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EIGENVALUE INTERVALS FOR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH THE p-LAPLACIAN OPERATOR

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Abstract: In this paper, we study a two-point boundary value problem of fractional differential equations with the p-Laplacian operator. By using a fixed-point theorem on cones, we establish eigenvalue intervals of the problem, which generalizes the conclusions in the case of integer-order boundary value problems.

Keywords: fractional differential equation; p-Laplacian operator; boundary value problem

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1 Introduction

Fractional calculus [1–2] developed since 17th century. In recent years, fractional differential equations have been of great interest. Both fractional differential equations and differential equations with the p-Laplacian operator are widely applied in different fields. For details, see [3–10] and references therein.

Goodrich [8] considered a class of fractional boundary value problems of the form

$$\begin{cases} -D_{0+}^{\nu}y(t) = f(t, y(t)), \ \ 0 < t < 1, \\ y^{(i)}(0) = 0, \left[D_{0+}^{\alpha}y(t)\right]_{t=1} = 0, \end{cases}$$

where $0 \le i \le n-2, 1 \le \alpha \le n-2, \nu > 3$ satisfying $n-1 < \nu \le n, n$ is a given integer, and $D_{0+}^{\nu}, D_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative. The author obtained the Green's function of this problem and proved that the Green's function satisfied a Harnack-like inequality. By using a fixed point theorem due to Krasnoselskii, the author established the existence results for at least one positive solution of the problem.

Yang, Zhang and Liu [9] studied the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f\left(t, u(t), D_{0+}^{\beta}u(t)\right) = 0, \ \ t \in (0,1), \\ u^{(i)}(0) = 0, 0 \le i \le n-2, \left[D_{0+}^{\delta}u(t)\right]_{t=1} = 0, 1 \le \delta \le n-2, \end{cases}$$

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where $f \in C([0,1] \times R^+ \times R, R^+), 0 < \beta \leq 1, n-1 < \alpha \leq n, n > 3$ is a given integer, and $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\delta}$ is the Riemann-Liouville fractional derivative. By means of a fixed point theorem in a cone, the author obtained the existence results for at least one positive solution.

There were many papers [5, 6] studying eigenvalue problems for boundary value problems of integer-order differential equations. But there are few papers discussing eigenvalue problems of fractional boundary value problems with the p-Laplacian operator. Motivated by these works, we study the the higher-order two-point boundary value problem of fractional order differential equations with the p-Laplacian operator

$$\begin{cases} \left[\varphi_p\left(D_{0+}^{\alpha}u(t)\right)\right]' + \lambda h(t) f(u(t)) = 0, \ \ 0 < t < 1, \\ u^{(i)}(0) = 0 (i = 0, 1, \cdots, N-2), D_{0+}^{\beta}u(1) = 0, \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s, p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 2, \lambda > 0, 1 \leq \beta \leq N-2, h \in C((0,1), [0,+\infty))$, $f \in C([0,+\infty), [0,+\infty)), N$ is the smallest integer greater than or equal to α, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ is the Riemann-Liouville fractional derivative.

2 Preliminaries

For the convenience of the reader, we list the necessary definitions from fractional calculus theory here.

Definition 2.1 [7] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to R$ is given by $I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$, provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 [7] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 [7] Assume that $u \in C(0,1) \bigcap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \bigcap L(0,1)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$$

for some $c_i \in R$ $(i = 1, 2, \cdots, N)$, where N is the smallest integer greater than or equal to α.

Lemma 2.2 [8] Given $y \in C[0,1]$. The problem

$$\begin{cases} \left[\varphi_p\left(D_{0+}^{\alpha}u(t)\right)\right]' + y(t) = 0, \ \ 0 < t < 1, \\ u^{(i)}(0) = 0 (i = 0, 1, \cdots, N-2), D_{0+}^{\beta}u(1) = 0 \end{cases} \tag{2.1}$$

is equivalent to $u(t)=\int_0^1 G(t,s)\varphi_q\left(\int_0^s y(r)dr\right)ds$, where

$$G(t,s)=\begin{cases}\dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1.\end{cases} \tag{2.2}$$

Lemma 2.3 [8] The function $G(t,s)$ in (2.2) satisfies

(1) $G(t,s)>0,\ t,s\in(0,1)$;

(2) $\max\limits_{0\le t\le 1} G(t,s)\le G(1,s)(s\in(0,1))$;

(3) $\min\limits_{\frac{1}{2}\le t\le 1} G(t,s)\ge \gamma_0 G(1,s)(s\in(0,1))$, where $0<\gamma_0=\min\left\{\frac{\left(\frac{1}{2}\right)^{\alpha-\beta-1}}{2^\beta-1},\left(\frac{1}{2}\right)^{\alpha-1}\right\}\le\frac{1}{2}$.

The following theorem is fundamental in the proofs of our main results.

Lemma 2.4 [6] Let P be a cone in a Banach space X. Assume Ω_1,Ω_2 are open subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$. If $F:P\to P$ is completely continuous such that either

(1) $\|Fu\|\le\|u\|,\forall u\in P\bigcap\partial\Omega_1,\|Fu\|\ge\|u\|,\forall u\in P\bigcap\partial\Omega_2$, or

(2) $\|Fu\|\ge\|u\|,\forall u\in P\bigcap\partial\Omega_1,\|Fu\|\le\|u\|,\forall u\in P\bigcap\partial\Omega_2$.

Then F has a fixed point in $P\bigcap\left(\overline{\Omega}_2\backslash\Omega_1\right)$.

3 Main Result

Let $E=C([0,1],R)$. Then E is a Banach space with the norm $\|u\|=\max\limits_{0\le t\le 1}|u(t)|$. Define the cone $P\subseteq E$ by $P=\left\{u\in E: u(t)\ge 0(t\in[0,1]),\ \min\limits_{\frac{1}{2}\le t\le 1}u(t)\ge\gamma_0\|u\|\right\}$. For any $u\in P$, define $F_\lambda:P\to E,(F_\lambda u)(t)=\lambda\int_0^1 G(t,s)\varphi_q\left(\int_0^s h(r)f(u(r))dr\right)ds$. For each $u\in P$, we get

$$\begin{aligned}\min_{\frac{1}{2}\le t\le 1}(F_\lambda u)(t) &= \min_{\frac{1}{2}\le t\le 1}\lambda\int_0^1 G(t,s)\varphi_q\left(\int_0^s h(r)f(u(r))dr\right)ds\\ &\ge \lambda\gamma_0\int_0^1 G(1,s)\varphi_q\left(\int_0^s h(r)f(u(r))dr\right)ds\ge\gamma_0\|F_\lambda u\|,\end{aligned}$$

$F_\lambda P\subseteq P$. Standard arguments show that $F_\lambda:P\to P$ is completely continuous. u is a positive solution of (1.1) if and only if $u\in P$ is a fixed point of F_λ.

For convenience, we denote

$$\begin{aligned}F_0 &= \lim_{u\to 0^+}\sup\frac{\varphi_q(f(u))}{u}, F_\infty=\lim_{u\to+\infty}\sup\frac{\varphi_q(f(u))}{u}, f_0=\lim_{u\to 0^+}\inf\frac{\varphi_q(f(u))}{u},\\ f_\infty &= \lim_{u\to+\infty}\inf\frac{\varphi_q(f(u))}{u}, C_1=\int_0^1 G(1,s)\varphi_q\left(\int_0^s h(r)dr\right)ds, C_2\\ &= \gamma_0^2\int_{\frac{1}{2}}^1 G(1,s)\varphi_q\left(\int_0^s h(r)dr\right)ds.\end{aligned}$$

Theorem 3.1 If $f_\infty C_2 > F_0 C_1$ holds, then for each $\lambda \in \left(\frac{1}{f_\infty C_2}, \frac{1}{F_0 C_1}\right)$, (1.1) has at least one positive solution. Here we impose $\frac{1}{f_\infty C_2} = 0$ if $f_\infty = +\infty$ and $\frac{1}{F_0 C_1} = +\infty$ if $F_0 = 0$.

Proof For $\lambda \in \left(\frac{1}{f_\infty C_2}, \frac{1}{F_0 C_1}\right)$, let $\varepsilon > 0$ be such that $\frac{1}{(f_\infty - \varepsilon) C_2} \le \lambda \le \frac{1}{(F_0 + \varepsilon) C_1}$. There exists $r_1 > 0$ such that $f(u) \le \varphi_p\left[(F_0 + \varepsilon) u\right]$, for $0 < u \le r_1$. If $u \in P$ with $\|u\| = r_1$,

$$\begin{aligned} \|F_\lambda u\| &\le \lambda \int_0^1 G(1,s) \varphi_q \left\{ \int_0^s h(r) \varphi_p \left[(F_0 + \varepsilon) u(r) \right] dr \right\} ds \\ &\le \lambda r_1 (F_0 + \varepsilon) \int_0^1 G(1,s) \varphi_q \left(\int_0^s h(r) dr \right) ds = \lambda r_1 (F_0 + \varepsilon) C_1 \le r_1 = \|u\|. \end{aligned}$$

If we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then $\|F_\lambda u\| \le \|u\|$, for $u \in P \bigcap \partial\Omega_1$. Let $r_3 > 0$ be such that $f(u) \ge \varphi_p\left[(f_\infty - \varepsilon) u\right]$, for $u \ge r_3$. If $u \in P$ with $\|u\| = r_2 = \max\left\{2r_1, \frac{r_3}{\gamma_0}\right\}$,

$$\begin{aligned} \|F_\lambda u\| &\ge \min_{\frac{1}{2} \le t \le 1} \lambda \int_{\frac{1}{2}}^1 G(t,s) \varphi_q \left(\int_0^s h(r) f(u(r)) dr \right) ds \\ &\ge \lambda \gamma_0 \int_{\frac{1}{2}}^1 G(1,s) \varphi_q \left\{ \int_0^s h(r) \varphi_p \left[(f_\infty - \varepsilon) u(r) \right] dr \right\} ds \\ &\ge \lambda (f_\infty - \varepsilon) \gamma_0^2 \|u\| \int_{\frac{1}{2}}^1 G(1,s) \varphi_q \left(\int_0^s h(r) dr \right) ds = \lambda (f_\infty - \varepsilon) C_2 \|u\| \ge \|u\|. \end{aligned}$$

If we choose $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then $\|F_\lambda u\| \ge \|u\|$, for $u \in P \bigcap \partial\Omega_2$. By Lemma 2.4, F_λ has a fixed point $u \in P \bigcap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \le \|u\| \le r_2$. The proof is completed.

Theorem 3.2 If $f_0 C_2 > F_\infty C_1$ holds, then for each $\lambda \in \left(\frac{1}{f_0 C_2}, \frac{1}{F_\infty C_1}\right)$, (1.1) has at least one positive solution. Here we impose $\frac{1}{f_0 C_2} = 0$ if $f_0 = +\infty$ and $\frac{1}{F_\infty C_1} = +\infty$ if $F_\infty = 0$.

Proof For $\lambda \in \left(\frac{1}{f_0 C_2}, \frac{1}{F_\infty C_1}\right)$, let $\varepsilon > 0$ be such that $\frac{1}{(f_0 - \varepsilon) C_2} \le \lambda \le \frac{1}{(F_\infty + \varepsilon) C_1}$. There exists $r_1 > 0$ such that $f(u) \ge \varphi_p\left[(f_0 - \varepsilon) u\right]$ for $0 < u \le r_1$. If $u \in P$ with $\|u\| = r_1$, then similar to the proof of Theorem 3.1, we can obtain that $\|F_\lambda u\| \ge \|u\|$.

If we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then $\|F_\lambda u\| \ge \|u\|$, for $u \in P \bigcap \partial\Omega_1$. Let $r_3 > 0$ be such that $f(u) \le \varphi_p\left[(F_\infty + \varepsilon) u\right]$, for $u \ge r_3$. We consider two cases:

Case 1 If f is bounded, there exists $M > 0$ such that $f(u) \le \varphi_p(M)(u \in (0, +\infty))$. Let $r_4 = \max\{2r_1, \lambda M C_1\}$. For $u \in P$ with $\|u\| = r_4$, $\|F_\lambda u\| \le \lambda M \int_0^1 G(1,s) \varphi_q \left(\int_0^s h(r) dr \right) ds \le \lambda M C_1 \le r_4 = \|u\|$. Thus $\|F_\lambda u\| \le \|u\|$, for $u \in \partial P_{r_4}$.

Case 2 If f is unbounded, there exists $r_5 > \max\{2r_1, r_3\}$ such that $f(u) \le f(r_5)$ for $0 < u \le r_5$. For $u \in P$ with $\|u\| = r_5$, $\|F_\lambda u\| \le \lambda r_5 (F_\infty + \varepsilon) \int_0^1 G(1,s) \varphi_q \left(\int_0^s h(r) dr \right) ds \le r_5 = \|u\|$. Thus $\|F_\lambda u\| \le \|u\|$, for $u \in \partial P_{r_5}$.

In both Cases 1 and 2, if we set $\Omega_2 = \{u \in E : \|u\| < r_2 = \max\{r_4, r_5\}\}$, then $\|F_\lambda u\| \le \|u\|$, for $u \in P \bigcap \partial\Omega_2$. By Lemma 2.4, F_λ has a fixed point $u \in P \bigcap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \le \|u\| \le r_2$. The proof is completed.

Theorem 3.3 Suppose there exist $r_2 > r_1 > 0$ or $\gamma_0 r_1 > r_2 > 0$ such that

$$\max_{0\le u\le r_2} f(u) \le \varphi_p\left(\frac{r_2}{\lambda C_1}\right), \min_{\gamma_0 r_1\le u\le r_1} f(u) \ge \varphi_p\left(\frac{r_1}{\lambda C_2}\right).$$

Then (1.1) has at least one positive solution $u \in P$.

The proof of Theorem 3.3 is similar to that of Theorem 3.1, we omit it here.

For the reminder of the paper, we will need condition (H_1) $\sup\limits_{r>0} \min\limits_{\gamma_0 r\le u\le r} f(u) > 0$. Denote

$$\lambda_1 = \sup_{r>0} \frac{r}{C_1 \max\limits_{0\le u\le r} \varphi_q(f(u))},\ \lambda_2 = \inf_{r>0} \frac{r}{C_2 \min\limits_{\gamma_0 r\le u\le r} \varphi_q(f(u))}.$$

In view of the continuity of $f(u)$ and (H_1), we have $0 < \lambda_1 \le +\infty, 0 \le \lambda_2 < +\infty$.

Theorem 3.4 Assume (H_1) holds. If $f_0 = +\infty$ and $f_\infty = +\infty$, then (1.1) has at least two positive solutions for each $\lambda \in (0, \lambda_1)$.

Proof Define $a(r) = \frac{r}{C_1 \max\limits_{0\le u\le r}\varphi_q(f(u))}$. $a(r) : (0,+\infty) \to (0,+\infty)$ is continuous and $\lim\limits_{r\to 0} a(r) = \lim\limits_{r\to+\infty} a(r) = 0$. There exists $r_0 \in (0,+\infty)$ such that $a(r_0) = \sup\limits_{r>0} a(r) = \lambda_1$. For $\lambda \in (0, \lambda_1)$, there exist $c_1, c_2 (0 < c_1 < r_0 < c_2 < +\infty)$ with $a(c_1) = a(c_2) = \lambda$,

$$f(u) \le \varphi_p\left(\frac{c_1}{\lambda C_1}\right)(u \in [0, c_1]), f(u) \le \varphi_p\left(\frac{c_2}{\lambda C_1}\right)(u \in [0, c_2]).$$

On the other hand, for $f_0 = +\infty$ and $f_\infty = +\infty$, there exist $d_1, d_2 (0 < d_1 < c_1 < r_0 < c_2 < \gamma_0 d_2 < +\infty)$ satisfying $\frac{\varphi_q(f(u))}{u} \ge \frac{1}{\gamma_0\lambda C_2}$, for $u \in (0, d_1] \bigcup [\gamma_0 d_2, +\infty)$. Thus

$$\min_{\gamma_0 d_1\le u\le d_1} f(u) \ge \varphi_p\left(\frac{d_1}{\lambda C_2}\right), \min_{\gamma_0 d_2\le u\le d_2} f(u) \ge \varphi_p\left(\frac{d_2}{\lambda C_2}\right).$$

By Theorem 3.3, (1.1) has at least two positive solutions for each $\lambda \in (0, \lambda_1)$. The proof is completed.

Corollary 3.1 Assume (H_1) holds. If $f_0 = +\infty$ or $f_\infty = +\infty$, then (1.1) has at least one positive solution for each $\lambda \in (0, \lambda_1)$.

Theorem 3.5 Assume (H_1) holds. If $F_0 = 0$ and $F_\infty = 0$, then (1.1) has at least two positive solutions for each $\lambda \in (\lambda_2, +\infty)$.

Proof Define $b(r) = \frac{r}{C_2 \min\limits_{\gamma_0 r\le u\le r}\varphi_q(f(u))}$. $b(r) : (0,+\infty) \to (0,+\infty)$ is continuous and $\lim\limits_{r\to 0} b(r) = \lim\limits_{r\to+\infty} b(r) = +\infty$. There exists $r_0 \in (0,+\infty)$ such that $b(r_0) = \inf\limits_{r>0} b(r) = \lambda_2$. For $\lambda \in (\lambda_2, +\infty)$, there exist $d_1, d_2 (0 < d_1 < r_0 < d_2 < +\infty)$ satisfying $b(d_1) = b(d_2) = \lambda$. Thus

$$f(u) \ge \varphi_p\left(\frac{d_1}{\lambda C_2}\right)(u \in [\gamma_0 d_1, d_1]), f(u) \ge \varphi_p\left(\frac{d_2}{\lambda C_2}\right)(u \in [\gamma_0 d_2, d_2]).$$

On the other hand, applying the condition $F_0 = 0$, there exist $c_1\,(0 < c_1 < \gamma_0 d_1)$ satisfying $\frac{\varphi_q(f(u))}{u} \le \frac{1}{\lambda C_1}$, for $u \in (0, c_1]$. Thus $\max\limits_{0\le u\le c_1} f(u) \le \varphi_p\left(\frac{c_1}{\lambda C_1}\right)$. For $F_\infty = 0$, there exists $c_3 (c_3 > d_2)$ satisfying $\frac{\varphi_q(f(u))}{u} \le \frac{1}{\lambda C_1}$, for $u \in (c_3, +\infty)$. Let

$$M = \max_{0\le u\le c_3} f(u),\ c_2 = \max\{2c_3, \lambda C_1 \varphi_q(M)\}.$$

Thus $\max\limits_{0\le u\le c_2} f(u) \le \varphi_p\left(\frac{c_2}{\lambda C_1}\right)$. By Theorem 3.3, (1.1) has at least two positive solutions for each $\lambda \in (\lambda_2, +\infty)$. The proof is completed.

Corollary 3.2 Assume (H_1) holds. If $F_0 = 0$ or $F_\infty = 0$, then (1.1) has at least one positive solution for each $\lambda \in (\lambda_2, +\infty)$.

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带有p-Laplace算子的分数阶边值问题的特征区间

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摘要: 本文研究了一类带有p-Laplace算子的分数阶微分方程两点边值问题. 利用锥上的不动点定理, 得到了这类边值问题的特征区间, 推广了整数阶边值问题情形的结论.

关键词: 分数阶微分方程; p-Laplace算子; 边值问题

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