

## LOCALIZATION IN HOM-COMPUTABLE COALGEBRAS

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**Abstract:** In this paper, the localization problems of computable comodules and Hom-computable coalgebras are studied. By applying some localization techniques, the equivalent conditions for computable comodules and Hom-computable coalgebras are obtained, which extend the developing of localization theory of coalgebras.

**Keywords:** coalgebra; comodule; Hom-computable coalgebra; localization

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### 1 Introduction

It is well known that the localization plays an important role in the theory of algebras. It was developed continuously from different aspects. The canonical procedure is actually the formulation of rings of fractions and the associated process of localization which are the most important technical tools in commutative algebra. Meanwhile, Goodearl, Warfield and others investigated the localization in the noncommutative case and obtain many nice results. Also, Gabriel described the localization abstractly in Abelian and Grothendieck category [1]. In the process of localization, one usually applies a functor onto a new category, the quotient category, which has a right adjoint, the section functor. Specifically, if  $T : \mathcal{A} \rightarrow \mathcal{A}'$  is an exact functor between Abelian categories, and  $S : \mathcal{A}' \rightarrow \mathcal{A}$  is a full and faithful right adjoint functor of  $T$ , then the dense subcategory  $\ker T$ , with object class  $\{X \in \mathcal{A} \mid T(X) = 0\}$ , is a localizing subcategory of  $\mathcal{A}$ , and the category  $\mathcal{A}'$  is equivalent to  $\mathcal{A}/\ker T$ . In particular, localization in which  $\mathcal{A}$  is a Grothendieck category is the same as in Abelian categories. Starting from the localization of rings, some mathematicians developed a theory of localization for coalgebras. For instance, Navarro elaborated Gabriel's ideas in comodule categories (Grothendieck categories of finite type) in [2–6]. The key point of the theory lies in the description that quotient category becomes a comodule category. In other words, a quotient category  $\mathcal{M}^C/T$  is a category of comodules  $\mathcal{M}^D$  for certain coalgebra  $D$ , where

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$C$  is a coalgebra, and  $\mathcal{T}$  is a localizing subcategory of  $\mathcal{M}^C$ . Indeed, this is because the category  $\mathcal{M}^C$  of right comodules over a coalgebra  $C$  is a locally finite Grothendieck category in which the theory of localization can be applied. The advantage of this is that it is better to understand than the case of modules over an arbitrary algebra. It is worth mentioning that the key point in most of such applications is the behaviour of simple comodules through the action of the section functor. Therefore, by studying on a set of localized coalgebras of any coalgebra  $C$ , we can obtain some information about  $C$  or its category of comodules  $\mathcal{M}^C$ .

In this paper, we give a description of the localization in Hom-computable coalgebras. The paper is organized as follows. In Section 2, we list some notations and basic facts about coalgebras, localization and quivers, in order to make the article self-contained. In Section 3, we characterize the localization in Hom-computable coalgebras, and obtain two localizing properties about computable comodules and Hom-computable coalgebras.

## 2 Preliminaries

Throughout this paper  $K$  will be a ground field, and  $C$  is a  $K$ -coalgebra. We denote by  $\mathcal{M}^C$  and  $\mathcal{M}_f^C$  the categories of right  $C$ -comodules and right  $C$ -comodules of finite  $K$ -dimension, respectively.

Following [2], a full subcategory  $\mathcal{T}$  of  $\mathcal{M}^C$  is said to be dense if each exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

in  $\mathcal{M}^C$  satisfies that  $M$  belongs to  $\mathcal{T}$  if and only if  $M_1$  and  $M_2$  belong to  $\mathcal{T}$ . For any dense subcategory  $\mathcal{T}$  of  $\mathcal{M}^C$ , there exists an Abelian category  $\mathcal{M}^C/\mathcal{T}$  and an exact functor  $T : \mathcal{M}^C \longrightarrow \mathcal{M}^C/\mathcal{T}$ , such that  $T(M) = 0$  for each  $M \in \mathcal{T}$ , satisfying the following universal property: for any exact functor  $F : \mathcal{M}^C \longrightarrow \mathcal{C}$  such that  $F(M) = 0$  for each  $M \in \mathcal{T}$ , there exists a unique functor  $\bar{F} : \mathcal{M}^C/\mathcal{T} \longrightarrow \mathcal{C}$  verifying that  $F = \bar{F}T$ , where  $\mathcal{C}$  is an arbitrary Abelian category. The category  $\mathcal{M}^C/\mathcal{T}$  is called the quotient category of  $\mathcal{M}^C$  with respect to  $\mathcal{T}$ , and  $T$  is known as the quotient functor. A dense subcategory  $\mathcal{T}$  of  $\mathcal{M}^C$  is said to be localizing if the quotient functor  $T : \mathcal{M}^C \longrightarrow \mathcal{M}^C/\mathcal{T}$  has a right adjoint functor  $S$ , called the section functor. If the section functor is exact, then  $\mathcal{T}$  is called perfect localizing,  $\mathcal{T}$  is said to be colocalizing if  $T$  has a left adjoint functor  $H$ , called the colocalizing functor,  $\mathcal{T}$  is called perfect colocalizing if the colocalizing functor is exact.

Let us list some properties of the (co)localizing functors (see [1] and [7]).

**Lemma 2.1** Let  $\mathcal{T}$  be a dense subcategory of the category of right comodules  $\mathcal{M}^C$  over a coalgebra  $C$ . The following statements hold:

- (a)  $T$  is exact.
- (b) If  $\mathcal{T}$  is localizing, then the section functor  $S$  is left exact and the equivalence  $TS \cong 1_{\mathcal{M}^C/\mathcal{T}}$  holds.
- (c) If  $\mathcal{T}$  is colocalizing, then the colocalizing functor  $H$  is right exact and the equivalence  $TH \cong 1_{\mathcal{M}^C/\mathcal{T}}$  holds.

In [2] and [8], localizing subcategories are described by means of idempotents in the dual algebra  $C^*$ . In particular, it is proved that the quotient category is the category of right comodules over the coalgebra  $eCe$ , where  $e$  is the idempotent associated to the localizing subcategory. The coalgebra structure of  $eCe$  is given by

$$\Delta_{eCe}(exe) = \Sigma_{(x)}ex_{(1)}e \otimes ex_{(2)}e \text{ and } \varepsilon_{eCe}(exe) = e(x),$$

where  $\Delta_C(x) = \Sigma_{(x)}x_{(1)} \otimes x_{(2)}$  for any  $x \in C$ . If  $M$  is a right  $C$ -comodule,  $eM$  has a natural structure of right  $eCe$ -comodule given by

$$\rho(ex) = \Sigma_{(x)}ex_{(0)} \otimes ex_{(1)}e,$$

where  $\rho_M(x) = \Sigma_{(x)}x_{(0)} \otimes x_{(1)}$  for any  $x \in M$ .

**Lemma 2.2** Let  $C$  be a coalgebra and  $e$  be an idempotent in  $C^*$ . Then the following statements hold:

- (a) The quotient functor  $T : \mathcal{M}^C \rightarrow \mathcal{M}^{eCe}$  is naturally equivalent to the functor  $e(-)$ .  $T$  is also equivalent to the cotensor functor  $-\square_C eC$ .
- (b) The section functor  $S : \mathcal{M}^{eCe} \rightarrow \mathcal{M}^C$  is naturally equivalent to the cotensor functor  $-\square_{eCe} Ce$ .
- (c) If  $\mathcal{T}$  is a colocalizing subcategory of  $\mathcal{M}^C$ , the colocalizing functor  $H : \mathcal{M}^{eCe} \rightarrow \mathcal{M}^C$  is naturally equivalent to the functor  $C \circ \text{hom}_{eCe}(eC, -)$ .

Next, for completeness, we remind some points about quivers and path (co)algebras. By a quiver,  $Q$ , we mean a quadruple  $(Q_0, Q_1, h, s)$  where  $Q_0$  is the set of vertices (points),  $Q_1$  is the set of arrows and for each arrow  $\alpha \in Q_1$ , the vertices  $h(\alpha)$  and  $s(\alpha)$  are the source (or start point) and the sink (or end point) of  $\alpha$ , respectively. If  $i$  and  $j$  are vertices in  $Q$ , an (oriented) path in  $Q$  of length  $m$  from  $i$  to  $j$  is a formal composition of arrows

$$p = \alpha_m \cdots \alpha_2 \alpha_1,$$

where  $h(\alpha_1) = i, s(\alpha_m) = j$  and  $s(\alpha_{k-1}) = h(\alpha_k)$  for  $k = 2, \dots, m$ . To any vertex  $i \in Q_0$  we attach a trivial path of length 0, say  $e_i$ , starting and ending at  $i$  such that  $\alpha e_i = \alpha$  (resp.  $e_i \beta = \beta$ ) for any arrow  $\alpha$  (resp.  $\beta$ ) with  $h(\alpha) = i$  (resp.  $s(\beta) = i$ ). We identify the set of vertices and the set of trivial paths. An (oriented) cycle is a path in  $Q$  which starts and ends at the same vertex.  $Q$  is said to be acyclic if there is no oriented cycle in  $Q$ .

Let  $KQ$  be the  $K$ -vector space generated by the set of all paths in  $Q$ . Then  $KQ$  can be endowed with the structure of a (not necessarily unitary)  $K$ -algebra with multiplication induced by concatenation of paths, that is

$$(\alpha_m \cdots \alpha_2 \alpha_1)(\beta_n \cdots \beta_2 \beta_1) = \begin{cases} \alpha_m \cdots \alpha_2 \alpha_1 \beta_n \cdots \beta_2 \beta_1, & \text{if } s(\beta_n) = h(\alpha_1), \\ 0, & \text{otherwise,} \end{cases}$$

$KQ$  is the path algebra of the quiver  $Q$ . The algebra  $KQ$  can be graded by

$$KQ = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_m \oplus \cdots,$$

where  $Q_m$  is the set of all paths of length  $m$ .

Following [8], the path algebra  $KQ$  can be viewed as a graded  $K$ -coalgebra with comultiplication induced by the decomposition of paths, that is, if  $p = \alpha_m \cdots \alpha_2 \alpha_1$  is a path from the vertex  $i$  to the vertex  $j$ , then

$$\Delta(p) = e_j \otimes p + p \otimes e_i + \sum_{i=1}^{m-1} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 = \sum_{\eta\tau=p} \eta \otimes \tau$$

and for a trivial path,  $e_i$ , we have  $\Delta(e_i) = e_i \otimes e_i$ . The counit of  $KQ$  is defined by the formula

$$\varepsilon(\alpha) = \begin{cases} 1, & \text{if } \alpha \in Q_0, \\ 0, & \text{if } \alpha \text{ is a path of length } \geq 1. \end{cases}$$

The coalgebra  $(KQ, \Delta, \varepsilon)$  (shortly  $KQ$ ) is called the path coalgebra of the quiver  $Q$ .

For any coalgebra  $C$ , we denote by  $\{S_j\}_{j \in I_C}$  and  $\{E_j\}_{j \in I_C}$  a complete set of pairwise nonisomorphic simple and indecomposable injective right  $C$ -comodules, respectively. From now on, we fix an idempotent element  $e \in C^*$ . We also denote by  $\mathcal{T}_e$  the localizing subcategory associated to  $e$  and by  $\{S_j\}_{j \in I_e \subset I_C}$  the subset of simple comodules of the quotient category. In what follows, we will denote by  $\{\overline{E}_j\}_{j \in I_e}$  a complete set of pairwise nonisomorphic indecomposable injective right  $eCe$ -comodules, and assume that  $\overline{E}_j$  is the injective envelope of the simple right  $eCe$ -comodule  $S_j$  for each  $j \in I_e$ .

The  $K$ -coalgebra  $C$  is said to be basic if the right  $C$ -comodule  $\text{soc}C_C$  has a direct sum decomposition  $\text{soc}C_C = \bigoplus_{j \in I_C} S_j$ , where  $I_C$  is a set,  $S_j$  are simple comodules, and  $S_i \not\cong S_j$  for all  $i \neq j$ . Given a basic  $K$ -coalgebra  $C$ , we fix right comodule decompositions

$$\text{soc}C_C = \bigoplus_{j \in I_C} S_j \quad \text{and} \quad C_C = \bigoplus_{j \in I_C} E_j,$$

where  $I_C$  is a set and  $S_j$ , with  $j \in I_C$ , are pairwise non-isomorphic simple comodules in  $\mathcal{M}_f^C$  and  $E_j$  is the injective envelope of  $S_j$  in  $\mathcal{M}^C$ , for each  $j \in I_C$ .

Following [9] and [10], for every  $M$  in  $\mathcal{M}_f^C$ , we define the composition length vector

$$\text{lgth}M = (l_j(M))_{j \in I_C} \in \mathbb{Z}^{(I_C)},$$

where  $\mathbb{Z}^{(I_C)}$  is the direct sum of  $I_C$  copies of the free Abelian group  $\mathbb{Z}$  and  $l_j(M) \in \mathbb{N}$  is the number of simple composition factors of  $M$  isomorphic to the simple comodule  $S_j$ . Now we extend the definition of  $\text{lgth}M$  to a class of infinite-dimensional  $C$ -comodules  $M$ . We recall that, given a right  $C$ -comodule, the socle filtration of  $M$  is the chain  $\text{soc}^0 M \subseteq \text{soc}^1 M \subseteq \cdots \subseteq \text{soc}^m M \subseteq \cdots \subseteq M$ , where  $\text{soc}^0 M = \text{soc}M$  and, given  $n \geq 0$ ,  $\text{soc}^{m+1} M$  is the preimage of  $\text{soc}(M/\text{soc}^m M)$  under the canonical projection  $M \rightarrow M/\text{soc}^m M$ . Note also that, by definition, the comodule

$$M_m = \text{soc}^m M / \text{soc}^{m-1} M = \text{soc}(M/\text{soc}^{m-1} M)$$

is semisimple, for each  $m \geq 0$ , where we set  $M_0 = \text{soc}M$ .

Following [10] and [11], we get the following definitions. Assume that  $C$  is a basic  $K$ -coalgebra.

(a) A comodule  $M$  in  $\mathcal{M}^C$  is defined to be computable if, for each  $j \in I_C$ , the sum

$$l_j(M) = \sum_{m=0}^{\infty} l_j(M_m)$$

called the composition  $S_j$ -length of  $M$ , is finite, where  $l_j(M_m)$  is the number of times the simple comodule  $S_j$  appears as a summand in a semisimple decomposition of  $M_m$ .

(b) An arbitrary coalgebra  $C$  is defined to be Hom-computable if every indecomposable injective right  $C$ -comodule is computable.

(c) If  $M$  is a computable  $C$ -comodule, the integral vector

$$\text{lgth}M = (l_j(M))_{j \in I_C} \in \mathbb{Z}^{I_C}$$

is called the composition length vector of  $M$ , where  $\mathbb{Z}^{I_C}$  is the direct product of  $I_C$  copies of the free Abelian group  $\mathbb{Z}$ . The composition length of  $M$  is the cardinal number (finite or infinite)

$$\text{lgth}(M) = \sum_{j \in I_C} l_j(M).$$

### 3 Main Results

**Definition 3.1** Given a coalgebra  $C$  and a comodule  $M$  with a subcomodule  $N$ , the comodule  $M$  is said to be an essential extension of  $N$  (or  $N$  is said to be an essential subcomodule of  $M$ ) if for every subcomodule  $H$  of  $M$ ,  $H \cap N = 0$  implies that  $H = 0$ .

**Proposition 3.2** Let  $e \in C^*$  be an idempotent. If  $C$  is a basic  $K$ -coalgebra, then  $eCe$  is also a basic  $K$ -coalgebra.

**Proof** Note that  $C$  is basic if and only if  $C_C = \bigoplus_{j \in I_C} E_j$  and that  $Ce = S(eCe)$  as right  $C$ -comodule. Moreover,  $Ce$  is a direct summand of  $C_C$ . Then  $eCe = \bigoplus_{j \in I_e} \bar{E}_j$  because  $S$  commutes with direct sums and sends  $\bar{E}_j$  to  $E_j$  for each  $j \in I_e$ . So,  $eCe$  is a basic  $K$ -coalgebra.

**Proposition 3.3** Let  $C$  be a basic  $K$ -coalgebra and  $e \in C^*$  be an idempotent. If  $M$  is a computable right  $C$ -comodule, then  $eM$  is a computable right  $eCe$ -comodule.

**Proof** Let us consider the socle filtration of  $M$

$$\text{soc}^0 M \subseteq \text{soc}^1 M \subseteq \cdots \subseteq \text{soc}^m M \subseteq \cdots \subseteq M,$$

where  $\text{soc}^0 M = \text{soc}M$  and give  $m \geq 0$ ,  $\text{soc}^{m+1}M$  is the preimage of  $\text{soc}(M/\text{soc}^m M) = \text{soc}^{m+1}M/\text{soc}^m M$  under the canonical projection  $M \rightarrow M/\text{soc}^m M$ . Since  $e(-)$  is an exact functor, we obtain the inclusions

$$e(\text{soc}^0 M) \subseteq e(\text{soc}^1 M) \subseteq \cdots \subseteq e(\text{soc}^m M) \subseteq \cdots \subseteq eM$$

and  $e(\text{soc}^{m+1}M)/e(\text{soc}^mM) \cong e(\text{soc}^{m+1}M/\text{soc}^mM)$  is semisimple. We consider the socle filtration of  $eM$

$$\text{soc}^0eM \subseteq \text{soc}^1eM \subseteq \dots \subseteq \text{soc}^meM \subseteq \dots \subseteq eM.$$

From [12], we have  $e(\text{soc}^mM) \subseteq \text{soc}^meM$ , because the socle filtration of  $eM$  is the “largest” series in  $eM$ .

Let

$$\begin{aligned} M_m &= \text{soc}^mM/\text{soc}^{m-1}M = \text{soc}(M/\text{soc}^{m-1}M), \\ eM_m &= \text{soc}^meM/\text{soc}^{m-1}eM = \text{soc}(eM/\text{soc}^{m-1}eM) \subseteq \text{soc}(eM/e\text{soc}^{m-1}M) \\ &\cong \text{soc } e(M/\text{soc}^{m-1}M) \subseteq \text{soc}(M/\text{soc}^{m-1}M) = M_m. \end{aligned}$$

Thus  $l_j(eM_m) \leq l_j(M_m)$ , and if  $M$  is computable right  $C$ -comodule, then  $eM$  is a computable right  $eCe$ -comodule.

**Corollary 3.4** Let  $C$  be a basic  $K$ -coalgebra and  $e \in C^*$  be an idempotent. If  $C$  is Hom-computable, then  $eCe$  is also Hom-computable.

**Proposition 3.5** Let  $C$  be a basic  $K$ -coalgebra and  $e \in C^*$  be an idempotent such that  $S(S_j) = S_j$  for all  $j \in I_e$ . If  $N$  is computable right  $eCe$ -comodule, then  $S(N)$  is a computable right  $C$ -comodule.

**Proof** Let us consider the socle filtration of  $N$

$$\text{soc}^0N \subseteq \text{soc}^1N \subseteq \dots \subseteq \text{soc}^mN \subseteq \dots \subseteq N,$$

where  $\text{soc}^0N = \text{soc}N$  and give  $m \geq 0$ ,  $\text{soc}^{m+1}N$  is the preimage of  $\text{soc}(N/\text{soc}^mN) = \text{soc}^{m+1}N/\text{soc}^mN$  under the canonical projection  $N \rightarrow N/\text{soc}^mN$ . Since  $S$  is a left exact functor, we obtain the inclusions

$$S(\text{soc}^0N) \subseteq S(\text{soc}^1N) \subseteq \dots \subseteq S(\text{soc}^mN) \subseteq \dots \subseteq S(N)$$

and

$$S(\text{soc}^{m+1}N)/S(\text{soc}^mN) \subseteq S(\text{soc}^{m+1}N/\text{soc}^mN)$$

is semisimple. We consider the socle filtration of  $S(N)$

$$\text{soc}^0S(N) \subseteq \text{soc}^1S(N) \subseteq \dots \subseteq \text{soc}^mS(N) \subseteq \dots \subseteq S(N).$$

From [12], we have  $S(\text{soc}^mN) \subseteq \text{soc}^mS(N)$ , because the socle filtration of  $S(N)$  is the “largest” series in  $S(N)$ .

Let

$$\begin{aligned} N_m &= \text{soc}^mN/\text{soc}^{m-1}N = \text{soc}(N/\text{soc}^{m-1}N), \\ S(N)_m &= \text{soc}^mS(N)/\text{soc}^{m-1}S(N) = \text{soc}(S(N)/\text{soc}^{m-1}S(N)) \subseteq \text{soc}(S(N)/S(\text{soc}^{m-1}N)) \\ &\subseteq \text{soc}S(N/\text{soc}^{m-1}N) = \text{soc}(N/\text{soc}^{m-1}N) = N_m. \end{aligned}$$

Thus  $l_j(S(N)_m) \leq l_j(N_m)$ , and if  $N$  is computable right  $eCe$ -comodule, then  $S(N)$  is a computable right  $C$ -comodule.

Following [2], the idempotent  $e \in C^*$  is said to be left (right) semicentral if  $eCe = eC(eCe = Ce)$ .

**Corollary 3.6** Let  $C$  be a basic  $K$ -coalgebra and  $e \in C^*$  be a right semicentral idempotent. If  $N$  is a computable right  $eCe$ -comodule, then  $S(N)$  is a computable right  $C$ -comodule.

**Proof** By [5],  $e$  is right semicentral if and only if  $S(S_j) = S_j$  for all  $j \in I_e$ . Then the statement follows from the former result.

**Example 1** Let  $KQ$  be the path coalgebra of the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and  $e \in C^*$  be the idempotent associated to the set  $X_e = \{3\}$ . Then  $E_1 = \langle 1 \rangle, E_2 = \langle 2, \alpha \rangle, E_3 = \langle 3, \beta, \beta\alpha \rangle, eE_1 = 0, eE_2 = 0, eE_3 = \overline{E}_3$ . Thus  $e$  is a left semicentral idempotent, by [5]. If  $e' \in C^*$  is an idempotent associated to the set  $X_{e'} = \{1\}$ , then the localized coalgebra  $e'Ce'$  is  $S_1$  and

$$S(S_1) = S_1 \square_{e'Ce'} Ce' = e'Ce' \square_{e'Ce'} Ce' \cong Ce' \cong \langle 1 \rangle \cong S_1.$$

Thus  $e'$  is a right semicentral idempotent, by [5].

The following notations are given in [13] and [14]. Let  $S_i$  be a simple  $C$ -comodule, we define the set

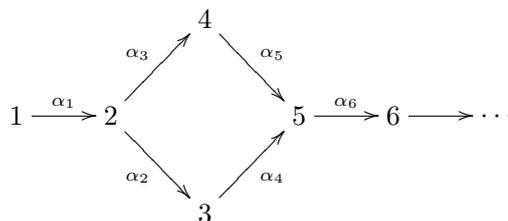
$$I(\blacktriangleright i) = \{S_j \text{ simple } C\text{-comodule} \mid \text{there is a path in } (Q_C, d_C) \text{ from } S_j \text{ to } S_i\}.$$

It is easy to see that  $I(\blacktriangleright i)$  is the set of predecessors of  $S_i$  in the Gabriel quiver of  $C$  (see [10]). In general, for some subset  $U \subseteq I_C$ , we set

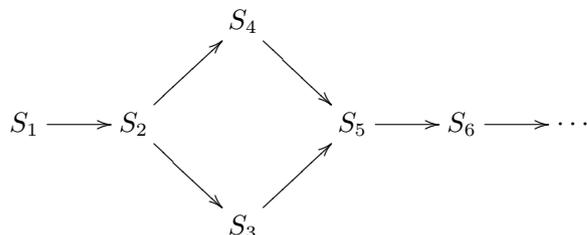
$$I(\blacktriangleright U) = \bigcup_{i \in U} I(\blacktriangleright i).$$

Observe that, for any subset  $U \subseteq I_C$ , the idempotent  $e_U$  associated to the set of simple comodules  $I(\blacktriangleright U)$  is right semicentral, by [5].

**Example 2** Consider the quiver  $Q$ ,



then the Gabriel quiver  $(Q_{KQ}, d_{KQ})$  of  $KQ$  is given by



$I_C = \{1, 2, 3, 4, 5, 6, \dots\}$ , if  $U = \{2, 4, 5\} \subseteq I_C$ ,  $I(\blacktriangleright U) = \{S_1, S_2, S_3, S_4\}$ , then  $e_U$  associated to the set of simple comodules  $I(\blacktriangleright U)$  is right semicentral; if  $U = \{3, 5, 6\} \subseteq I_C$ ,  $I(\blacktriangleright U) = \{S_1, S_2, S_3, S_4, S_5\}$ , then  $e_U$  associated to the set of simple comodules  $I(\blacktriangleright U)$  is right semicentral.

By above results, we get the following localizing properties.

**Theorem 3.7** Let  $C$  be a basic  $K$ -coalgebra.  $M$  is a computable right  $C$ -comodule if and only if  $e_U M$  is a computable right  $e_U C e_U$ -comodule for each finite set  $U \subseteq I_C$ .

**Proof** It is only to prove if  $S(e_U M)$  is computable then  $M$  is computable. Let  $S(e_U M) = N$ , we have

$$\text{soc}^0 N \subseteq \text{soc}^1 N \subseteq \cdots \subseteq \text{soc}^m N \subseteq \cdots \subseteq N.$$

Since  $M$  is an essential subcomodule of  $N$ , we obtain the inclusions

$$\text{soc}^0 N \cap M \subseteq \text{soc}^1 N \cap M \subseteq \cdots \subseteq \text{soc}^m N \cap M \subseteq \cdots \subseteq N \cap M = M.$$

We consider the socle filtration of  $M$

$$\text{soc}^0 M \subseteq \text{soc}^1 M \subseteq \cdots \subseteq \text{soc}^m M \subseteq \cdots \subseteq M.$$

From [12], we have  $\text{soc}^m N \cap M \subseteq \text{soc}^m M$ , because the socle filtration of  $M$  is the “largest” series in  $M$ .

Let

$$\begin{aligned}
 M_m &= \text{soc}^m M / \text{soc}^{m-1} M = \text{soc}(M / \text{soc}^{m-1} M), \\
 M / \text{soc}^{m-1} M &\subseteq M / \text{soc}^{m-1} N \cap M \cong (M + \text{soc}^{m-1} N) / \text{soc}^{m-1} N \subseteq N / \text{soc}^{m-1} N.
 \end{aligned}$$

Thus  $l_j(M_m) \leq l_j(N_m)$ , and if  $N$  is computable right  $e C e$ -comodule, then  $M$  is a computable right  $C$ -comodule.

By Theorem 3.7 and the definition of Hom-computable coalgebras, we obtain the following corollary.

**Corollary 3.8** Let  $C$  be a basic  $K$ -coalgebra.  $C$  is a Hom-computable coalgebra if and only if  $e_U C e_U$  is a Hom-computable coalgebra for each finite set  $U \subseteq I_C$ .

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## Hom -可计算余代数的局部化

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**摘要:** 本文研究了可计算余模和Hom -可计算余代数的局部化问题. 利用局部化方法, 得到了可计算余模和Hom -可计算余代数的等价条件, 推广了余代数上局部化理论的发展.

**关键词:** 余代数; 余模; Hom -可计算余代数; 局部化

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