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样条的机械化求解

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摘　要

本文在逐次分解法的基础上, 给出一种样条机械化求解方法. 该方法对多项式样条, 有理样条乃至更一般样条的研究都是十分有效的. 它适用于三角剖分, 矩形剖分乃至更一般的代数曲线剖分.

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The Mechanical Solution of Splines*

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Abstract In this paper, a kind of mechnical solution of splines is presented. This method bases on the decomposition method proposed in [9] and is efficient for polynomial splines, rational splines, and even more general splines. This method can be also used for triangulations, rectilinear partitions, and even more general algebraic curve partitions.

Keywords splines, dimensions, decomposition, mechanical solution.
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1. Introduction

In recent years, there were considerable work on polynomial splines (cf. the references). Most of them are concerning on the dimensions. In [9] a so-called decomposition method for studying multivariate splines is presented, this method is suitable for polynomial splines, rational splines, and even more general splines. In this paper, we will use this method to study mechanical solution of splines. The concerning results in [9] used in this paper will be reproved briefly in the following for the sake of self-compledness. Let $\triangle = \{\Omega_i;\ 1 \le i \le \omega\}$ be a rectilinear partition of a simply connected domain Ω, i.e., for each i, $\partial\Omega_i$ is homeomorphic to a circle and $\partial\Omega_i \cap \Omega$ is a piecewise linear curve. The spline space of smoothness r and degree k on $\triangle$, concerning a 1-Bezout function set F, is defined as

$$S^r(\triangle) = \{f \in C^r(\Omega);\quad f|_{\Omega_i} \in F, \forall \Omega_i \in \triangle\},$$

where i-Bezout function set means a function set F that i) for a function $f \in F$ and an irreducible polynomial p, if f vanishes on the curve of $p = 0$, then there exists an $f \in F$ such that $f = f'p$ and, ii) $\alpha f + \beta g \in F$ for all scalars α and β if f, $g \in F$. Usually, F is taken as polynomial

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space, rational function space, and analytic function space. Especially, $S^r(\triangle)$ is denoted by $S_k^r(\triangle)$ if F is polynomial space of total degree k. In recent years there has been considerable work on determining the dimension of $S_k^r(\triangle)$. For $\triangle$, let $v_i=(x_i, y_i)$, $1\le i\le \theta$ be its vertices and v_i, $1\le i\le \theta_0$ be the inner vertices. We denote by $I_i=\{j;\ v_j \text{ is adjacent to } v_i\}$ and

$$l_{i,j}=\frac{(x_j-x_i)(y-y_i)-(y_j-y_i)(x-x_i)}{(x_j-x_i)^2+(y_j-y_i)^2}.$$

It is well-know ([10]) that to study multivariate splines in $S^{r-1}(\triangle)$ one needs only to study conformality conditions

$$\sum_{j\in I_i} q_{i,j}\, l_{i,j}^r=0,\quad 1\le i\le \theta_0, \tag{1}$$

where $q_{i,j}=(-1)^{r+1}q_{j,i}$ are called smoothness cofactors. For (1), it holds[9]

Lemma 1 *Let* F *be a* 1- *Bezout function set, Then* (1) *is equivalent to*

$$C_r^m\sum_{j\in I_i}\alpha_{i,j}^{r-m}\beta_{i,j}^m q_{i,j}=-p_{i,m+1}(y-y_i)+p_{i,m}(x-x_i), \tag{2}$$

$$0\le m\le r,\ 1\le i\le \theta_0,$$

where $p_{i,m}$, $1\le m\le r$ *are some functions in* F $(p_{i,0}=p_{i,r+1}\equiv 0)$, *and* $\alpha_{i,j}=y_i-y_j$ *and* $\beta_{i,j}=x_j-x_i$, *and* $C_m^n=\dfrac{m!}{n!\,(m-n)!}$ *if* $m\ge n\ge 0$; *otherwise it equals zero.* (2) *is called the first decomposition of* (1).

(1) and (2) show that in essential splines are the kernel of a module homomorphism if F is a ring; otherwise it is the kernel of a transformation. By using decomposition method, one can study this problem by only analysing matrices with scalar entries instead of dealing with matrices with polynomial entries as done in [2].

Proof

$$\begin{aligned}\sum_{j\in I_i} q_{i,j}\, l_{i,j}^r &= \sum_{j\in I_i} q_{i,j}(\alpha_{i,j}(x-x_i)+\beta_{i,j}(y-y_i))^r\\ &=\sum_{j\in I_i} q_{i,j}\sum_{m=0}^{r}C_r^m\alpha_{i,j}^{r-m}\beta_{i,j}^m(x-x_i)^{r-m}(y-y_i)^m\\ &=\sum_{m=0}^{r}C_r^m\sum_{j\in I_i} q_{i,j}\alpha_{i,j}^{r-m}\beta_{i,j}^m(x-x_i)^{r-m}(y-y_i)^m.\end{aligned}$$

From above and (1) **we know that** $y-y_i$ **is a divisor of** $\sum_{j\in I_i} q_{i,j}\alpha_{i,j}^r$. *So there exists* $p_{i,1}\in F$ *such that*

$$\begin{cases}\sum_{j\in I_i} q_{i,j}\alpha_{i,j}^r=-p_{i,1}(y-y_i),\\[2mm] -p_{i,1}(x-x_i)^r+\sum_{m=1}^{r}C_r^m\sum_{j\in I_i} q_{i,j}\alpha_{i,j}^{r-m}\beta_{i,j}^m(x-x_i)^{r-m}(y-y_i)^{m-1}.\end{cases}$$

Decomposing continuously, we may find that there also exist $p_{i,2}$, $p_{i,3}$, $\cdots$, $p_{i,r}$ such that

$$C_r^m \sum_{j\in I_i} q_{i,j}\alpha_{i,j}^{r-m}\beta_{i,j}^m = p_{i,m}(x - x_i) - p_{i,m+1}(y - y_i), \quad 0 \le m \le r,$$

where $p_{i,0} = p_{i,r+1} \equiv 0$. □

Comparing each other (1) and (2), it is found that the solution space of (1) is the kernel of module homomorphism (or transformation if F is not a ring)

$$f: \quad F^{\delta} \to F^{\theta_0},$$

where δ is the number of inner edges of $\triangle$, and $f = \{f_1, f_2, \cdots, f_{\theta_0}\}$, $f_i = \sum_{j\in I_i} q_{i,j}\, l_{i,j}^r$, while the solution space of (2) is the kernel of module homomorphism

$$f: \quad F^{\delta + r\theta_0} \to F^{(r+1)\theta_0},$$

where

$$f = \{f_{i,m};\ 1 \le i \le \theta_0,\quad 1 \le m \le r\}$$

and

$$f_{i,m} = C_r^m \sum_{j\in I_i} q_{i,j}\alpha_{i,j}^{r-m}\beta_{i,j}^m - p_{i,m}(x - x_i) + p_{i,m+1}(y - y_i),$$

where $p_{i,0} = p_{i,r+1} \equiv 0$. □

2 Main Results

In this section, we will describe the approach of mechanical solution of splines. To this end, we write (2) in matrix form

$$AQ = C, \tag{3}$$

where Q is the function vector formed by all $q_{i,j}$, C is the function vector formed by the right hand side of (2), and A is the corresponding scalar coefficient matrix. Now we give out the process of mechanical solution of (3).

I) If A is of full rank and the number of its columns is no less than that of its rows, sl such a matrix is called a good matrix and (2) is then called a good system of linear equations for Q, (3) can be solved directly by dealing with scalar matrix A. In this case, obviously, the free function variants of (3) are functions $p_{i,t}$ in C and a number of δ- rank(A) $q_{i,j}$ in Q. Other functions $q_{i,j}$ in Q are fixed consequently and it then stops the process of solution.

II) If A is not a good matrix, i.e., A is not of full rank or the number of its rows is great than that of its columns, it can be deduced from (3) that

$$\bar{A}Q = \bar{C} \tag{4}$$

and a number $N_1 = (1 + r)\theta_0 -$ rank(A) of equations about $p_{i,j}$

$$\sum_{i=1}^{\theta_0}\sum_{j=0}^{r} a_{t,i,j}^{(1)}\left(p_{i,j}(x - x_i) - p_{i,j+1}(y - y_i)\right), \quad 1 \le t \le N_1, \tag{5}$$

where $\bar{A}$ is the matrix formed by some rows of A such that rank $(\bar{A})$ = rank (A) and $\bar{A}$ is a good matrix and $a^{(1)}_{t,i,j}$ are some known scalars, obtained by dealing with matrix A. Noting $p_{i,0} = p_{i,r+1} \equiv 0$, (5) becomes

$$(\sum_{i=1}^{\theta_0}\sum_{j=1}^{r} a^{(1)}_{t,i,j} p_{i,j}) x - (\sum_{i=1}^{\theta_0}\sum_{j=1}^{r} a^{(1)}_{t,i,j-1} p_{i,j}) y - \sum_{i=1}^{\theta_0}\sum_{j=1}^{r} (x_i a^{(1)}_{t,i,j} - y_i a^{(1)}_{t,i,j-1}) p_{i,j} = 0, \qquad (6)$$

$$1 \leq t \leq N_1.$$

Thus, to solve (3), it needs only to solve (6). For a 1-Bezout function set F, we have

Lemma 2 If $g \in F$, there exist uniquely a function $g_3 \in F$ in two variants and two functions g_1, $g_2 \in F$ in one variant such that

$$g = g(0,0) + g_1(x)\, x + g_2(y)\, y + 2g_3\, xy.$$

The proof of this lemma is obvious. In this note, the variant of a function are omitted if it is in two variants, and it will be written out if the function is in one variant. Such decomposition of a function in F as Lemma 2 is called symmetric. To obtain the minimal set of generators of the solution space of a system of equations such as (3), we have to decomposize functions into its symmetric forms. For example, for polynomials g_1, g_2 and g_3 of degree k, we solve

$$g_1 x + g_2 y + g_3 = 0. \qquad (7)$$

It is easy to get that the dimension of the solution space of this equation is $k(k+2)$. For a general decomposition of g_3 that $g_3 = g_{3,1} x + g_{3,2} y$, (7) is equivalent to

$$\begin{cases} g_1 = g_4 y - g_{3,1}, \\ g_2 = -g_4 x - g_{3,2}, \\ g_3 = g_{3,1} x + g_{3,2} y, \end{cases}$$

where $g_{3,1}$, $g_{3,2}$ and g_4 are polynomials of degree $k-1$. It is easy to check that 3$\{x^i y^j;\ i, j \geq 0,\ i + j \leq k-1\}$ is not a minimal set of generators of the solution space of (7), where $n\{\underbrace{M, \cdots, M}_{n}\}$ is a Multi-set, i.e., a group of naturally connected things in which it is allowed that two things are the same. But for the symmetric decomposition $g_3 = g_{3,1}(x)\, x + g_{3,2}(y)\, y + 2g_{3,3}\, xy$, (7) is equivalent to

$$\begin{cases} g_1 = g_4 y - g_{3,1}(x) - g_{3,3} y, \\ g_2 = -g_4 x - g_{3,2}(y) - g_{3,3} x, \\ g_3 = g_{3,1}(x)\, x + g_{3,2}(y)\, y + 2g_{3,3}\, xy, \end{cases} \qquad (8)$$

where $g_{3,1}$, $g_{3,2}$, g_4 are polynomials of degree $k-1$, while $g_{3,3}$ is a polynomial of degree $k-2$. It is easy to check that multi-set

$$\{x^i;\ 0 \leq i \leq k-1\} \cup \{y^j;\ 0 \leq j \leq k-1\} \cup \{x^i y^j;\ i, j \geq 0,\ i + j \leq k-2\}$$
$$\cup \{x^i y^j;\ i, j \geq 0,\ i + j \leq k-1\}$$

is a minimal set of generators of the solution space of (7).

Similar to (8), there exist $p_{t,2}^{(1)}$, $p_{t,1}^{(2)}$, $q_{t,1}^{(1)}(x), q_{t,2}^{(1)}(y) \in F$, such that (6) is equivalent to

$$\begin{cases} \sum_{i=1}^{\theta_0} \sum_{j=1}^{r} a_{t,i,j}^{(1)} p_{i,j} = p_{t,2}^{(1)} y + p_{t,1}^{(2)} y + q_{t,1}^{(1)}(x), \\ \sum_{i=1}^{\theta_0} \sum_{j=1}^{r} a_{t,i,j-1}^{(1)} p_{i,j} = p_{t,2}^{(1)} x - p_{t,1}^{(2)} x - q_{t,2}^{(1)}(y), & 1 \le t \le N_1. \\ \sum_{i=1}^{\theta_0} \sum_{j=1}^{r} (x_i a_{t,i,j}^{(1)} - y_i a_{t,i,j-1}^{(1)}) p_{i,j} = q_{t,1}^{(1)}(x)\, x + q_{t,2}^{(1)}(y)\, y + 2p_{t,1}^{(2)} xy, \end{cases} \quad (9)$$

Lemma 2 shows that (9) will give out the minimal set of generatorsof the solution space of (6).

It is not difficut to see that the symmetric decomposition will result in a minimal set of generators of the solution space of (6). If (9) is a good system of equations for $p_{i,j}$, (6) can be solved by considering only the scalar coefficient matrix A_1 of (9).

III) If (9) is not a good system of equations for $p_{i,j}$, then, except a good systemof equations for $p_{i,j}$, a number of $N_2 = 3N_1 -$ rnak(A_1) equations

$$\sum_{i=1}^{N_1} (a_{t,i,1}^{(2)} (p_{i,2}^{(1)} y + p_{i,1}^{(2)} y + q_{i,1}^{(1)}(x)) + a_{t,i,2}^{(2)} (p_{i,2}^{(1)} x - p_{i,1}^{(2)} x - q_{i,2}^{(1)}(y))$$
$$+ a_{t,i,3}^{(2)} (q_{i,1}^{(1)}(x)\, x + q_{i,2}^{(1)}(y)\, y + 2p_{i,1}^{(2)} xy))$$
$$= y \sum_{i=1}^{N_1} (a_{t,i,1}^{(2)} (p_{i,2}^{(1)} y + p_{i,1}^{(2)}) + a_{t,i,3}^{(2)} (q_{i,2}^{(1)}(y) + p_{i,1}^{(2)} x))$$
$$+ x \sum_{i=1}^{N_1} (a_{t,i,2}^{(2)} (p_{i,2}^{(1)} - p_{i,1}^{(2)}) + a_{t,i,3}^{(2)} (q_{i,1}^{(1)}(x) + p_{i,1}^{(2)} y)) + \sum_{i=1}^{N_1} (a_{t,i,1}^{(2)} q_{i,1}^{(1)}(x) - a_{t,i,2}^{(2)} q_{i,2}^{(1)}(y))$$
$$= 0, \quad 1 \le t \le N_2, \quad (10)$$

will be obtained, where, similar to $a_{t,i,j}^{(1)}$, $a_{t,i,j}^{(2)}$ are known scalars To solve (10) continuously, there exist $p_{t,2}^{(2)} \in F$ in two variants, $q_{t,1}^{(2)}(x), q_{t,2}^{(2)}(y) \in F$ in one variant, and scalar variants $c_t^{(2)}$, such that(10) is equivalent to

$$\begin{cases} \sum_{i=1}^{N_1} a_{t,i,2}^{(2)} p_{i,2}^{(1)} = (p_{t,2}^{(2)} - \sum_{i=1}^{N_1} a_{t,i,3}^{(2)} p_{i,1}^{(2)})\, y + \sum_{i=1}^{N_1} a_{t,i,2}^{(2)} p_{i,1}^{(2)} \\ \qquad + q_{t,1}^{(2)}(x) - \sum_{i=1}^{N_1} a_{t,i,3}^{(2)} q_{i,1}^{(1)}(x), & 1 \le t \le N_2, \\ \sum_{i=1}^{N_1} a_{t,i,1}^{(2)} p_{i,2}^{(1)} = -(p_{t,2}^{(2)} + \sum_{i=1}^{N_1} a_{t,i,3}^{(2)} p_{i,1}^{(2)})\, y - \sum_{i=1}^{N_1} a_{t,i,1}^{(2)} p_{i,1}^{(2)} \\ \qquad + q_{t,2}^{(2)}(y) - \sum_{i=1}^{N_1} a_{t,i,3}^{(2)} q_{i,2}^{(1)}(y), \end{cases} \quad (11)$$

and

$$\begin{cases} \sum_{i=1}^{N_1} a_{t,i,1}^{(2)} q_{i,1}^{(1)}(x) = -x\, q_{t,1}^{(2)}(x) + c_t^{(2)}, \\ \sum_{i=1}^{N_1} a_{t,i,1}^{(2)} q_{i,2}^{(1)}(y) = y\, q_{t,2}^{(2)}(y) + c_t^{(2)}, \end{cases} \quad 1 \le t \le N_2. \tag{12}$$

Ⅳ) If (11) is a good system of equatios for $p_{i,2}^{(1)}$, $1 \le i \le N_1$, and then (12) is obviously also a good system of equations for $q_{i,1}^{(1)}(x)$, $q_{i,2}^{(1)}(y)$, $1 \le i \le N_1$, the process can be finished. If (11) and/or (12) are not good systems of equations, we repeat the above processes continuously. Noting that $p_{t,j}^{(i)}$ and $q_{t,j}^{(i)}$ $(i \ge 1)$ are polynomials of degree $\le k - r - i$ if F is the polynomial space of total degree $\le k$, it holds the following

Theorem 1 *If* F *is the polynomial space of total degree* $\le$ k, *then the above processes of mechanical solution will be teminated within finite steps and, according to Lemma* 2, *will result to the minimal set of generators of the solution space of* (6).

The following conjecture seems right.

Conjecture 1 *Theorem* 1 *keeps right for all* 1- *Bezout function sets.*

Note 1: The scalar matrices obtained from above mechanical solution processes are all independent of F.

3 Example

Up to now, there are only few papers to consider $S^r(\triangle)$. In this section we will study some spline spaces and spline rings by using decomposition method. Our results are represented in explicit form which is very usful for studying both spline spaces and spline rings. Further study may be found in our other papers.

3.1 Continuous splines on triangulation

Let $\triangle$ be a triangulation and $S^0(\triangle)$ is continuous spline space on $\triangle$, concerning 1- Bezout function set F. For $\triangle$ being a trigulation, there exists at least an inner vertex denoted by v, and two other boundary vertices denotedby v_1 and v_2 such that v, v_1, and v_2 are the vertices of a triangle in $\triangle$. Denote by $v_3, \cdots, v_\epsilon$ the vertices in $\triangle$ ofjoining with v, the conformality condition for v is

$$\sum_{j=1}^{\epsilon} q_j l_j = 0. \tag{13}$$

By setting $l_j = \alpha_j l_1 + \beta_j l_2$, $3 \le j \le \epsilon$, the above equation becomes

$$\begin{cases} q_1 = -\sum_{j=1}^{\epsilon} q_j\alpha_j + p\,l_1, \\ q_2 = -\sum_{j=1}^{\epsilon} q_j\beta_j - p\,l_2, \end{cases} \tag{14}$$

where $p \in F$. If v is the only inner vetex of $\triangle$, (14) is an explict solution of (13). By induction, noting that q_1 and q_2 do not appear in other conformality conditions, (14) together with an explicit solution of the conformality conditions of $S^0(\triangle\{\delta\})$ will be an explicit solution of (13), the conformality conditions of $S^0(\triangle)$. At the same time, the minimal set of generators of spline ring $S^0(\triangle)$ is also obtained. In particular, since there are $\delta - 2\theta_0$ free polynomials of degree of $k-1$, and θ_0 free polynomials of degree of $k-2$, the dimension of polynomial spline space $S_k^0(\triangle)$ is

$$\begin{aligned}\dim S_k^0(\triangle) &= \frac{1}{2}(k+1)(k+2) + \frac{1}{2}k(k+1)(\delta - 2\theta_0) + \frac{1}{2}k(k-1)\theta_0 \\ &= \frac{1}{2}(k+1)(k+2) + k^2\theta_0 + \frac{1}{2}k(k+1)(\partial\theta - 3),\end{aligned}$$

where δ, θ_0 and $\partial\theta$ are the numbers of inner edges, inner vertices and boundary vertices, respectively.

3.2 Smooth Splines on Triangulations

Let $\triangle$ and F be the same as above and $S^1(\triangle)$ be smooth spline space on $\triangle$. Then (2) has the form

$$C_r^m \sum_{j \in I_i} \alpha_{i,j}^{r-m}\beta_{i,j}^m q_{i,j} = -p_{i,m+1}(y - y_i) + p_{i,m}(x - x_i),\ 0 \le m \le 2,\ 1 \le i \le \theta_0, \tag{15}$$

where $q_{i,j} = -q_{j,i}$, $p_{i,m}$, $1 \le m \le 2$ are some functions in F, and $p_{i,0} = p_{i,3} \equiv 0$. Write (8) in matrix form

$$AQ = B, \tag{16}$$

where Q is the function vector formed by all $q_{i,j}$, B is the function vector formed by the left hand side of (15), and A is the corresponding coefficient matrix. In particular, if taking $S^1(\triangle)$ as $S_2^1(\triangle)$, then $B = 0$, Q is a scalar vector, and A remains unchanged. Thus, for almost all triangulations, A is not singular and its column number is larger than or equal to its row number ([1]). This shows that for almost all triangulations, we can derive from (16) an explicit solution of (2), the conformality conditions of $S^1(\triangle)$, by dealing only with scalar matrix A.

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