Article ID: 1000-341X(2007)01-0001-06 **Document code**: A

On Jackson Estimate for Müntz Rational Approximation in $L^p_{[0,1]}$ Spaces

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Abstract: Let $\Lambda=\{\lambda_n\}_{n=1}^\infty$ be a sequence of real numbers, and $\lambda_n\searrow 0$ as $n\to\infty$. Suppose that $\lambda_n\le Mn^{-\frac12}$ for $n=1,2,\cdots$, where $M>0$ is an absolute constant. The present paper considers the Müntz rational approximation rate in $L^p_{[0,1]}$ spaces and gets

$$R_n(f,\Lambda)_{L^p}\le C_M\omega(f,n^{-\frac12})_{L^p}$$

for $1\le p\le\infty$.

Key words: Müntz rational functions; L^p spaces; approximation rate.
MSC(2000): 41A20; 41A30
CLC number: O174.41

1. Introduction

Let $L^p[0,1]$ be the space of all p-power integrable functions on [0,1], $1\le p\le\infty$. When $p=\infty$, it can be understood as $C_{[0,1]}$, that is, the space of all continuous functions on [0,1]. For any given real sequence $\{\lambda_n\}_{n=1}^\infty$, denote by $\Pi_n(\Lambda)$ the set of Müntz polynomials of degree n, that is, all linear combinations of $\{x^{\lambda_1},x^{\lambda_2},\cdots,x^{\lambda_n}\}$, and let $R_n(\Lambda)$ be the Müntz rational functions of degree n, that is,

$$R_n(\Lambda)=\left\{\frac{P(x)}{Q(x)}:P(x),Q(x)\in\Pi_n(\Lambda),Q(x)\ge0,x\in[0,1]\right\}.$$

If $Q(0)=0$, we require that $\lim_{x\to0+}\frac{P(x)}{Q(x)}$ exist and be finite. For $f\in L^p[0,1],1\le p\le\infty$, define

$$R_n(f,\Lambda)_{L^p}=\inf_{r\in R_n(\Lambda)}\|f-r\|_{L^p},$$

$$\omega(f,t)_{L^p}=\sup_{|h|\le t}\|f(x+t)-f(x)\|_{L^p},$$

where $\|\cdot\|_{L^p}$ is the usual L^p- norm, that is

$$\|f\|_{L^p}=\left\{\int_0^1|f(x)|^p\,\mathrm{d}x\right\}^{\frac1p},\quad 1\le p<\infty,$$

$$\|f\|_{L^\infty}=\|f\|_C=\max_{0\le x\le1}|f(x)|.$$

Received date: 2005-03-04; **Accepted date**: 2005-10-25
Foundation item: the National Natural Science Foundation of China (10471130)

In the past dozens of years, there have been a few good works on rational Müntz approximation rate[1−3,5−9]. Here, we want to remind the readers of a special Müntz system $\{x^{\lambda_n}\}$, that is, $\{\lambda_n\}$ satisfy $\lambda_n \searrow 0$ ($\lambda_n \searrow 0$ means λ_n strictly decreasing to 0), and $\lambda_n \le Mn^{-\frac{1}{2}}$ for every $n \ge 1$. The second named author did the original work on this subject in [8]:

Theorem 1 *Given $M > 0$. Let $\{\lambda_n\}$ be a sequence with $\lambda_n \searrow 0$, suppose that $\lambda_n \le Mn^{-\frac{1}{2}}$ for $n = 1, 2, \cdots$, then for any $f \in C_{[0,1]}$, we have*

$$R_n(f, \Lambda) \le C_M \omega(f, n^{-\frac{1}{2}}),$$

where C_M is a positive constant only depending on M.

Recently, we[7] generalized the above result to include the general $L^p[0,1]$ spaces and established the following theorem

Theorem 2 *Given $M > 0$. Let $\{\lambda_n\}$ be a sequence with $\lambda_n \searrow 0$, and let $\lambda_n \le Mn^{-\frac{1}{2}}$ for $n = 1, 2, \cdots$. Then for any $f \in L^p_{[0,1]}, 1 < p \le \infty$, there is a positive constant $C_{M,p}$ only depending on M and p such that*

$$R_n(f, \Lambda)_{L^p} \le C_{M,p} \omega(f, n^{-\frac{1}{2}})_{L^p}.$$

However, it is not a satisfactory result yet. First, $C_{M,p}$ is a positive constant depending not only on M but also on p. So a natural problem comes: whether can $C_{M,p}$ be replaced by C_M only depending on M? Secondly, the method used in [7] cannot give any result for $p = 1$. These problems could be hard without using the efficient tool, the Hardy-Littlewood maximum functions, which played a very important role in [7]. The present paper will give positive answers to these problems by employing a new method and by constructing a new type of Müntz rational functions. We obtain the following theorem

Theorem 3 *Given $M > 0$. Let $\{\lambda_n\}$ be a real sequence with $\lambda_n \searrow 0$, and let $\lambda_n \le Mn^{-\frac{1}{2}}$ for $n = 1, 2, \cdots$. Then for any $f \in L^p_{[0,1]}, 1 \le p \le \infty$, there is a positive constant C_M only depending on M such that*

$$R_n(f, \Lambda)_{L^p} \le C_M \omega(f, n^{-\frac{1}{2}})_{L^p}.$$

Throughout the paper, C always denotes an absolute constant, and C_M a positive constant only depending on M. Their values may be different in different circumstances.

2. Auxiliary lemmas

Let $P_k(x, a_0, a_1, \cdots, a_k)$ denote the k-th divided difference of $(\frac{x}{e})^\alpha$ with respect to α at $\alpha = a_0, a_1, \cdots, a_k$, that is,

$$P_0(x, a_0) = (\frac{x}{e})^{a_0},$$

$$P_k(x, a_0, a_1, \cdots, a_k) = \frac{P_{k-1}(x, a_0, a_1, \cdots, a_{k-1}) - P_{k-1}(x, a_1, a_2, \cdots, a_k)}{a_0 - a_k}.$$

Set

$$P_k(x) = P_{(n+k)^2}(x, \lambda_{n^2}, \lambda_{n^2+1}, \cdots, \lambda_{n^2+(n+k)^2})$$

for $k = 1, 2, \cdots, n-1$, and

$$P_n(x) = P_{(2n)^2}(x, \lambda_{5n^2}, \lambda_{5n^2+1}, \cdots, \lambda_{9n^2}).$$

By the mean value theorem,

$$P_k(x) = \left(\frac{x}{e}\right)^{\eta_k} \frac{\log^{(n+k)^2}(\frac{x}{e})}{((n+k)^2)!}, \tag{1}$$
$$\lambda_{n^2+(n+k)^2} \le \eta_k \le \lambda_{n^2}, \quad k = 1, 2, \cdots, n-1,$$

and

$$P_n(x) = \left(\frac{x}{e}\right)^{\eta_n} \frac{\log^{4n^2}(\frac{x}{e})}{(4n^2)!}, \tag{2}$$
$$\lambda_{9n^2} \le \eta_n \le \lambda_{5n^2}.$$

Define

$$x_j = \frac{n+1}{n+1-j}, \quad t_j = e^{1-x_j}, \quad j = 1, 2, \cdots, n.$$

In particular, let $t_0 = 1$ and $t_{n+1} = 0$. For any $f \in C_{[0,1]}$, we define

$$\begin{aligned} L_n(f,x) &= \frac{\sum_{j=1}^n (-1)^{(n+j)^2}((n+j)^2)! f(t_j) \prod_{i=1}^j x_i^{-((n+i)^2-(n+i-1)^2)} P_j(x)}{\sum_{j=1}^n (-1)^{(n+j)^2}((n+j)^2)! \prod_{i=1}^j x_i^{-((n+i)^2-(n+i-1)^2)} P_j(x)} \\ &= \frac{\sum_{j=1}^n f(t_j) \prod_{i=1}^j x_i^{-((n+i)^2-(n+i-1)^2)} (-\log\frac{x}{e})^{(n+j)^2} \left(\frac{x}{e}\right)^{\eta_j}}{\sum_{j=1}^n \prod_{i=1}^j x_i^{-((n+i)^2-(n+i-1)^2)} \left(-\log\frac{x}{e}\right)^{(n+j)^2} \left(\frac{x}{e}\right)^{\eta_j}} \qquad \text{(from (1), (2))} \\ &:= \frac{\sum_{j=1}^n f(t_j) Q_j(x)}{\sum_{j=1}^n Q_j(x)} := \sum_{j=1}^n f(t_j) r_j(x), \end{aligned}$$

where

$$Q_j(x) := \prod_{i=1}^j x_i^{-\left((n+i)^2-(n+i-1)^2\right)} \left(-\log\frac{x}{e}\right)^{(n+j)^2} \left(\frac{x}{e}\right)^{\eta_j},$$
$$r_j(x) := \frac{Q_j(x)}{\sum_{j=1}^n Q_j(x)}, \quad j = 1, 2, \cdots, n.$$

We estimate $r_j(x)$ first.

Lemma 1 *For any $x \in [t_{j+1}, t_j], 0 \le j \le n$, we have*

$$r_k(x) \le C_M e^{-C_M|j-k|}, \quad k = 1, 2, \cdots, n. \tag{3}$$

Proof In fact, from Zhou[8], we have

$$Q_j^{-1}(x) Q_k(x) \le C_M e^{-C_M|j-k|}.$$

By the definition of $r_k(x)$, (3) holds.

Lemma 2 *For any $f \in L^p_{[0,1]}, 1 \le p \le \infty$, define*

$$K(f,h)_{L^p} = \inf_g \{\|f-g\|_{L^p} + h\|g'\|_{L^p}\},$$

where $g \in AC_{[0,1]}$, that is, g is an absolute continuous function on $[0,1]$, then

$$K(f,h)_{L^p} \sim \omega(f,h)_{L^p}, \quad 1 \le p \le \infty.$$

It is a well-known result in [4]. From Lemma 2, we have the following Lemma 3 immediately.

Lemma 3 *For any $f \in L^p_{[0,1]}, 1 \le p \le \infty$, there is a $g \in AC_{[0,1]}$ such that*

$$\|f-g\|_{L^p} \le C\omega(f,\frac{1}{n})_{L^p}, \quad \|g'\|_{L^p} \le Cn\omega(f,\frac{1}{n})_{L^p}.$$

3. Proof of Theorem 3

In view of Theorem 2, we only need to show Theorem 3 in the case $1 \le p < \infty$. For any $f \in L^p_{[0,1]}$, take a function g satisfying the properties of Lemma 3. Furthermore, set

$$L_n(g,x) = \sum_{k=1}^n r_k(x) \frac{1}{t_{k-1}-t_k} \int_{t_k}^{t_{k-1}} g(u)\mathrm{d}u.$$

By the definition of $r_k(x)$, we have $L_n(1,x)=1$, and $L_n(g,x) \in R(\Lambda_{9n^2})$. In some sense, $L_n(g,x)$ is very similar to the usual Kantorovich-type operators in $L^p[0,1]$ spaces although there is still some minor difference. Here we use g instead of f directly to construct the rational function. It enables us to avoid the proof of the boundness of operators, which may be very difficult to do. Note $L_n(g,x) \in R(\Lambda_{9n^2})$. Theorem 3 will be proved if the following inequality holds:

$$\|L_n(g,x) - f(x)\|_{L^p} \le C_M \omega(f,\frac{1}{n})_{L^p}.$$

Applying Lemma 3, we have

$$\begin{aligned}\|L_n(g,x) - f(x)\|_{L^p} &\le \|L_n(g,x) - g(x)\|_{L^p} + \|f(x) - g(x)\|_{L^p} \\ &\le \|L_n(g,x) - g(x)\|_{L^p} + C\omega(f,\frac{1}{n})_{L^p},\end{aligned}$$

so that we only need to show that

$$\|L_n(g,x) - g(x)\|_{L^p} \le C\omega(f,\frac{1}{n})_{L^p}. \tag{4}$$

Assume that $1 < p < \infty$. The case $p=1$ can be treated in a similar and easier way. It is not difficult to verify that, for $1 \le j \le n-1$, it holds that

$$|t_j - t_{j+1}| = |e^{1-x_j} - e^{1-x_{j+1}}| \le Cn^{-1}e^{-x_j}x_j^2 \le Cn^{-1},$$

and

$$|t_0 - t_1| = |1 - e^{-\frac{1}{n}}| \le \frac{1}{n}, \quad |t_n - t_{n+1}| = e^{-n} \le \frac{1}{n}.$$

Hence, for any $x \in [t_{k+1}, t_k]$, $k = 0, 1, 2, \cdots, n$, we have

$$|x - t_j| \le C\left(|k-j|) + 1\right) n^{-1}, \quad j = 0, 1, 2, \cdots, n. \tag{5}$$

Without loss of generality, we always assume that $C_M \ge 1$. From Lemma 1, for any $x \in [t_{j+1}, t_j], 0 \le j \le n$, we have

$$\begin{aligned}\sum_{k=1}^{n} r_k^{\frac{p}{2p-2}}(x) &\le C_M^{\frac{p}{2p-2}} \sum_{k=1}^{n} \exp\left(\frac{-pC_M|j-k|}{2p-2}\right) \\ &\le C_M^{\frac{p}{2p-2}} \left(\sum_{s=1}^{\infty} e^{-Cs/2}\right)^{\frac{p}{p-1}} \le C_M^{\frac{p}{p-1}}.\end{aligned} \tag{6}$$

Using (6) and applying the Hölder's inequality repeatedly, we have

$$\begin{aligned}&\|L_n(g,x) - g(x)\|_{L^p}^p \le \int_0^1 \left|\sum_{k=1}^{n} r_k(x) \frac{1}{t_{k-1} - t_k} \int_{t_k}^{t_{k-1}} |g(u) - g(x)| \mathrm{d}u\right|^p \mathrm{d}x \\ &\le \int_0^1 \left(\sum_{k=1}^{n} r_k^{\frac{p}{2p-2}}(x)\right)^{p-1} \left(\sum_{k=1}^{n} r_k^{\frac{p}{2}}(x) \left|\frac{1}{t_{k-1} - t_k}\right|^p \left|\int_{t_k}^{t_{k-1}} \left|\int_x^u |g'(t)| \mathrm{d}t\right| \mathrm{d}u\right|^p \mathrm{d}x\right) \\ &\le C_M^p \int_0^1 \sum_{k=1}^{n} r_k^{\frac{p}{2}}(x) \left|\frac{1}{t_{k-1} - t_k}\right|^p \left|\int_{t_k}^{t_{k-1}} \left|\int_x^u |g'(t)| \mathrm{d}t\right| \mathrm{d}u\right|^p \mathrm{d}x \\ &\le C_M^p \sum_{j=1}^{n+1} \sum_{k=1}^{n} \int_{t_j}^{t_{j-1}} r_k^{\frac{p}{2}}(x) \left|\frac{1}{t_{k-1} - t_k}\right|^p |t_k - t_{k-1}|^p \left|\int_{x^*}^{t^*} |g'(t)| \mathrm{d}t\right|^p \mathrm{d}x,\end{aligned}$$

where we take $x^* = t_{j-1}, \quad t^* = t_k$, when $j < k$, and take $x^* = t_j, \quad t^* = t_{k-1}$, when $j \ge k$. We continue the above process. By using (3), (5) and Hölder's inequality again, we have

$$\begin{aligned}&\|L_n(g,x) - g(x)\|_{L^p}^p \le C_M^p \sum_{j=1}^{n+1} \sum_{k=1}^{n} e^{-C_M|j-k|p/2} |x^* - t^*|^{p-1} \int_{t_j}^{t_{j-1}} \left|\int_{x^*}^{t^*} |g'(t)|^p \mathrm{d}t\right| \mathrm{d}x \\ &\le C_M^p \sum_{j=1}^{n+1} \sum_{k=1}^{n} e^{-C_M|j-k|p/2} |x^* - t^*|^{p-1} |t_{j-1} - t_j| \left|\int_{x^*}^{t^*} |g'(t)|^p \mathrm{d}t\right| \\ &\le C_M^p n^{-p} \sum_{j=1}^{n+1} \sum_{k=1}^{n} e^{-C|j-k|p/2} (|j-k|^p + 1) \left|\int_{x^*}^{t^*} |g'(t)|^p \mathrm{d}t\right| \\ &\le C_M^p n^{-p} \left\{\sum_{j=1}^{n+1} \sum_{k=1, k \ne j}^{n} e^{-C_M|j-k|p/2} |j-k|^p \left|\int_{x^*}^{t^*} |g'(t)|^p \mathrm{d}t\right| + \sum_{j=1}^{n} \left|\int_{x^*}^{t^*} |g'(t)|^p \mathrm{d}t\right|\right\} \\ &:= I_1 + I_2.\end{aligned}$$

It is easy to verify that

$$\begin{aligned}I_2 &\le C_M^p n^{-p} \sum_{j=1}^{n} \int_{t_j}^{t_{j-1}} |g'(t)|^p \mathrm{d}t \le C_M^p n^{-p} \int_0^1 |g'(t)|^p \mathrm{d}t \\ &\le C_M^p n^{-p} \|g'(t)\|_{L^p}^p,\end{aligned}$$

while

$$\begin{aligned} I_1 &\le C_M^p n^{-p}\left\{\sum_{j=1}^{n+1}\sum_{k=1}^{n} e^{-C_M|j-k|p/2}|j-k|^p\left|\int_{x^*}^{t^*}|g'(t)|^p\mathrm{d}t\right|\right\} \\ &\le C_M^p n^{-p}\sum_{j=1}^{n+1}\sum_{k=1}^{n} e^{-C_M|j-k|p/2}|j-k|^p\left[\left|\int_{t_{j+1}}^{t_k}|g'(t)|^p\mathrm{d}t\right|+\left|\int_{t_{k+1}}^{t_j}|g'(t)|^p\mathrm{d}t\right|\right] \\ &\le C_M^p n^{-p}\sum_{m=1}^{n} e^{-C_M mp/2}m^p\sum_{|j-k|=m}\left[\left|\int_{t_{j-1}}^{t_k}|g'(t)|^p\mathrm{d}t\right|+\left|\int_{t_{k-1}}^{t_j}|g'(t)|^p\mathrm{d}t\right|\right] \\ &\le C_M^p n^{-p}\sum_{m=1}^{n} e^{-C_M mp/2}m^{p+1}\int_0^1|g'(t)|^p\mathrm{d}t \\ &\le C_M^p n^{-p}\left(\sum_{m=1}^{n} e^{-C_M m/2}m^2\right)^p\|g'(t)\|_{L^p}^p \le C_M^p n^{-p}\|g'(t)\|_{L^p}^p. \end{aligned}$$

Altogether with Lemma 3, we get

$$\|L_n(g,x)-g(x)\|_{L^p}^p \le C_M^p n^{-p}\|g'(t)\|_{L^p}^p \le C_M^p\omega(f,\frac{1}{n})_{L^p}^p,$$

and thus Inequality (4), consequently. Theorem 3 is proved.

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$L^p_{[0,1]}$ 空间 Müntz 有理逼近的 Jackson 型估计

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摘要: 设 $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ 为正的实数数列，且当 $n\to\infty$ 时，有 $\lambda_n\searrow 0$. 本文给出了当 $\lambda_n\le Mn^{-\frac{1}{2}}$, $n=1,2,\cdots$, (其中 $M>0$ 为一正常数) 时 Müntz 系统 $\{x^{\lambda_n}\}$ 的有理函数在 $L^p_{[0,1]}$ 空间的逼近速度，主要结论为 $R_n(f,\Lambda)_{L^p}\le C_M\omega(f,n^{-\frac{1}{2}})_{L^p}$, $1\le p\le\infty$.

关键词: Müntz 有理函数; L^p 空间; 逼近速度.