is an alternating series with

$$\frac{(j-1)}{2j(j+1)} > \frac{j}{2(j+1)(j+2)}, \quad j = 3, 4, 5, \cdots.$$

Thus it is clear that $\xi - \xi_n > 0$, and the LHS of (2) follows from (6). Moreover, we observe that

$$\begin{aligned}
\xi - \xi_n &= \sum_{k=n}^{\infty}\left\{\frac{1}{12k^2}\left(1-\frac{1}{k}+\frac{1}{k^2}-\frac{1}{k^3}+\cdots\right)+\sum_{j=4}^{\infty}\left(\frac{j-1}{2j(j+1)}-\frac{1}{12}\right)\left(\frac{-1}{k}\right)^j\right\} \\
&< \sum_{k=n}^{\infty}\left\{\frac{1}{12k(k+1)}-\frac{1}{120}\left(\frac{1}{k}\right)^4+\frac{1}{12}\left(\frac{1}{k}\right)^5\left(1-\frac{1}{k}\right)^{-1}\right\} \\
&= \frac{1}{12}\sum_{k=n}^{\infty}\left\{\left(\frac{1}{k}-\frac{1}{k+1}\right)-\left(\frac{1}{10}-\frac{1}{k-1}\right)\left(\frac{1}{k}\right)^4\right\} \\
&\le \frac{1}{12n} < \log\left(1+\frac{1}{12n-1}\right), \quad (\text{for } n > 10),
\end{aligned}$$

where the last inequality may be checked at once by the logarithmic expansion in powers of $1/(12n-1)$. Hence the RHS of (2) is proved via (1) and (6). □

3. A few remarks (i) It should be possible to transform the double summation involved in RHS of (1) into the classical form containing Bernoulli numbers (ii) Proof of (1) implies the evaluation of $\sum a_k$ and $\sum c_k$, viz.

$$\sum_{v=2}^{\infty}(-1)^v\frac{\zeta(v)}{v+1} = 1+\frac{\gamma}{2}-\log\sqrt{2\pi},$$

$$\sum_{v=2}^{\infty}(-1)^v\frac{\zeta(v)}{v} = \gamma.$$

They are well-known series involving Riemann's Zeta-function.

Stirling 渐进公式的一个新的构造证明

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摘　要

本文用简易的分析工具, 对 n! 给出了一个精确等式, 从而导出 Stirling 渐近公式(2)的一个新的简短证明

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A New Constructive Proof of the Stirling Formula*

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0. Introduction The object of this note is to prove the identity

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\exp\left(\sum_{k=n}^{\infty}\sum_{j=2}^{\infty}\frac{j-1}{2j(j+1)}\left(\frac{-1}{k}\right)^j\right) \tag{1}$$

that implies the useful asymptotic relation

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} < n! < \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+\frac{1}{12n-1}\right) \tag{2}$$

for all $n > 10$. The proof I will present is a by-product of my teaching in analysis at the Nanjing University of Aeronautics and Astronautics in 1995.

2 Proof of (1) Our proof is elementary and simple in nature, and consists of there main steps of construction. Throughout we shall employ two convergent series $\sum a_k$ and $\sum b_k$, where

$$a_k = \frac{1}{3k^2} - \frac{1}{4k^3} + \frac{1}{5k^4} - \cdots,$$
$$c_k = \frac{1}{2k^2} - \frac{1}{3k^3} + \frac{1}{4k^4} - \cdots,$$

so that $0 < a_k < 1/3k^2$ and $0 < c_k < 1/2k^2$, $(k = 1, 2, 3, \cdots)$. We denote

$$\sum_{k=1}^{\infty} a_k = a, \quad \sum_{k=1}^{\infty} c_k = \gamma \quad \text{(Euler's constant)}.$$

Starting with the expression

$$\frac{n^n}{n!} = \left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n}{n-1}\right)^{n-1}$$

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and taking logarithm, we find

$$
\begin{aligned}
n\log n - \log n! &= \sum_{k=1}^{n-1} k\log\left(1+\frac{1}{k}\right) \\
&= \sum_{k=1}^{n-1}\left(1-\frac{1}{2k}\right) + \sum_{k=1}^{n-1} a_k \\
&= (n-1) - \frac{1}{2}\log n - \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{k} - \log\frac{k+1}{k}\right) + \sum_{k=1}^{n-1} a_k \\
&= n - \frac{1}{2}\log n - 1 - \frac{1}{2}\sum_{k=1}^{n-1} c_k + \sum_{k=1}^{n-1} a_k \\
&\overset{def}{=} n - \frac{1}{2}\log n + \xi_n.
\end{aligned}
$$

Here we have $\lim_n \xi_n = -1 - \frac{1}{2}\gamma + a = \xi$

In order to determine ξ, let us take anti-logarithm of the above. We have

$$
\begin{aligned}
n^n/n! &= e^n n^{-1/2} e^{\xi_n}, \\
n! &= (n/e)^n \sqrt{n}\, e^{-\xi_n}.
\end{aligned}
\tag{3}
$$

Similarly we have

$$(2n)! = (2n/e)^{2n}\sqrt{2n}\, e^{-\xi_{2n}}. \tag{4}$$

Since $\lim_n \xi_{2n} = \lim_n \xi_n = \xi$, we may substitute (3) and (4) into Wallis' product formula $\lim_{n\to\infty}\left[(n!)^2 2^{2n}\right] / \left[(2n)!\sqrt{n}\right] = \sqrt{\pi}$ to get

$$\lim_{n\to\infty} e^{-\xi_n} = e^{-\xi} = \sqrt{2\pi} \tag{5}$$

Thus (3) may be rewritten in the form

$$n! = (n/e)^n \sqrt{2\pi n}\exp(\xi - \xi_n). \tag{6}$$

Finally, notice that

$$
\begin{aligned}
\xi_n &= -1 - \frac{1}{2}\left(\sum_{k=1}^{\infty} c_k - \sum_{k=n}^{\infty} c_k\right) + \left(\sum_{k=1}^{\infty} a_k - \sum_{k=n}^{\infty} a_k\right) \\
&= -1 - \frac{1}{2}\gamma + a - \sum_{k=n}^{\infty}\left(a_k - \frac{1}{2}c_k\right) \\
&= \xi - \sum_{k=n}^{\infty}\sum_{j=2}^{\infty}\frac{(j-1)}{2j(j+1)}\left(\frac{-1}{k}\right)^j
\end{aligned}
$$

so that (1) is obtained via (6) and the above relation. □

2 Proof of (2). Notice that

$$\sum_{j=2}^{\infty}\frac{(j-1)}{2j(j+1)}\left(\frac{-1}{k}\right)^j = \frac{1}{12k^2} - \frac{1}{12k^3} + \frac{3}{40k^4} - \cdots$$