Omega$. The relevant jumps are given by

$$\mu^+-\mu^-=\rho(x,y,k)e^{i\beta(kz-\frac{\overline{z}}{k})},\quad k\in L, \tag{2.13}$$

where $L=\sum_{j=1}^{4} l_j$ depicted in Figure 2, and $\rho(x,y,k)=\rho_j(x,y,k)(j=1,2,3,4)$ will be determined below.

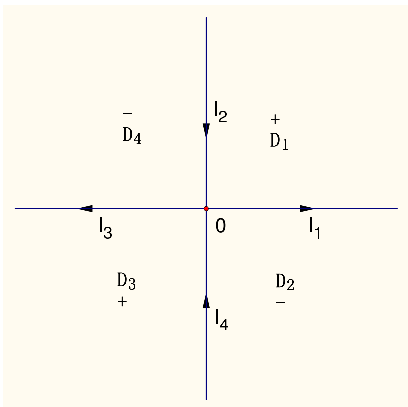

Figure 4 The domains $D_j(j=1,2,3,4)$ in the k-plane

In the following we will derive $\rho(x,y,k)$. Since $\mu_j(x,y,k)$ satisfy Lax pair equations (2.10) and (2.11) for $j=1,2,3,4$, the difference of any two solutions of (2.10) and (2.11) satisfies $p(k)e^{i\beta(kz-\frac{\overline{z}}{k})}$. Hence, when $k\in l_1$, we have

$$\mu^+-\mu^-=\mu_1-\mu_2=p_{12}(k)e^{i\beta(kz-\frac{\overline{z}}{k})},$$

which implies that

$$\begin{aligned}p_{12}(k) &= \int_{z_1}^{z_2} e^{-i\beta(kz-\frac{\overline{z}}{k})}[(q_z + ik\beta q)dz - (q_{\overline{z}} + \frac{\beta}{ik}q)d\overline{z}] \\ &= \int_{-1}^{1} e^{-i\beta(k-\frac{1}{k})x}[-iq_y(x,0) + i\beta(k+\frac{1}{k})q(x,0)]dx = \rho_1(x,y,k),\end{aligned}$$

that is, the equality (2.2) holds; when $k \in l_2$, we get

$$\mu^+ - \mu^- = \mu_1 - \mu_4^* = \mu_1 - (\mu_4 - \mu_4^0) = (\mu_1 - \mu_4) + \mu_4^0 = p_{14}(k)e^{i\beta(kz-\frac{\overline{z}}{k})} + \mu_4^0,$$

which gives

$$\begin{aligned}p_{14}(k)e^{i\beta(kz-\frac{\overline{z}}{k})} + \mu_4^0 =& e^{i\beta(kz-\frac{\overline{z}}{k})} \int_{\widehat{z_1 z_4}} e^{-i\beta(k\zeta-\frac{\overline{\zeta}}{k})}[(q_\zeta + ik\beta q)d\zeta - (q_{\overline{\zeta}} + \frac{\beta}{ik}q)d\overline{\zeta}] \\ =& e^{i\beta(kz-\frac{\overline{z}}{k})} \int_{\pi}^{\theta_2} e^{-i\beta(ke^{i\theta}-\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta) \\ & - \beta(ke^{i\theta} - \frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta \\ =& \rho_2(x,y,k)e^{i\beta(kz-\frac{\overline{z}}{k})},\end{aligned}$$

which yields that (2.3) holds; similarly, we can conclude that spectral functions ρ_3 and ρ_4 possess the expressions (2.4) and (2.5), respectively.

The solution of the Riemann-Hilbert problem with the estimate (2.12) and the jump (2.13) along L can be expressed by

$$\mu = -q + \frac{1}{2\pi i}\int_L \rho(x,y,k)e^{i\beta(sz-\frac{\overline{z}}{s})}\frac{ds}{s-k}, \quad k \in \mathbb{C} \setminus L. \tag{2.14}$$

Thus the Lax pair (2.11) where μ is replaced by (2.14) implies that (2.1) holds. This completes the proof of Theorem 1.

Similar to the proof of Theorem 1, we easily obtain

Corollary 1 Under the conditions of Theorem 1, the solution $q(z,\overline{z})$ of Laplace equation (see (1.1) with $\alpha = 0$) can be expressed by

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi}\sum_{j=1}^{4}\int_{l_j} e^{ikz}\rho_j(x,y,k)dk,$$

where

$$\begin{aligned}\rho_1(x,y,k) =& \frac{1}{2}[q(1,0)e^{-ik} - q(-1,0)e^{ik}] + \frac{i}{2}\int_{-1}^{1} e^{-ikx}[kq(x,0) - iq_y(x,0)]dx, \\ \rho_2(x,y,k) =& \frac{1}{2}[q(\cos\theta_2,\sin\theta_2)e^{-ike^{i\theta_2}} - q(-1,0)e^{ik}] \\ & + \frac{1}{2}\int_{\pi}^{\theta_2} e^{-ike^{i\theta}}[iq_r(\cos\theta,\sin\theta) - ke^{i\theta}q(\cos\theta,\sin\theta)]d\theta,\end{aligned}$$

$$\rho_3(x,y,k) = \frac{1}{2}[q(\cos\theta_2,\sin\theta_2)e^{-ike^{i\theta_2}} - q(\cos\theta_1,\sin\theta_1)e^{-ike^{i\theta_1}}] + \frac{1}{2}\int_{\theta_1}^{\theta_2} e^{-ike^{i\theta}}[iq_r(\cos\theta,\sin\theta) - ke^{i\theta}q(\cos\theta,\sin\theta)]d\theta,$$

$$\rho_4(x,y,k) = \frac{1}{2}[q(1,0)e^{-ik} - q(\cos\theta_1,\sin\theta_1)e^{-ike^{i\theta_1}}] + \frac{1}{2}\int_{\theta_1}^{0} e^{-ike^{i\theta}}[iq_r(\cos\theta,\sin\theta) - ke^{i\theta}q(\cos\theta,\sin\theta)]d\theta$$

and θ_1, θ_2 , $l_j (j = 1, 2, 3, 4)$ are defined as in Theorem 1. Moreover, the following global relation holds

$$i\int_{-1}^{1} e^{-ikx}[kq(x,0) - iq_y(x,0)]dx = \int_{\pi}^{0} e^{-ike^{i\theta}}[iq_r(\cos\theta,\sin\theta) - ke^{i\theta}q(\cos\theta,\sin\theta)]d\theta.$$

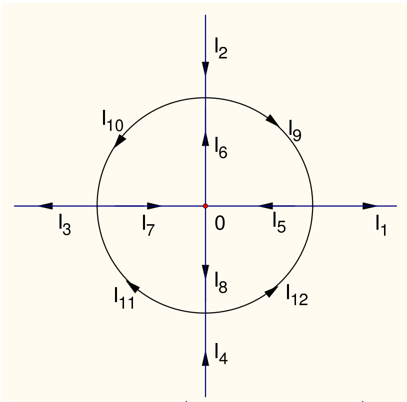

Figure 5 The integral curves $l_j (j = 1, 2, \ldots, 12)$ for the Helmholtz equation in the k-plane

Corollary 2 Under the conditions of Theorem 1, the solution $q(z,\bar{z})$ of Helmholtz equation (see (1.1) with $\alpha = \beta^2$) can be expressed as

$$q(x,y) = \frac{1}{4\pi i}\sum_{j=1}^{12}\int_{l_j} e^{i\beta(kz+\frac{\bar{z}}{k})}\rho_j(x,y,k)\frac{dk}{k}, \tag{2.15}$$

where

$$\rho_1(x,y,k) = \int_{-1}^{1} e^{-i\beta(k+\frac{1}{k})x}[-iq_y(x,0) + i\beta(k-\frac{1}{k})q(x,0)]dx,$$

$$\rho_2(x,y,k) = \int_{\pi}^{\theta_2} e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta) - \beta(ke^{i\theta} + \frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta,$$

$$\rho_3(x,y,k) = \int_{\theta_1}^{\theta_2} e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta) - \beta(ke^{i\theta} + \frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta,$$

$$\rho_4(x,y,k) = \int_{\theta_1}^{0} e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta) - \beta(ke^{i\theta} + \frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta,$$

$$\rho_9(x,y,k) = \int_{\pi}^{\theta_1} e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta) - \beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta,$$
$$\rho_{10}(x,y,k) = \int_{0}^{\theta_2} e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta) - \beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta,$$
$$\rho_5(x,y,k) = -\rho_3(x,y,k), \quad \rho_6(x,y,k) = -\rho_4(x,y,k), \quad \rho_7(x,y,k) = -\rho_1(x,y,k),$$
$$\rho_8(x,y,k) = -\rho_2(x,y,k), \quad \rho_{11}(x,y,k) = -\rho_9(x,y,k), \quad \rho_{12}(x,y,k) = -\rho_{10}(x,y,k),$$

and θ_1, θ_2 are defined as in Theorem 1, and l_j $(j = 1, 2, \ldots, 12)$ are shown as in Figure 5. Moreover, there exists the following global relation

$$\begin{aligned}&\int_{-1}^{1} e^{-i\beta(k+\frac{1}{k})x}[-iq_y(x,0) + i\beta(k-\frac{1}{k})q(x,0)]dx \\ =&\int_{\pi}^{0} e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})}[iq_r(\cos\theta,\sin\theta) - \beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})q(\cos\theta,\sin\theta)]d\theta. \end{aligned} \tag{2.16}$$

3 The Generalized Dirichlet to Neumann Map

In Section 2, we see that the integral representations of the solutions for basic linear elliptic PDEs can be expressed in terms of spectral functions $\rho_j(x,y,k)$. However, $\rho_j(x,y,k)$ are determined by boundary values $q(x,0), q_y(x,0)$ and $q(\cos\theta,\sin\theta), q_r(\cos\theta,\sin\theta)$. For some boundary conditions such as Dirichlet boundary conditions, boundary values $q_y(x,0)$ and $q_r(\cos\theta,\sin\theta)$ are unknown. The goal of generalized Dirichlet to Neumann map is to determine these unknown boundary values, which can be accomplished through the global relation and some symmetric properties.

For illustration, we now discuss the Helmholtz equation in a semi-disk domain Ω with the following Dirichlet boundary condition

$$q(x,y) = f(x,y), (x,y) \in \partial\Omega = L_1 \cup L_2,$$

where function $f(x,y)$ has appropriate smoothness.

Theorem 2 Let Ω be a semi-disk domain in the complex plane $\mathbb{C}$ described in Figure 1. Assume that the boundary value $f(x,y)$ has appropriate smoothness and that the Dirichlet boundary problem of the Helmholtz equation (see (2.1) with $\alpha = \beta^2$) has the solution with form (2.15). Then all spectral function $\rho_j(x,y,k)(j = 1,2,3,4)$ can be determined by the boundary value $f(x,y)$.

Proof It follows from the global relation (2.16) that

$$G_1(k) - G_2(k) = F(k), \qquad k \in \mathbb{C}, \tag{3.1}$$

where

$$
\begin{aligned}
G_1(k) =& \int_{-1}^{1} q_y(x,0) e^{-i\beta(k+\frac{1}{k})x} dx, \quad G_2(k) = \int_0^{\pi} q_r(\cos\theta, \sin\theta) e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})} d\theta, \\
F(k) =& \beta(k-\frac{1}{k}) \int_{-1}^{1} f(x,0) e^{-i\beta(k+\frac{1}{k})x} dx \\
&+ i\beta \int_0^{\pi} f_r(\cos\theta, \sin\theta)(ke^{i\theta} + \frac{1}{k}e^{-i\theta}) e^{-i\beta(ke^{i\theta}+\frac{1}{k}e^{-i\theta})} d\theta.
\end{aligned}
$$

It can be verified directly that $G_1(k), G_2(k)$ have following symmetric relation

$$
\overline{G_1(\overline{k})} = G_1(-k), \quad \overline{G_2(\overline{k})} = G_2(-ke^{-2i\theta}), \qquad k \in \mathbb{C}. \tag{3.2}
$$

We can deduce from (3.1) and (3.2) that

$$
G_2(k) - G_2(ke^{-2i\theta}) = \overline{F(-\overline{k})} - F(k),
$$

that is,

$$
\int_0^{\pi} q_r(\cos\theta, \sin\theta) \cdot K_1(\theta, k) d\theta = \overline{F(-\overline{k})} - F(k), \tag{3.3}
$$

where

$$
K_1(\theta, k) = 2\sinh[\beta(k-\frac{1}{k})\sin\theta] e^{-i\beta(k+\frac{1}{k})\cos\theta}.
$$

This is the first kind of Fredholm integral equation. So by (3.1) and (3.3), we may determine the functions $G_1(k)$ and $q_r(\cos\theta, \sin\theta)$, which implies that all spectral functions $\rho_j(x,y,k)$ can be characterized by a given boundary value $f(x,y)$.

We remark that spectral functions $\rho_j(x,y,k)$ can be expressed by $G_1(k)$ and $q_r(\cos\theta, \sin\theta)$, so we only determine $q_r(\cos\theta, \sin\theta)$ from equation (3.3), which results in $G_2(k)$, and hence determine $G_1(k)$ from equation (3.1) and needn't calculate $q_y(x,0)$.

4 Conclusions

We improve Fokas method to study the Laplace, modified Helmholtz and Helmholtz equations in a semi-disk region Ω. The integral representations of the solutions for these basic linear elliptic PDEs are derived in terms of spectral functions $\rho_j(x,y,k)$, where the spectral functions $\rho_j(x,y,k)$ depend on the boundary values $q(x,0), q(\cos\theta, \sin\theta)$ and its derivatives $q_y(x,0), q_r(\cos\theta, \sin\theta)$. For some specific boundary value problems, using the global relation and symmetric properties, all spectral functions $\rho_j(x,y,k)$ can be determined through the solution to a Fredholm integral equation of the first kind. The results here can be further applied to discuss numerical solutions or asymptotic analysis.

The method here can also be used to solve boundary value problems for linear elliptic PDEs on sector domains or disk ones.

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半圆域内的二维线性椭圆偏微分方程

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摘要: 本文研究了半圆域内的二维线性椭圆偏微分方程. 利用Fokas提出的求解凸多边形区域内的线性椭圆偏微分方程的变换方法, 我们改进了这个方法来研究半圆域内Laplace方程, 修改Helmholtz方程和Helmholtz方程的解, 并且导出了这些方程解的积分表达式, 讨论了Helmholtz方程的广义Dirichlet到Neumann映射.

关键词: 边值问题; Fokas变换方法; Riemann-Hilbert技术; 广义Dirichlet 到Neumann映射

MR(2000)**主题分类号**: 35C15; 35J05; 35J25 **中图分类号**: O175.23