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Limiting Behavior of Weighted Sums of NOD Random Variables

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Abstract The strong laws of large numbers and laws of the single logarithm for weighted sums of NOD random variables are established. The results presented generalize the corresponding results of Chen and Gan [5] in independent sequence case.

Keywords NOD random variables; strong laws of large numbers; laws of single logarithm; weighted sums.

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1. Introduction

Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants. The limiting behavior of weighted sums $\sum_{i=1}^n a_{ni}X_i$ has been studied by many authors. For the strong laws of large numbers, see Bai and Cheng [1], Chen and Gan [5], Choi and Sung [6], Cuzick [7], Sung [15], Teicher [18], Wu [19] and others. For the laws of the single logarithm, see Bai et al. [2], Chen and Gan [5], Li et al. [10], Li and Tomkins [11] and others.

Bai and Cheng [1] and Cuzick [7] ($1 < p < 2$ and $p = 1$, respectively) proved the Marcinkiewicz-Zygmund strong laws of large numbers $n^{-1/p}\sum_{i=1}^n a_{ni}X_i \to 0$ a.s. when $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables with $EX = 0$ and $E|X|^\beta < \infty$, and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of constants satisfying

$$A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n}^\alpha = n^{-1}\sum_{i=1}^n |a_{ni}|^\alpha, \tag{1}$$

where $0 < \alpha$, $\beta < \infty$, and $1/p = 1/\alpha + 1/\beta$.

Bai et al. [2] proved the following laws of the single logarithm

$$\limsup_{n\to\infty} \frac{|\sum_{i=1}^n a_{ni}X_i|}{\sqrt{n\log n}} \le \sqrt{2A_2^2EX^2} \quad \text{a.s.}$$

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when $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $EX = 0$, $E|X|^\beta < \infty$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying (1) for $\alpha > 0, \beta > 0$ and $1/\alpha + 1/\beta = 1/2$.

Chen and Gan [5] proved the above result of Bai et al. [2] under a weaker moment condition $E(|X|^\beta/(\log|X|)^{\beta/2}) < \infty$. Sung [16] obtained a version of the result of Chen and Gan [5] in a Banach space setting.

The main purpose of this paper is to establish the strong laws of large numbers and the law of the single logarithm for weighted sums of identically distributed NOD random variables (its definition is given below). These results extend the corresponding results of Chen and Gan [5] from independent case to NOD setting.

Definition *A finite family of random variables $\{X_i, 1 \leq i \leq k\}$ is said to be*

(a) Negatively upper orthant dependent (NUOD) if

$$P(X_i > x_i, i = 1, 2, \ldots, k) \leq \prod_{i=1}^{k} P(X_i > x_i) \tag{2}$$

for $\forall x_1, x_2, \ldots, x_k \in R$;

(b) Negatively lower orthant dependent(NLOD) if

$$P(X_i \leq x_i, i = 1, 2, \ldots, k) \leq \prod_{i=1}^{k} P(X_i \leq x_i) \tag{3}$$

for $\forall x_1, x_2, \ldots, x_k \in R$;

(c) Negatively orthant dependent (NOD) if both (2) and (3) hold.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be NOD if for each n, $X_1, X_2, \ldots, X_n$ are NOD.

This definition was introduced by Joag-Dev and Proschan [9]. Obviously, an independent random variables sequence is NOD. Joag-Dev and Proschan [9] pointed out that NA must be NOD and NOD is not necessarily NA. This shows that NOD is strictly weaker than NA. Since NA sequences have wide applications in multivariate statistical analysis and reliability, the notion of NA random variables has received more and more attention in recent years. There are many papers about NA random variables, while papers about NOD random variables are too few. The following is not necessarily an exhaustive list of such papers: Bozorgnia et al. [4], Gan and Chen [8]), Joag-Dev and Proschan [9], Qiu [12], Qiu et al. [13], Taylor et al. [17].

For the proof of the theorems in this paper, we need the following lemmas:

Lemma 1 ([4]) *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables.*

1) If $\{f_n, n \geq 1\}$ is a sequence of real measurable functions all of which are monotone increasing (or all monotone decreasing), then $\{f_n(X_n), n \geq 1\}$ is a sequence of NOD random variables.

2) If $\{X_n, n \geq 1\}$ is a sequence of nonnegative NOD random variables, then $E(\prod_{j=1}^n X_j) \leq \prod_{j=1}^n E(X_j), \forall n \geq 2$ provided the expectations are finite.

Lemma 2 ([13]) *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise NOD random variables with $EX_{ni} = 0$ for all $1 \le i \le k_n,\ n \ge 1$. Let $\{a_n, n \ge 1\}$ be a sequence of positive constants and $\phi(x)$ be a real function such that for some $\delta > 0$*

$$\sup_{x>\delta} \frac{x}{\phi(x)} < \infty \text{ and } \sup_{0\le x\le\delta} \frac{x^2}{\phi(x)} < \infty.$$

Suppose that

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty$ *for all* $\varepsilon > 0$,

(ii) $\sum_{i=1}^{k_n} E\phi(|X_{ni}|) \to 0$ *as* $n \to \infty$,

(iii) $\sum_{n=1}^{\infty} a_n (\sum_{i=1}^{k_n} E\phi(|X_{ni}|))^J < \infty$ *for some* $J \ge 1$.

Then

$$\sum_{n=1}^{\infty} a_n P\Big(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon\Big) < \infty \quad \text{for any } \ \varepsilon > 0.$$

Lemma 3 ([14]) *If $X \le 1$ a.s., then $E\exp(X) \le \exp(EX + EX^2)$.*

Throughout this paper, C will represent positive constants whose value may change at each occurrence.

2. Main results

Theorem 1 *Let $0 < p < 2$, $0 < \alpha$, $\beta < \infty$, and $1/p = 1/\alpha + 1/\beta$. Assume that $\varphi(x) = x^{1/\beta} l(x)$, where $l(x) > 0$ $(x > 0)$ is a slowly varying function. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed NOD random variables and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1). If $E\varphi^-(|X|) < \infty$, where φ^- is the inverse of φ, and if $1 \le p < 2$, we moreover assume that $EX = 0$, then*

$$\lim_{n\to\infty} n^{-1/p}(l(n))^{-1} \sum_{i=1}^{n} a_{ni} X_i = 0 \quad \text{a.s.} \tag{4}$$

Conversely if (4) is true for any coefficient arrays satisfying (1), then $E\varphi^-(|X|) < \infty$ and if $1 \le p < 2$, we further have $EX = 0$.

Proof Without loss of generality, we may assume that $a_{ni} \ge 0$ for $1 \le i \le n$ and $n \ge 1$. For $\forall \gamma : 0 < \gamma < \alpha$, by (1) and Hölder's inequality, we have

$$\sum_{i=1}^{n} |a_{ni}|^\gamma = (\sum_{i=1}^{n} |a_{ni}|^\alpha)^{\gamma/\alpha} (\sum_{i=1}^{n} 1)^{1-\gamma/\alpha} \le Cn. \tag{5}$$

For $\forall \gamma : \gamma \ge \alpha$, by (1) and the C_r-inequality, we have

$$\sum_{i=1}^{n} |a_{ni}|^\gamma \le \sum_{i=1}^{n} (|a_{ni}|^\alpha)^{\gamma/\alpha} \le (\sum_{i=1}^{n} |a_{ni}|^\alpha)^{\gamma/\alpha} \le Cn^{\gamma/\alpha}. \tag{6}$$

Define $X_{ni} = n^{1/\beta} l(n) I(X_i > n^{1/\beta} l(n)) + X_i I(|X_i| \le n^{1/\beta} l(n)) - n^{1/\beta} l(n) I(X_i < -n^{1/\beta} l(n))$, $Y_{ni} = X_i - X_{ni}$ for $1 \le i \le n$ and $n \ge 1$. Note that $E\varphi^-(|X|) < \infty$ is equivalent to

$\sum_{n=1}^{\infty} P(|X_n| > n^{1/\beta} l(n)) < \infty$, hence we have by the Borel-Cantelli lemma that

$$\begin{aligned} n^{-1/p}(l(n))^{-1}|\sum_{i=1}^{n} a_{ni} Y_{ni}| &\le n^{-1/p}(l(n))^{-1} \max_{1\le i\le n} |a_{ni}| \sum_{i=1}^{n} |Y_{ni}| \\ &\le A_{\alpha,n} n^{-1/\beta}(l(n))^{-1} \sum_{i=1}^{n} |Y_{ni}| \to 0 \ \text{ a.s. } \ n \to \infty. \end{aligned} \tag{7}$$

Since $\varphi(x)$ $(x > 0)$ is a regularly varying function with exponent $1/\beta$, by Theorem 1.5.12 of Bingham et al. [3], φ^{-} is a regularly varying function with exponent β, then $E\varphi^{-}(|X|) < \infty$ implies

$$E|X|^{v} < \infty \ \text{ for all } \ v \in (0, \beta). \tag{8}$$

Next we will prove that

$$D_n \stackrel{\text{def}}{=} n^{-1/p}(l(n))^{-1} \sum_{i=1}^{n} E a_{ni} X_{ni} \to 0, \quad n \to \infty. \tag{9}$$

If $0 < p < 1$ and $0 < \alpha \le 1$, by (6) and (8), we have

$$\begin{aligned} |D_n| &\le n^{-1/p}(l(n))^{-1} \sum_{i=1}^{n} a_{ni} E|X_{ni}|^{p}|X_{ni}|^{1-p} \\ &\le C n^{-1/p+1/\alpha}(l(n))^{-1}(n^{1/\beta} l(n))^{1-p} E|X|^{p} = C n^{-p/\beta}(l(n))^{-p} \to 0, \quad n \to \infty. \end{aligned}$$

If $0 < p < 1$ and $\alpha > 1$, by (5) and (8), we have

$$\begin{aligned} |D_n| &\le n^{-1/p}(l(n))^{-1} \sum_{i=1}^{n} a_{ni} E|X_{ni}|^{p}|X_{ni}|^{1-p} \\ &\le C n^{-1/p+1}(l(n))^{-1}(n^{1/\beta} l(n))^{1-p} E|X|^{p} = C n^{(p-1)/\alpha}(l(n))^{-p} \to 0, \quad n \to \infty. \end{aligned}$$

If $1 \le p < 2$, we have that $\alpha, \beta > p$ from $1/p = 1/\alpha + 1/\beta$. We take $v \in (\max\{1, (1-1/\alpha)\beta\}, \beta)$. By (5) and (8) and $EX = 0$, we have

$$\begin{aligned} |D_n| &\le 2n^{-1/p}(l(n))^{-1} \sum_{i=1}^{n} a_{ni} E|X| I(|X| > n^{1/\beta} l(n)) \\ &= 2n^{-1/p}(l(n))^{-1} \sum_{i=1}^{n} a_{ni} E|X|^{v}|X|^{1-v} I(|X| > n^{1/\beta} l(n)) \\ &\le C n^{-1/p}(l(n))^{-1} \cdot n \cdot n^{(1-v)/\beta}(l(n))^{1-v} E|X|^{v} \\ &\le C n^{1-1/\alpha-v/\beta}(l(n))^{-v} \to 0, \quad n \to \infty. \end{aligned}$$

Therefore (9) holds. To prove (4), by (7) and (9), it is enough to show that

$$\lim_{n\to\infty} n^{-1/p}(l(n))^{-1} \sum_{i=1}^{n} a_{ni}(X_{ni} - EX_{ni}) = 0 \quad a.s.$$

Hence it suffices to prove that

$$\sum_{n=1}^{\infty} P\Big(n^{-1/p}(l(n))^{-1}\Big|\sum_{i=1}^{n} a_{ni}(X_{ni} - EX_{ni})\Big| > \epsilon\Big) < \infty, \quad \forall \epsilon > 0. \tag{10}$$

By Lemma 1 we can conclude that for each $n \geq 1, \{n^{-1/p}(l(n))^{-1}a_{ni}(X_{ni} - EX_{ni}), 1 \leq i \leq n\}$ is a sequence of NOD random variables. To prove (10), we will apply Lemma 2 with $a_n = 1$ and $\phi(x) = x^2$. Take q such that $q > \max\{\alpha, \beta\}$. By Markov's inequality, (6), C_r-inequality and Jensen's inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|a_{ni}(X_{ni} - EX_{ni})| > \epsilon n^{1/p} l(n)) \\
&\leq C\sum_{n=1}^{\infty} n^{-q/p}(l(n))^{-q}\sum_{i=1}^{n} E|a_{ni}(X_{ni} - EX_{ni})|^q \\
&\leq C\sum_{n=1}^{\infty} n^{-q/p+q/\alpha}(l(n))^{-q}\{E|X|^q I(|X| \leq n^{1/\beta}l(n)) + n^{q/\beta}(l(n))^q P(|X| > n^{1/\beta}l(n))\} \\
&\leq \sum_{n=1}^{\infty} n^{-q/\beta}(l(n))^{-q} E|X|^q I(|X| \leq n^{1/\beta}l(n)) + C\sum_{n=1}^{\infty} P(|X| > n^{1/\beta}l(n)) \\
&\leq CE\varphi^{-}(|X|) < \infty. \qquad (11)
\end{aligned}$$

Thus condition (i) of Lemma 2 is satisfied. Taking $v \in (\max\{0, \beta(1 - 2/\alpha)\}, \beta)$, by (5), (6) and (8) and C_r-inequality, we have

$$\begin{aligned}
&\sum_{i=1}^{n} n^{-2/p}(l(n))^{-2}a_{ni}^2 E(X_{ni} - EX_{ni})^2 \\
&\leq n^{-2/p}(l(n))^{-2}\sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2 \\
&\leq Cn^{-2/p}(l(n))^{-2}\sum_{i=1}^{n} a_{ni}^2\{EX^2 I(|X| \leq n^{1/\beta}l(n)) + n^{2/\beta}(l(n))^2 P(|X| > n^{1/\beta}l(n))\} \\
&\leq \begin{cases} Cn^{-v/\beta}(l(n))^{-v}, \ 0 < \alpha < 2, \ 0 < \beta \leq 2 \\ Cn^{-2/\beta}(l(n))^{-2}, \ 0 < \alpha < 2, \ \beta > 2 \\ Cn^{-v/\beta-2/\alpha+1}(l(n))^{-v}, \ \alpha \geq 2, \ 0 < \beta \leq 2 \\ Cn^{1-2/p}(l(n))^{-2}, \ \alpha \geq 2, \ \beta > 2 \end{cases} \\
&\longrightarrow 0, \quad n \to \infty.
\end{aligned}$$

Thus condition (ii) of Lemma 2 is satisfied. Taking J such that $J > \max\{\beta/v, \beta/2, 1/(v/\beta + 2/\alpha - 1), 1/(2/p - 1)\}$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\sum_{i=1}^{n} a_{ni}^2 n^{-2/p}(l(n))^{-2}E(X_{ni} - EX_{ni})^2\Big)^J \\
&\leq \begin{cases} C\sum_{n=1}^{\infty} n^{-Jv/\beta}(l(n))^{-Jv}, \ 0 < \alpha < 2, \ 0 < \beta \leq 2 \\ C\sum_{n=1}^{\infty} n^{-2J/\beta}(l(n))^{-2J}, \ 0 < \alpha < 2, \ \beta > 2 \\ C\sum_{n=1}^{\infty} n^{-J(v/\beta+2/\alpha-1)}(l(n))^{-Jv}, \ \alpha \geq 2, \ 0 < \beta \leq 2 \\ C\sum_{n=1}^{\infty} n^{-J(2/p-1)}(l(n))^{-2J}, \ \alpha \geq 2, \ \beta > 2 \end{cases} \\
&< \infty.
\end{aligned}$$

Thus condition (iii) of Lemma 2 is satisfied, and (10) holds.

Necessity. See the proof of Bai and Cheng [1]. □

Theorem 2 *Let $0 < p < 2$, $0 < \alpha \le 2$, $\alpha < \beta < \infty$, and $1/p = 1/\alpha + 1/\beta$. Assume that $\varphi(x) = x^{1/\beta} l(x)$, where $l(x) > 0$ $(x > 0)$ is a slowly varying function. Let $\{X, X_n, -\infty < n < \infty\}$ be a sequence of identically distributed NOD random variables and let $\{b_{ni}, -\infty < i < \infty, n \ge 1\}$ be a double infinite array of constants satisfying*

$$B_\alpha = \limsup_{n\to\infty} B_{\alpha,n} < \infty, \quad B_{\alpha,n}^\alpha = n^{-1} \sum_{i=-\infty}^{\infty} |b_{ni}|^\alpha. \tag{12}$$

If $E\varphi^-(|X|) < \infty$, where φ^- is the inverse of φ, and if $1 < \alpha \le 2$, we moreover assume that $EX = 0$, then

$$\lim_{n\to\infty} n^{-1/p}(l(n))^{-1} \sum_{i=-\infty}^{\infty} b_{ni} X_i = 0 \quad \text{a.s.} \tag{13}$$

Conversely, if (13) is true for any coefficient arrays satisfying (12), then $E\varphi^-(|X|) < \infty$ and if $1 < \alpha \le 2$, we further have $EX = 0$.

Proof Without loss of generality, we may assume that $b_{ni} \ge 0$ for $-\infty < i < \infty$ and $n \ge 1$.

Case 1 $0 < \alpha \le 1$.

We define X_{ni}, Y_{ni} as in Theorem 1. Similarly to the proof of Theorem 1, we have

$$n^{-1/p}(l(n))^{-1} \left| \sum_{i=-\infty}^{\infty} b_{ni} Y_{ni} \right| \to 0 \quad \text{a.s.}$$

Since $0 < \alpha < \beta$, by (8) and (12) we have

$$\begin{aligned} &\left| n^{-1/p}(l(n))^{-1} \sum_{i=-\infty}^{\infty} E b_{ni} X_{ni} \right| \le n^{-1/p}(l(n))^{-1} \sum_{i=-\infty}^{\infty} b_{ni}^\alpha b_{ni}^{1-\alpha} E|X_{ni}|^\alpha |X_{ni}|^{1-\alpha} \\ &\quad \le C n^{-1/p}(l(n))^{-1} n (n^{1/\alpha})^{1-\alpha} (n^{1/\beta} l(n))^{1-\alpha} E|X|^\alpha \le C n^{-\alpha/\beta}(l(n))^{-\alpha} \to 0, \quad n \to \infty. \end{aligned}$$

Therefore, to prove (13), it is enough to show that

$$\sum_{n=1}^{\infty} P\Big(n^{-1/p}(l(n))^{-1} \Big| \sum_{i=-\infty}^{\infty} b_{ni}(X_{ni} - EX_{ni}) \Big| > \epsilon \Big) < \infty, \quad \forall \epsilon > 0. \tag{14}$$

Similarly to the proof of (11), we have

$$\sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} P(|b_{ni}(X_{ni} - EX_{ni})| > \epsilon n^{1/p} l(n)) \le C E\varphi^-(|X|) < \infty.$$

Taking $v \in (\max\{0, \beta(1-2/\alpha)\}, \beta)$, by (8) and C_r-inequality, we have

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty} n^{-2/p}(l(n))^{-2} b_{ni}^2 E(X_{ni} - EX_{ni})^2 \\
&\le n^{-2/p}(l(n))^{-2} \sum_{i=-\infty}^{\infty} b_{ni}^2 EX_{ni}^2 \\
&\le Cn^{-2/p}(l(n))^{-2} \sum_{i=-\infty}^{\infty} b_{ni}^{\alpha} b_{ni}^{2-\alpha} \{EX^2 I(|X| \le n^{1/\beta} l(n)) + n^{2/\beta}(l(n))^2 P(|X| > n^{1/\beta} l(n))\} \\
&\le Cn^{-2/p}(l(n))^{-2} n (n^{1/\alpha})^{2-\alpha} (n^{1/\beta} l(n))^{2-\alpha} E|X|^{\alpha} = Cn^{-\alpha/\beta}(l(n))^{-\alpha} \to 0, \quad n \to \infty.
\end{aligned}$$

Taking J such that $J\alpha/\beta > 1$, then we have

$$\sum_{n=1}^{\infty} \Big(\sum_{i=-\infty}^{\infty} b_{ni}^2 n^{-2/p}(l(n))^{-2} E(X_{ni} - EX_{ni})^2 \Big)^J < \infty.$$

Therefore, (14) holds in the case $0 < \alpha \le 1$.

Case 2 $1 < \alpha \le 2$.

To prove (13), we will apply Lemma 2 with $a_n = 1$ and $\phi(x) = x^{\alpha}$. Similarly to the proof of (2.10) and (2.11) of Chen and Gan [5], we have

$$\sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} P(|n^{-1/p}(l(n))^{-1} b_{ni} X_i| > \epsilon) < \infty.$$

By (8) and $\alpha > p$, we have

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty} E(n^{-1/p}(l(n))^{-1} b_{ni} X_i)^{\alpha} = E|X|^{\alpha} \sum_{i=-\infty}^{\infty} n^{-\alpha/p}(l(n))^{-\alpha} b_{ni}^{\alpha} \\
&\le Cn^{1-\alpha/p}(l(n))^{-\alpha} \to 0, \quad n \to \infty.
\end{aligned}$$

Taking J such that $J(1-\alpha/p) < -1$, then we have

$$\sum_{n=1}^{\infty} \Big(\sum_{i=-\infty}^{\infty} E(n^{-1/p}(l(n))^{-1} b_{ni} X_i)^{\alpha} \Big)^J \le C \sum_{n=1}^{\infty} n^{J(1-\alpha/p)}(l(n))^{-J\alpha} < \infty.$$

Therefore, by Lemma 2, (13) holds.

Necessity. See the proof of Bai and Cheng [1]. □

Theorem 3 *Let $0 < \alpha$, $\beta < \infty$ and $1/2 = 1/\alpha + 1/\beta$. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed NOD random variables with $EX = 0$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (1). If $E(|X|^{\beta}/(\log|X|)^{\beta/2}) < \infty$, then*

$$\limsup_{n\to\infty} \frac{|\sum_{i=1}^{n} a_{ni} X_i|}{\sqrt{n \log n}} \le 4\sqrt{A_2^2 EX^2}. \tag{15}$$

Conversely, if (15) is true for any coefficient arrays satisfying (1), then $E(|X|^{\beta}/(\log|X|)^{\beta/2}) < \infty$ and $EX = 0$.

Proof Define $b = \sqrt{A_2^2 E|X|^2}/4, X_{ni} = bn^{1/\beta}(\log n)^{1/2}(X_i > bn^{1/\beta}(\log n)^{1/2}) + X_i I(|X_i| \le$

$bn^{1/\beta}(\log n)^{1/2}) - bn^{1/\beta}(\log n)^{1/2}I(X_i < -bn^{1/\beta}(\log n)^{1/2}), Y_{ni} = X_n - X_{ni}$ for $1 \le i \le n$ and $n \ge 1$. Note that $E(|X|^\beta/(\log |X|)^{\beta/2}) < \infty$ is equivalent to $\sum_{n=1}^\infty P(|X| > \epsilon n^{1/\beta}(\log n)^{1/2}) < \infty$ for any $\epsilon > 0$. Then by the Borel-Cantelli Lemma, we obtain that

$$\frac{|\sum_{i=1}^n a_{ni}Y_{ni}|}{\sqrt{n\log n}} \le \frac{Cn^{1/\alpha}\sum_{i=1}^n |Y_{ni}|}{\sqrt{n\log n}} = \frac{C\sum_{i=1}^n |Y_{ni}|}{n^{1/\beta}\sqrt{\log n}} \to 0 \text{ a.s.} \tag{16}$$

Since $EX = 0$ and $x^{1-\beta}(\log x)^{\beta/2}$ $(x \ge 3)$ is a decreasing function, for any $A \subseteq \{1, 2, \ldots, n\}$, by (5), we have

$$\begin{aligned}
&\frac{1}{\sqrt{n\log n}}\Big|\sum_{i\in A} a_{ni}EX_{ni}\Big| \le \frac{1}{\sqrt{n\log n}}\Big|\sum_{i\in A} a_{ni}EXI(|X| > bn^{1/\beta}(\log n)^{1/2})\Big| + \\
&\quad \frac{1}{\sqrt{n\log n}}\sum_{i\in A} |a_{ni}|bn^{1/\beta}(\log n)^{1/2}P(|X| > bn^{1/\beta}(\log n)^{1/2}) \\
&\le 2\frac{1}{\sqrt{n\log n}}\sum_{i=1}^n |a_{ni}|E|X|I(|X| > bn^{1/\beta}(\log n)^{1/2}) \\
&\le \frac{Cn}{\sqrt{n\log n}}E\frac{|X|^\beta}{(\log |X|)^{\beta/2}} \cdot |X|^{1-\beta}(\log |X|)^{\beta/2}I(|X| > bn^{1/\beta}(\log n)^{1/2}) \\
&\le Cn^{-1/\alpha}E\frac{|X|^\beta}{(\log |X|)^{\beta/2}} \to 0, \quad n \to \infty.
\end{aligned} \tag{17}$$

For any $n \ge 1$, let

$$S_n = \{i : 1 \le i \le n \text{ and } |a_{ni}| > n^{1/\alpha}/\log n\}, \quad T_n = \{i : 1 \le i \le n \text{ and } |a_{ni}| \le n^{1/\alpha}/\log n\}.$$

To prove (15), by (16) and (17), it is enough to show that

$$\lim_{n\to\infty} \frac{|\sum_{i\in S_n} a_{ni}(X_{ni} - EX_{ni})|}{\sqrt{n\log n}} = 0 \text{ a.s.} \tag{18}$$

and

$$\limsup_{n\to\infty} \frac{|\sum_{i\in Tn} a_{ni}(X_{ni} - EX_{ni})|}{\sqrt{n\log n}} \le 4\sqrt{A_2^2EX^2} \text{ a.s.} \tag{19}$$

Firstly we prove (18). It suffices to prove that

$$\lim_{n\to\infty} \frac{|\sum_{i\in S_n} a_{ni}^+(X_{ni} - EX_{ni})|}{\sqrt{n\log n}} = 0 \text{ a.s.} \tag{20}$$

and

$$\lim_{n\to\infty} \frac{|\sum_{i\in S_n} a_{ni}^-(X_{ni} - EX_{ni})|}{\sqrt{n\log n}} = 0 \text{ a.s.} \tag{21}$$

Obviously, $\{a_{ni}^+(X_{ni} - EX_{ni})/\sqrt{n\log n}, i \in S_n\}$ and $\{a_{ni}^-(X_{ni} - EX_{ni})/\sqrt{n\log n}, i \in S_n\}$ are sequences of NOD random variables for each $n \ge 1$. To prove (20), let $a_n = 1$ in Lemma 2.

Taking $q > \max\{\alpha, \beta\}$, we get by Markov inequality, C_r-inequality and Jensen inequality that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{i\in S_n} P\left(\frac{|a_{ni}^{+}(X_{ni}-EX_{ni})|}{\sqrt{n\log n}} > \epsilon\right) \le \sum_{n=2}^{\infty}\frac{2^{q-1}}{\epsilon^q}\sum_{i=1}^{n}\frac{E(|a_{ni}^{+}X_{ni}|)^q}{(n\log n)^{q/2}} \\
&\le C\sum_{n=2}^{\infty}\frac{1}{(n\log n)^{q/2}}\sum_{i=1}^{n}|a_{ni}|^q E|X|^q I(|X|\le bn^{1/\beta}(\log n)^{1/2})+ \\
&\quad C\sum_{n=2}^{\infty}\frac{1}{(n\log n)^{q/2}}\sum_{i=1}^{n}|a_{ni}|^q\left(n^{1/\beta}(\log n)^{1/2}\right)^q P(|X|>bn^{1/\beta}(\log n)^{1/2}) \\
&\le C\sum_{i=2}^{\infty}E|X|^q I\left(b(i-1)^{1/\beta}(\log(i-1))^{1/2}<|X|\le bi^{1/\beta}(\log i)^{1/2}\right)\sum_{n=i}^{\infty}\frac{n^{q/\alpha}}{(n\log n)^{q/2}}+ \\
&\quad C\sum_{n=1}^{\infty}P(|X|>bn^{1/\beta}(\log n)^{1/2}) \\
&\le C\sum_{i=2}^{\infty}E|X|^q I\left(b(i-1)^{1/\beta}(\log(i-1))^{1/2}<|X|\le bi^{1/\beta}(\log i)^{1/2}\right)\cdot \\
&\quad \frac{1}{i^{q/2-q/\alpha-1}(\log i)^{q/2}} + CE\frac{|X|^\beta}{(\log|X|)^{\beta/2}} \\
&\le CE\left(|X|^\beta/(\log|X|)^{\beta/2}\right) < \infty.
\end{aligned}$$

Note that $\alpha > 2$, $\sharp S_n \le C(\log n)^\alpha$ and $EX^2 < \infty$, we have that

$$\begin{aligned}
\sum_{i\in S_n} E\Big(\frac{a_{ni}^{+}(X_{ni}-EX_{ni})}{\sqrt{n\log n}}\Big)^2 \quad &\le \frac{\sum_{i\in S_n} a_{ni}^2 E|X|^2}{n\log n} \\
&\le \frac{Cn^{2/\alpha}(\log n)^\alpha}{n\log n} \le C\frac{(\log n)^\alpha}{n^{2/\beta}\log n} \to 0, \quad n\to\infty.
\end{aligned}$$

We can choose J such that $2J/\beta > 1$, then

$$\sum_{n=1}^{\infty}\Big(\sum_{i\in S_n} E\Big(\frac{a_{ni}^{+}(X_{ni}-EX_{ni})}{\sqrt{2n\log n}}\Big)^2\Big)^J \le C\sum_{n=1}^{\infty}\Big(\frac{(\log n)^\alpha}{n^{2/\beta}\log n}\Big)^J < \infty.$$

Thus, (20) holds. Similarly to the proof of (20), (21) holds.

Secondly, we prove (19). By the Borel-Cantelli Lemma, it is enough to prove that for any $\eta > 0$, we have

$$\sum_{n=1}^{\infty} P\Big(\frac{|\sum_{i\in Tn} a_{ni}(X_{ni}-EX_{ni})|}{\sqrt{n\log n}} > 4\sqrt{A_2^2E|X|^2}+2\eta\Big) < \infty. \tag{22}$$

Define $\delta = 2\sqrt{A_2^2E|X|^2}+\eta$. To prove (22), it is enough to show that

$$\sum_{n=1}^{\infty} P\Big(\frac{|\sum_{i\in Tn} a_{ni}^{+}(X_{ni}-EX_{ni})|}{\sqrt{n\log n}} > \delta\Big) < \infty, \tag{23}$$

and

$$\sum_{n=1}^{\infty} P\Big(\frac{|\sum_{i\in Tn} a_{ni}^{-}(X_{ni}-EX_{ni})|}{\sqrt{n\log n}} > \delta\Big) < \infty. \tag{24}$$

Note that $\{a_{ni}^+(X_{ni}-EX_{ni})/\sqrt{n\log n}, i\in T_n\}$ and $\{a_{ni}^-(X_{ni}-EX_{ni})/\sqrt{n\log n}, i\in T_n\}$ are sequences of NOD random variables for each $n\geq 1$. Since $|a_{ni}(X_{ni}-EX_{ni})/\sqrt{n\log n}|\leq 2b/\log n$ for any $i\in T_n$ and $n\geq 1$, by Lemmas 1 and 3, for any $0<t_n\leq \log n/(2b)=2\log n/\sqrt{A_2^2E|X|^2}$, we can get that

$$\begin{aligned}
&P\Big(\frac{\sum_{i\in Tn}a_{ni}^+(X_{ni}-EX_{ni})}{\sqrt{n\log n}}>\delta\Big)=P\Big(\exp\Big(\frac{t_n\sum_{i\in Tn}a_{ni}^+(X_{ni}-EX_{ni})}{\sqrt{n\log n}}\Big)>\exp(t_n\delta)\Big)\\
&\leq \exp(-t_n\delta)\prod_{i\in T_n}E\exp\Big(t_n\frac{a_{ni}^+(X_{ni}-EX_{ni})}{\sqrt{n\log n}}\Big)\\
&\leq \exp(-t_n\delta)\prod_{i\in T_n}\exp\Big(t_n^2\frac{(a_{ni}^+)^2E(X_{ni}-EX_{ni})^2}{n\log n}\Big)\\
&\leq \exp\Big(-t_n\delta+t_n^2\frac{\sum_{i=1}^n a_{ni}^2EX_{ni}^2}{n\log n}\Big)\leq \exp\left(-t_n\delta+t_n^2A_2^2E|X|^2/\log n\right).
\end{aligned}$$

Taking $t_n=\log n/\sqrt{A_2^2E|X|^2}$, we get that

$$\sum_{n=1}^{\infty}P\Big(\frac{\sum_{i\in Tn}a_{ni}^+(X_{ni}-EX_{ni})}{\sqrt{n\log n}}>\delta\Big)\leq \sum_{n=1}^{\infty}n^{-1-\eta/\sqrt{A_2^2E|X|^2}}<\infty.$$

Considering $\{-a_{ni}^+(X_{ni}-EX_{ni})/\sqrt{n\log n}, i\in T_n\}$ and using the above result gives

$$\sum_{n=1}^{\infty}P\Big(\frac{\sum_{i\in Tn}-a_{ni}^+(X_{ni}-EX_{ni})}{\sqrt{n\log n}}>\delta\Big)\leq \sum_{n=1}^{\infty}n^{-1-\eta/\sqrt{A_2^2E|X|^2}}<\infty.$$

Therefore, (23) holds. Similarly to the proof of (23), (24) holds.

Necessity. See the proof of Bai and Cheng [1].

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