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Nonlinear Maps Satisfying Derivability on the Parabolic Subalgebras of the Full Matrix Algebras

Zheng Xin CHEN[1,*], **Yu E ZHAO**[2]
1. *School of Mathematics and Computer Science, Fujian Normal University, Fujian* 350007*, P. R. China;*
2. *School of Mathematics Science, Qingdao University, Shandong* 266071*, P. R. China*

Abstract Let $\mathbb{F}$ be a field of characteristic 0, $M_n(\mathbb{F})$ the full matrix algebra over $\mathbb{F}$, $\mathbf{t}$ the subalgebra of $M_n(\mathbb{F})$ consisting of all upper triangular matrices. Any subalgebra of $M_n(\mathbb{F})$ containing $\mathbf{t}$ is called a parabolic subalgebra of $M_n(\mathbb{F})$. Let $\mathbf{P}$ be a parabolic subalgebra of $M_n(\mathbb{F})$. A map φ on $\mathbf{P}$ is said to satisfy derivability if $\varphi(x\cdot y)=\varphi(x)\cdot y+x\cdot\varphi(y)$ for all $x,y\in\mathbf{P}$, where φ is not necessarily linear. Note that a map satisfying derivability on $\mathbf{P}$ is not necessarily a derivation on $\mathbf{P}$. In this paper, we prove that a map φ on $\mathbf{P}$ satisfies derivability if and only if φ is a sum of an inner derivation and an additive quasi-derivation on $\mathbf{P}$. In particular, any derivation of parabolic subalgebras of $M_n(\mathbb{F})$ is an inner derivation.

Keywords maps satisfying derivability; parabolic subalgebras; inner derivations; quasi-derivations.

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1. Introduction

Significant research has been done in studying automorphisms and derivations of matrix algebras and their subalgebras [1–4, 6–9]. Let $M_n(\mathbb{F})$ be the associative algebra consisting of all $n\times n$ matrices over a field $\mathbb{F}$ and with the matrix multiplication, $\mathbf{t}$ the subalgebra of $M_n(\mathbb{F})$ consisting of all upper triangular matrices. It is well-known that any derivation of $M_n(\mathbb{F})$ or $\mathbf{t}$ over a field $\mathbb{F}$ is an inner derivation. However, we could not find any reference about derivations of non-trivial parabolic subalgebras of $M_n(\mathbb{F})$, or about nonlinear maps on parabolic subalgebras of $M_n(\mathbb{F})$. In this paper, we determine the parabolic subalgebras of the full matrix algebras over a commutative ring, then prove that any map satisfying derivability on the parabolic subalgebras of the full matrix algebras over a field is a sum of an inner derivation and an additive quasi-derivation (see Theorem 3.2). In particular, we obtain a corollary that any derivation of the parabolic subalgebras of the full matrix algebras is an inner derivation (see Corollary 3.3).

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* Corresponding author
E-mail address: czxing@163.com (Z. X. CHEN)

Let us give an explicit description of the parabolic algebras of $M_n(R)$ over a commutative ring R. For the associative ring $M_n(R)$, there is a corresponding general linear Lie algebra $\mathrm{gl}(n,R)$ also consisting of the $n \times n$ matrices over R and with the bracket operation

$$[x,y] = x \cdot y - y \cdot x.$$

Any subalgebra of $\mathrm{gl}(n,R)$ containing $\mathbf{t}$ is also called a parabolic subalgebra of $\mathrm{gl}(n,R)$. We will prove that the set of the parabolic subalgebras of the full matrix algebra $M_n(R)$ coincides with the set of the parabolic subalgebras of the general linear Lie algebra $\mathrm{gl}(n,R)$.

At first we recall some results in [10] about the parabolic subalgebras of the general linear Lie algebra $\mathrm{gl}(n,R)$ over a commutative ring R. We denote by E the identity matrix in $M_n(R)$ and by E_{ij} the matrix in $M_n(R)$ whose sole nonzero entry 1 is in the (i,j)-position. Let $\mathcal{D}$ be the set of all diagonal matrices in $M_n(R)$. Let $I(R)$ be the set consisting of all ideals of R,

$$\Phi = \{A_{ji} \in I(R) | 1 \leq i < j \leq n\}$$

a subset of $I(R)$ consisting of $n(n-1)/2$ ideals of R. If

$$A_{jk}A_{ki} \subseteq A_{ji} \subseteq A_{jk} \cap A_{ki}$$

for any $1 \leq i < j \leq n$ and any k (if exists) for which $i < k < j$, then Φ is called a flag of ideals of R. By [10, Theorem 2.5], $\mathbf{P}$ is a parabolic subalgebra of $\mathrm{gl}(n,R)$ if and only if there exists a flag $\Phi = \{A_{ji} | 1 \leq i < j \leq n\}$ of ideals of R such that

$$\mathbf{P} = \mathbf{t} + \sum_{1 \leq i < j \leq n} A_{ji}E_{ji}.$$

Taking a proof similar to that in [10, Theorem 2.5], we can prove that the parabolic subalgebras of $M_n(R)$ also have the form of the above $\mathbf{P}$. See the following lemma:

Lemma 1.1 $\mathbf{P}$ *is a parabolic subalgebra of* $M_n(R)$ *if and only if there exists a flag* $\Phi = \{A_{ji} | 1 \leq i < j \leq n\}$ *of ideals of* R *such that* $\mathbf{P} = \mathbf{t} + \sum_{1 \leq i < j \leq n} A_{ji}E_{ji}$.

If $R = \mathbb{F}$ is a field, then there are only two different ideals of $\mathbb{F}$, i.e., 0 and $\mathbb{F}$. For any $1 \leq i < k < j \leq n$, it is easily checked that

$$A_{jk}A_{ki} = A_{jk} \cap A_{ki}$$

for any A_{ki} and A_{jk} in the flag Φ, and so $A_{ji} = A_{jk} \cap A_{ki}$ is determined by A_{jk} and A_{ki} in the flag Φ. Thus a flag Φ is determined by $A_{i+1,i}$, $i = 1, 2, \ldots, n-1$. Let

$$S = \{i | 1 \leq i \leq n-1, A_{i+1,i} = \mathbb{F}\}.$$

Then the subalgebra $\sum_{1 \leq i < j \leq n} A_{ji}E_{ji}$ of $\mathbf{P}$ is generated by $\{E_{i+1,i} | i \in S\}$. Let S_k be a subset of $\mathcal{I} = \{1, 2, \ldots, n\}$, l_k (resp., s_k) the largest (resp. smallest) number in S_k. S_k is called a piecewise subset if S_k consists of the continuous natural numbers between s_k and l_k, i.e.,

$$S_k = \{m \in \mathbb{N} | s_k \leq m \leq l_k\}.$$

For any piecewise subset S_k, there is a subalgebra $\mathbf{p}_k$ associated with S_k, spanned by all elements

E_{rt}, $s_k \le t < r \le l_k$. Or equivalently,

$$\mathbf{p}_k = \sum_{s_k \le t < r \le l_k} \mathbb{F} E_{rt}.$$

The following corollary is easily obtained from Lemma 1.1.

Corollary 1.2 $\mathbf{P}$ *is a parabolic subalgebra of* $M_n(\mathbb{F})$ *if and only if there are some pairwise disjoint piecewise subsets of* $S_1, S_2, \ldots, S_l$ *of* $\mathcal{I} = \{1, 2, \ldots, n\}$ *such that*

$$\mathbf{P} = \mathbf{t} + \sum_{j=1}^{l} \mathbf{p}_j,$$

where $\mathbf{p}_j$ *is the subalgebra associated with the piecewise subset* S_j.

Remark If $\sum_{j=1}^{l} \mathbf{p}_j$ is the set of all strictly low triangular matrices, then $\mathbf{P}$ is the full matrix algebra $M_n(\mathbb{F})$. If $\sum_{j=1}^{l} \mathbf{p}_j = 0$, then $\mathbf{P}$ is the upper triangular matrix algebra $\mathbf{t}$.

Let us give an example for parabolic subalgebras. For $n = 10$, in the parabolic subalgebra $\mathbf{P}$ as above, if $A_{i+1,i} = 0$ for $i = 2, 5, 6$, and $A_{i+1,i} = \mathbb{F}$ for any $i \neq 2, 5, 6$. Let $S_1 = \{1, 2\}$, $S_2 = \{3, 4, 5\}$, $S_3 = \{7, 8, 9, 10\}$. Then

$$\mathbf{P} = \mathbf{t} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3,$$

where $\mathbf{p}_j$ is the subalgebra of $\mathbf{P}$ associated with the piecewise subset S_j, $j = 1, 2, 3$. More explicitly, $\mathbf{p}_1 = \{aE_{21} | a \in \mathbb{F}\}$, $\mathbf{p}_2 = \{aE_{43} + bE_{54} + cE_{53} | a, b, c \in \mathbb{F}\}$, $\mathbf{p}_3 = \{aE_{87} + bE_{98} + cE_{10,9} + dE_{97} + eE_{10,8} + fE_{10,7} | a, b, c, d, e, f \in \mathbb{F}\}$.

2. Certain maps satisfying derivability on parabolic subalgebras

A map φ on an associative $\mathbb{F}$-algebra A is said to satisfy derivability if

$$\varphi(x \cdot y) = \varphi(x) \cdot y + x \cdot \varphi(y)$$

for all $x, y \in A$. The map φ is not necessarily linear. If φ is linear, then φ is a usual derivation on the associative algebra A. For any map φ satisfying derivability on A, it is easy to see that

$$\varphi(0) = 0$$

and

$$\varphi(x \cdot y \cdot z) = \varphi(x) \cdot y \cdot z + x \cdot \varphi(y) \cdot z + x \cdot y \cdot \varphi(z)$$

for any $x, y, z \in A$. In this section, we construct certain standard maps satisfying derivability on a parabolic subalgebra $\mathbf{P}$ of the full matrix algebra $M_n(\mathbb{F})$, which will be used to describe arbitrary maps satisfying derivability on $\mathbf{P}$.

(A) Inner derivations:

For any $A = (a_{ij})_{n \times n} \in \mathbf{P}$, the map

$$\text{ad } A : \ \mathbf{P} \to \mathbf{P}, \ B \mapsto A \cdot B - B \cdot A$$

is called an inner derivation of $\mathbf{P}$. Obviously, any inner derivation is a usual derivation, and so satisfies derivability on $\mathbf{P}$.

(B) Additive quasi-derivations:

Let f be a map on a field $\mathbb{F}$ satisfying the following two conditions:

(i) $f(a+b) = f(a) + f(b)$ for any $a, b \in \mathbb{F}$;

(ii) $f(ab) = f(a)b + af(b)$ for any $a, b \in \mathbb{F}$.

We call such a map f an additive quasi-derivation of $\mathbb{F}$.

Let f be an additive quasi-derivation of $\mathbb{F}$. Define a map φ_f on $\mathbf{P}$ in the way that

$$A = (a_{ij})_{n\times n} \mapsto A_f = (f(a_{ij}))_{n\times n}.$$

Then it is easy to verify that φ_f satisfies derivability. We call such a map φ_f an additive quasi-derivation on $\mathbf{P}$.

It should be pointed out that if f is not a zero map, then φ_f fails to preserve $\mathbb{F}$-scalar multiplication, and so φ_f is not linear, i.e., φ_f is not a derivation on $\mathbf{P}$. Here we give an example of an additive quasi-derivation which is not a derivation. Let $\mathbb{Q}(\pi)$ be the simple transcendental extension of the rational number field $\mathbb{Q}$ by the circular frequency π, i.e.,

$$\mathbb{Q}(\pi) = \Big\{\frac{a_0 + a_1\pi + \cdots + a_m\pi^m}{b_0 + b_1\pi + \cdots + b_n\pi^n} | m, n \in \mathbb{Z}_{\geq 0}, a_i, b_j \in \mathbb{Q}, 0 \leq i \leq m, 0 \leq j \leq n\Big\}.$$

Define an additive quasi-derivation on $\mathbb{Q}(\pi)$ by

$$\begin{aligned} f : \mathbb{Q}(\pi) &\to \mathbb{Q}(\pi), \\ \frac{a_0 + a_1\pi + \cdots + a_m\pi^m}{b_0 + b_1\pi + \cdots + b_n\pi^n} &\mapsto \frac{\partial}{\partial x}\Big(\frac{a_0 + a_1x + \cdots + a_mx^m}{b_0 + b_1x + \cdots + b_nx^n}\Big)\Big|_{x=\pi}, \end{aligned}$$

where $\frac{\partial}{\partial x}\, g(x)$ denotes the derived function of a function $g(x)$. It is easily checked that f is a nonzero additive quasi-derivation on $\mathbb{Q}(\pi)$. So φ_f is an additive quasi-derivation on $\mathbf{P}$ which is not a derivation.

3. Maps satisfying derivability on P

In this section,

$$\mathbf{P} = \mathbf{t} + \sum_{j=1}^{l} \mathbf{p}_j$$

always denotes a parabolic subalgebra of the full matrix algebra $M_n(\mathbb{F})$, where $\mathbf{p}_j$ is the subalgebra associated with the piecewise subset S_j of $\mathcal{I} = \{1, 2, \ldots, n\}$. Let l_j (resp., s_j) be the largest (resp., smallest) number in the piecewise subset S_j of $\mathcal{I}$. For $i, j \in \mathcal{I}$, let $\mathcal{L}_{ij} = \{aE_{ij} | a \in \mathbb{F}\}$ if $E_{ij} \in \mathbf{P}$, and let $\mathcal{L}_{ij} = 0$ if $E_{ij} \notin \mathbf{P}$. Set

$$\mathcal{P} = \{(i, j) \in \mathcal{I} \times \mathcal{I} | i \neq j, E_{ij} \in \mathbf{P}\}.$$

Lemma 3.1 *Let $\mathbf{P}$ be a parabolic subalgebra of the full matrix algebra $M_n(\mathbb{F})$ over a field $\mathbb{F}$, where $n \geq 2$, φ a map satisfying derivability on $\mathbf{P}$. If $\varphi(\mathcal{L}_{ij}) = 0$ for any $i, j \in \mathcal{I}$ with $(i, j) \in \mathcal{P}$, and $\varphi(\mathcal{L}_{ii}) = 0$ for any $i = 1, 2, \ldots, n$, then $\varphi = 0$.*

Proof For any

$$B = (b_{rs})_{n\times n} = \sum_{r,s=1}^{n} b_{rs}E_{rs} \in \mathbf{P},$$

let

$$\varphi(B) = (b'_{rs})_{n\times n} = \sum_{r,s=1}^{n} b'_{rs}E_{rs} \in \mathbf{P}.$$

For any $(k,l) \in \mathbf{P}$ or $k = l \in \{1,2,\ldots,n\}$,

$$b'_{kl}E_{kl} = E_{kk}\cdot\varphi(B)\cdot E_{ll} = \varphi(E_{kk}\cdot B\cdot E_{ll}) = \varphi(b_{kl}E_{kl}) = 0.$$

So $b'_{kl} = 0$. Thus $\varphi(B) = 0$. Therefore $\varphi = 0$. $\Box$

Theorem 3.2 *Let $\mathbf{P}$ be a parabolic subalgebra of the full matrix algebra $M_n(\mathbb{F})$ over a field $\mathbb{F}$ of characteristic 0, where $n \geq 2$. Then a map (without linearity assumption) φ on $\mathbf{P}$ satisfies derivability if and only if it is a sum of an inner derivation and an additive quasi-derivation.*

Proof It is easy to verify that a sum of several maps satisfying derivability on $\mathbf{P}$ still satisfies derivability. Thus the sufficient direction of the theorem is obvious. Now we prove the essential direction of the theorem.

Let φ be a map satisfying derivability on $\mathbf{P}$. Choose a fixed diagonal matrix

$$D_0 = \mathrm{diag}\{1,2,\ldots,n\}.$$

Let

$$\varphi(D_0) = (b_{ij})_{n\times n} \in \mathbf{P}.$$

For any $(i,j) \in \mathcal{P}$,

$$(\mathrm{ad}(b_{ij}(i-j)^{-1}E_{ij}))D_0 = -b_{ij}E_{ij}.$$

Let

$$\varphi_1 = \varphi + \sum_{(i,j)\in\mathcal{P}} \mathrm{ad}(b_{ij}(i-j)^{-1}E_{ij}).$$

Then $\varphi_1(D_0) = \mathrm{diag}\{b_{11}, b_{22}, \ldots, b_{nn}\} \in \mathcal{D}$.

For any diagonal matrix $D' = \mathrm{diag}\{t_1, t_2, \ldots, t_n\} \in \mathcal{D}$,

$$D_0 \cdot D' = D' \cdot D_0,$$

then

$$\varphi_1(D_0 \cdot D') = \varphi_1(D' \cdot D_0),$$

i.e.,

$$\varphi_1(D_0)\cdot D' + D_0 \cdot \varphi_1(D') = \varphi_1(D')\cdot D_0 + D'\cdot\varphi_1(D_0).$$

Since $\varphi_1(D_0)$, D' are diagonal matrices, we have

$$\varphi_1(D_0)\cdot D' = D'\cdot\varphi_1(D_0),$$

and so

$$D_0 \cdot \varphi_1(D') = \varphi_1(D') \cdot D_0. \tag{1}$$

Set

$$\varphi_1(D') = (c_{st})_{n\times n} \in \mathbf{P}.$$

By the equality (1), we have

$$c_{st}(s-t)E_{st} = 0$$

for any s, t. If $s \neq t$ with $(s,t) \in \mathcal{P}$, then $c_{st} = 0$. Thus $\varphi_1(D') = \mathrm{diag}\{c_{11}, c_{22}, \ldots, c_{nn}\}$ is a diagonal matrix. Therefore,

$$\varphi_1(\mathcal{D}) \subseteq \mathcal{D}.$$

For any $a \in \mathbb{F}$, $(i,j) \in \mathcal{P}$, we write aE_{ij} in the form that

$$aE_{ij} = E_{ii} \cdot (aE_{ij}). \tag{2}$$

Applying φ_1 on the both sides of the equality (2), we have

$$\varphi_1(aE_{ij}) = \varphi_1(E_{ii}) \cdot aE_{ij} + E_{ii} \cdot \varphi_1(aE_{ij}). \tag{3}$$

Let

$$\varphi_1(aE_{ij}) = (c_{kl})_{n\times n} \in \mathbf{P}, \varphi_1(E_{ii}) = \mathrm{diag}\{d_1, d_2, \ldots, d_n\},$$

where $d_s \in \mathbb{F}$, $s = 1, 2, \ldots, n$. By the equality (3), if $k \neq i$, then $c_{kl}E_{kl} = 0$ for any l, and so $c_{kl} = 0$ for any l. On the other hand, we write

$$aE_{ij} = aE_{ij} \cdot E_{jj}.$$

Similarly, if $l \neq j$, then $c_{kl} = 0$ for any k. Thus, for any $a \in \mathbb{F}$ and $(i,j) \in \mathcal{P}$, we have

$$\varphi_1(aE_{ij}) = c_{ij}E_{ij} \in \mathcal{L}_{ij}.$$

Or equivalently, $\varphi_1(\mathcal{L}_{ij}) \subseteq \mathcal{L}_{ij}$.

Assume that

$$\varphi_1(E_{i,i+1}) = \bar{b}_i E_{i,i+1},$$

$\bar{b}_i \in \mathbb{F}$, $i = 1, 2, \ldots, n-1$. Choosing $b_1, b_2, \ldots, b_n \in \mathbb{F}$ such that

$$b_i - b_{i+1} = \bar{b}_i, i = 1, 2, \ldots, n-1,$$

we can construct a diagonal matrix

$$h_0 = \mathrm{diag}\{b_1, b_2, \ldots, b_n\}.$$

Then $(\varphi_1 - \mathrm{ad}\, h_0)(E_{i,i+1}) = 0$ for any $i = 1, 2, \ldots, n-1$. Denote

$$\varphi_2 = \varphi_1 - \mathrm{ad}\, h_0.$$

Thus

$$\varphi_2(E_{i,i+1}) = 0$$

for any $i = 1, 2, \ldots, n-1$, and $\varphi_2(\mathcal{D}) \subseteq \mathcal{D}$, $\varphi_2(\mathcal{L}_{ij}) \subseteq \mathcal{L}_{ij}$ for any $(i,j) \in \mathcal{P}$.

Now for $1 \le i \le n-1$, we may define a map $f_i : \mathbb{F} \to \mathbb{F}$ in such a way that

$$\varphi_2(aE_{i,i+1}) = f_i(a)E_{i,i+1}$$

for any $a \in \mathbb{F}$. At first we show that all f_i are the same function. For any $a \in \mathbb{F}$, $i = 1, 2, \dots, n-2$, applying φ_2 on the equality

$$(aE_{i,i+1}) \cdot E_{i+1,i+2} = E_{i,i+1} \cdot aE_{i+1,i+2},$$

we have

$$(f_i(a)E_{i,i+1}) \cdot E_{i+1,i+2} = E_{i,i+1} \cdot (f_{i+1}(a)E_{i+1,i+2}),$$

which forces that $f_i(a) = f_{i+1}(a)$ for all $a \in \mathbb{F}$. So $f_i = f_{i+1}$. It follows that $f_1 = f_2 = \cdots = f_{n-1}$. Now we denote f_1 by f.

Next we show that the same function f is just an additive quasi-derivation of the field $\mathbb{F}$. Let $a, b \in \mathbb{F}$. Since

$$aE_{11} \cdot E_{12} = aE_{12},$$

we have

$$f(a)E_{12} = \varphi_2(aE_{12}) = \varphi_2(aE_{11}) \cdot E_{12},$$

which implies that the coefficient of E_{11} in $\varphi_2(aE_{11})$ is $f(a)$. Applying φ_2 on the equality

$$(aE_{11}) \cdot (bE_{11}) = abE_{11},$$

we have

$$\varphi_2(aE_{11}) \cdot (bE_{11}) + (aE_{11}) \cdot \varphi_2(bE_{11}) = \varphi_2(abE_{11}). \tag{4}$$

Comparing the coefficients of E_{11} on both sides of the equality (4), we have

$$f(ab) = af(b) + f(a)b.$$

So

$$f(1) = f(-1) = 0, f(-b) = -f(b).$$

In particular, the coefficient of E_{11} in $\varphi_2(E_{11})$ is $f(1) = 0$. Applying φ_2 on the equality

$$E_{11} \cdot (aE_{12} + E_{11}) = aE_{12},$$

we have

$$E_{11} \cdot \varphi_2(aE_{12} + E_{11}) + \varphi_2(E_{11}) \cdot (aE_{12} + E_{11}) = f(a)E_{12}. \tag{5}$$

Thus, by the equality (5), the coefficient of E_{12} in $\varphi_2(aE_{12}+E_{11})$ is $f(a)$. Similarly, applying φ_2 on the equality

$$(aE_{12} + E_{11}) \cdot E_{12} = E_{12},$$

we have

$$\varphi_2(aE_{12} + E_{11}) \cdot E_{12} = 0. \tag{6}$$

By the equality (6), the coefficient of E_{11} in $\varphi_2(aE_{12}+E_{11})$ is 0. By the same way, we obtain that the coefficient of E_{22} (resp., E_{12}) in $\varphi_2(E_{22}+bE_{12})$ is 0 (resp., $f(b)$). Then, applying φ_2 on the equality

$$(aE_{12}+E_{11})\cdot(E_{22}+bE_{12})=(a+b)E_{12},$$

we have

$$\begin{aligned}&\varphi_2(aE_{12}+E_{11})\cdot(E_{22}+bE_{12})+(aE_{12}+E_{11})\cdot\varphi_2(E_{22}+bE_{12})\\&\quad=f(a+b)E_{12}.\end{aligned}\tag{7}$$

By the preceding results, the equality (7) leads to $f(a+b)=f(a)+f(b)$. Thus the map f is an additive quasi-derivation of $\mathbb{F}$.

Therefore, we can construct an additive quasi-derivation φ_f of $\mathbf{P}$ extended by f as in Section 2. Denote

$$\varphi_3=\varphi_2-\varphi_f.$$

Thus

$$\varphi_3(aE_{i,i+1})=\varphi_2(aE_{i,i+1})-\varphi_f(aE_{i,i+1})=f(a)E_{i,i+1}-f(a)E_{i,i+1}=0$$

for any $a\in\mathbb{F}$ and any $i=1,2,\ldots,n-1$, i.e., $\varphi_3(\mathcal{L}_{i,i+1})=0$ for any $i=1,2,\ldots,n-1$.

For any diagonal matrix

$$D'=\operatorname{diag}\{t_1,t_2,\ldots,t_n\},$$

and any $i=1,2,\ldots,n-1$, applying φ_3 on

$$D'\cdot E_{i,i+1}=t_iE_{i,i+1},$$

we have

$$\varphi_3(D')\cdot E_{i,i+1}=0.$$

Let

$$\varphi_3(D')=\operatorname{diag}\{t'_1,t'_2,\ldots,t'_n\}.$$

Then $t'_iE_{i,i+1}=0$, which implies that $t'_i=0$ for any $i=1,2,\ldots,n-1$. Similarly, applying φ_3 on

$$E_{i,i+1}\cdot D'=t'_{i+1}E_{i,i+1},$$

we have $t'_{i+1}=0$ for any $i=1,2,\ldots,n-1$. Thus

$$\varphi_3(D')=0.$$

Or equivalently, $\varphi_3(\mathcal{D})=0$.

For any $a\in\mathbb{F}$ and $1\le i<j\le n$, applying φ_3 on

$$aE_{ij}=aE_{i,i+1}\cdot E_{i+1,i+2}\cdot E_{i+2,i+3}\cdots E_{j-1,j},$$

we have

$$\varphi_3(aE_{ij})=0,$$

since $\varphi_3(\mathcal{L}_{k,k+1}) = 0$ for any $k = 1, 2, \ldots, n-1$. If $(j, i) \in \mathcal{P}$, applying φ_3 on

$$E_{ij} \cdot (aE_{ji}) = aE_{ii},$$

we have

$$E_{ij} \cdot \varphi_3(aE_{ji}) = 0.$$

By construction of φ_3,

$$\varphi_3(\mathcal{L}_{ji}) \subseteq \mathcal{L}_{ji}.$$

Let $\varphi_3(aE_{ji}) = a'E_{ji}$, where $a' \in \mathbb{F}$. Then

$$E_{ij} \cdot \varphi_3(aE_{ji}) = a'E_{ii},$$

which implies that $a' = 0$, and so $\varphi_3(aE_{ji}) = 0$ for any $i < j$ with $(j, i) \in \mathcal{P}$, $a \in \mathbb{F}$. Thus

$$\varphi_3(aE_{ji}) = 0$$

for any $a \in \mathbb{F}$ and any $(i, j) \in \mathcal{P}$. Or equivalently, $\varphi_3(\mathcal{L}_{ij}) = 0$ for any $(i, j) \in \mathcal{P}$.

By Lemma 3.1, we know that φ_3 is a zero map on $\mathbf{P}$, i.e.,

$$0 = \varphi + \sum_{(i,j)\in\mathcal{P}} \operatorname{ad}(b_{ij}(i-j)^{-1}E_{ij}) - \operatorname{ad} h_0 - \varphi_f.$$

Thus φ is a sum of an inner derivation

$$-\sum_{(i,j)\in\mathcal{P}} \operatorname{ad}(b_{ij}(i-j)^{-1}E_{ij}) + \operatorname{ad} h_0$$

and an additive quasi-derivation φ_f on $\mathbf{P}$. □

Remark From Theorem 3.2, it is interesting to see that a map on a parabolic subalgebra of the full matrix algebra preserves the additive operation if it satisfies derivability.

It is well-known that any (usual) derivation on the full matrix algebra $M_n(\mathbb{F})$ or the upper triangular matrix algebra $\mathbf{t}$ is an inner derivation. The following corollary generalizes the result to any parabolic subalgebra $\mathbf{P}$ of the full matrix algebra $M_n(\mathbb{F})$.

Corollary 3.3 *Let $\mathbf{P}$ be a parabolic subalgebra of the full matrix algebra over a field $\mathbb{F}$ of characteristic 0, where $n \geq 2$. Then any (usual) derivation φ on $\mathbf{P}$ is an inner derivation.*

Proof For a usual derivation φ, φ is a linear map satisfying derivability. By Theorem 3.2, we can write φ as the following form

$$\varphi = \operatorname{ad} x + \varphi_f,$$

where $\operatorname{ad} x$ is an inner derivation associated with some $x \in \mathbf{P}$, and φ_f is an additive quasi-derivation on $\mathbf{P}$ induced by an additive quasi-derivation f on the field $\mathbb{F}$. Since φ and $\operatorname{ad} x$ are linear, φ_f is also linear. For any $a \in \mathbb{F}$, $0 \neq b \in \mathbb{F}$, then, by linearity of φ_f,

$$\varphi_f(a \cdot bE_{11}) = a \cdot \varphi_f(bE_{11}) = af(b)E_{11}.$$

On the other hand,

$$\varphi_f(a \cdot bE_{11}) = \varphi_f(abE_{11}) = f(ab)E_{11}.$$

Since f is an additive quasi-derivation on the field $\mathbb{F}$, we have

$$\varphi_f(a \cdot bE_{11}) = (af(b) + f(a)b)E_{11}.$$

Therefore,

$$af(b) = af(b) + f(a)b,$$

which leads to $f(a)b = 0$. Since $b \neq 0$, we have $f(a) = 0$. Thus $f = 0$. Or equivalently, $\varphi_f = 0$. It follows that $\varphi = \operatorname{ad} x$ is an inner derivation. □

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