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# Species Permanence Analysis of an Ecological Model with an Impulsive Control Strategy

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**Abstract** In this paper, on the basis of the theories and methods of ecology and ordinary differential equations, an ecological model with an impulsive control strategy is established. By using the theories of impulsive equations, small amplitude perturbation skills and comparison technique, we get the condition which guarantees the global asymptotical stability of the prey-$x$-eradication and predator-$y$-eradication periodic solution. It is proved that the system is permanent. Furthermore, numerical simulations are also illustrated which agree well with our theoretical analysis. All these results may be useful in study of the dynamic complexity of ecosystems.

**Keywords** impulsive control strategy; locally asymptotically stable; complex dynamics; periodic solution.

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## 1. Introduction

The research of the last two decades demonstrates that very complex dynamics can arise in continuous time food chain models with three or more species [1–4], while similar results are obtained for multi-species food models with specialist and generalist top-predators [5–7].

Many evolution processes are characterized by the fact at certain moments of time when they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulse. It is well known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulate systems do exhibit impulsive effects. Thus impulsive differential equations,

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differential equations involving impulsive effects, appear as a natural description of observed evolution phenomena of several real world problems [8–10]. Furthermore, a new approach was presented in [11] via variational methods and critical point theory to obtain the existence of solutions to impulsive problems. It was pointed out in [12] that there exist Li-York chaos in the system with impacts. The paper [13, 14] studied the qualitative behavior of a class of ratio-dependent predator-prey system at the origin of the first quadrant, and it was shown that the origin is indeed a critical point of higher order. The heteroclinic bifurcation of predator-prey system was investigated in [15–17], where parametric conditions of the existence of the heteroclinic loop were given analytically and the heteroclinic bifurcation surface in the parameter space was described. The field of research of chaotic impulsive differential equations about biological and chemical control seems to be a new growing interesting area in the recent years, which some scholars have paid attention to [18–35].

In this paper, we consider a three-species ecological model with an impulsive control strategy. The model can be described by the following differential equations:

$$\left\{\begin{array}{ll}\left.\begin{array}{l}\frac{\mathrm{d}x(t)}{\mathrm{d}t}=r_1x(t)(1-\frac{x(t)}{k_1})-\frac{a_1x(t)y(t)}{b_1+x(t)}-\frac{a_2x(t)z(t)}{b_2+x(t)}\\ \frac{\mathrm{d}y(t)}{\mathrm{d}t}=\frac{e_1a_1x(t)y(t)}{b_1+x(t)}-d_1y(t)z(t)-m_1y(t)\\ \frac{\mathrm{d}z(t)}{\mathrm{d}t}=\frac{a_2e_2x(t)z(t)}{b_2+x(t)}-d_2y(t)z(t)-m_2z(t)\end{array}\right\} & t\neq nT\\ \left.\begin{array}{l}\Delta x(t)=0\\ \Delta y(t)=0\\ \Delta z(t)=p\end{array}\right\} & t=nT\end{array}\right. \tag{1.1}$$

where $x(t), y(t), z(t)$ are the densities of one prey and two predators at time $t$, respectively, $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$, $\Delta z(t) = z(t^+) - z(t)$, $r_1$ is the intrinsic growth rate, $a_i (i = 1, 2)$ and $b_i (i = 1, 2)$ measure the efficience of the prey in evading a predator attack, $e_i$ $(i = 1, 2)$ denote the efficiency with which resources are converted to new consumers, $k_1$ is carrying capacity in the absence of predator, $d_i$ $(i = 1, 2)$ are competing parameters, $m_i$ $(i = 1, 2)$ are the mortality rates for the predator, $T$ is the period of the impulsive effect, $n \in N$, $N$ is the set of all non-negative integers, and $p > 0$ is the release amount of predator at $t = nT$. In order to get some conditions to guarantee species permanence, we will relaese a certain amount of predator $z$ at $t = nT$.

## 2. Mathematical analysis

Let $R_+ = [0, \infty)$, $R^3_+ = \{X \in R^3 \mid X \geq 0\}$. Denote by $f = (f_1, f_2, f_3)$ the map defined by the right hand sides of the first three equations of system (1.1). Let $V : R_+ \times R^3_+ \to R_+$. Then $V$ is said to belong to class $V_0$ if:

(1) $V$ is continuous in $(nT, (n+1)T] \times R^3_+$, and for each $X \in R^3_+$, $n \in N$, $\lim_{(t,y)\to(nT^+,X)} V(t, y) = V(nT^+, X)$ exists.

(2) $V$ is locally Lipschitzian in $X$.

**Definition 2.1** *Let $V \in V_0$. Then for $(t, x) \in (nT, (n + 1)T] \times R^3_+$, the upper right derivative* *of $V(t,X)$ with respect to the impulsive differential system (1.1) is defined as*

$$D^+V(t,X)=\lim_{h\to 0^+}\sup\frac{1}{h}[V(t+h,X+hf(t,x))-V(t,X)].$$

*The solution of system* (1.1) *is a piecewise continuous function $X:R_+\to R^3_+$, $X(t)$ is continuous on $(nT,(n+1)T]$, $n\in N$ and $X(nT^+)=\lim_{t\to nT^+}X(t)$ exists. The smoothness properties of $f$ guarantee the global existence and uniqueness of solution of system (1.1) (refering to [8–10] for details).*

**Definition 2.2** *System (1.1) is said to be permanent if there exists a compact $\Omega\subset \operatorname{int} R^3_+$ such that every solution $(x(t),y(t),z(t))$ of system (1.1) will eventually enter and remain in the region $\Omega$.*

The following lemma is obvious.

**Lemma 2.1** *Let $X(t)$ be a solution of system (1.1) with $X(0^+)\geq 0$. Then $X(t)\geq 0$ for all $t\geq 0$. And further $X(t)>0$, $t>0$ if $X(0^+)>0$.*

We will use an important comparison theorem on impulsive differential equation.

**Lemma 2.2** ([8]) *Suppose $V\in V_0$. Assume that*

$$\begin{cases} D^+V(t,X)\leq g(t,V(t,X)), & t\neq nT\\ V(t,X(t^+))\leq \psi_n(V(t,X)), & t=nT\end{cases}\tag{2.1}$$

*where $g: R_+\times R_+\to R$ is continuous in $(nT,(n+1)T]\times R_+$ and for $u\in R_+, n\in N$, $\lim_{(t,v)\to(nT^+,u)}g(t,v)=g(nT^+,u)$ exists, $\psi_n: R_+\to R_+$ is non-decreasing. Let $r(t)$ be maximal solution of the scalar impulsive differential equation*

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t}=g(t,u(t)), & t\neq nT\\ u(t^+)=\psi_n(u(t)), & t=nT\\ u(0^+)=u_0\end{cases}\tag{2.2}$$

*existing on $[0,\infty)$. Then $V(0^+,X_0)\leq u_0$, implies that $V(t,X(t))\leq r(t), t\geq 0$, where $X(t)$ is any solution of system (1.1).*

If the prey $x$ and the predator $y$ become extinct, the system (1.1) changes into the subsystem (2.3)

$$\begin{cases} \frac{\mathrm{d}z(t)}{\mathrm{d}t}=-m_2z(t), & t\neq nT,\\ z(t^+)=z(t)+p, & t=nT,\\ z(0^+)=z_0.\end{cases}\tag{2.3}$$

Clearly, if the initial value is $z^*(0^+)=\frac{p}{1-\exp(-m_2T)}>0$, for any $t\in(nT,(n+1)T]$, $n\in N$, the subsystem (2.3) has a positive periodic solution which can be described as follows

$$z^*(t)=\frac{p\exp(-m_2(t-nT))}{1-\exp(-m_2T)}.$$

If the initial value is $z_0=z(0^+)\geq 0$, for any $t\in(nT,(n+1)T]$, $n\in N$, the subsystem (2.3) has a general solution which can be described as follows

$$z(t) = (z(0^+) - \frac{p}{1 - \exp(-m_2 T)}) \exp(-m_2 t) + z^*(t).$$

It is obvious to get Lemma 2.3.

**Lemma 2.3** *For a positive periodic solution $z^*(t)$ of system (2.3) and a general solution $z(t)$ of system (2.3), there holds $\mid z(t) - z^*(t) \mid \to 0$, $t \to \infty$.*

Therefore, we obtain the complete expression for the prey-$x$-eradication and predator-$y$-eradication periodic solution $(0, 0, z^*(t))$ of system (1.1), where the prey-$x$-eradication means the prey $x$ becomes extinct and the predator-$y$-eradication means the predator $y$ becomes extinct.

Now, we study the stability of the prey-$x$-eradication and predator-$y$-eradication periodic solution

**Theorem 2.1** *Let $(x(t), y(t), z(t))$ be any solution of system (1.1). Then $(0, 0, z^*(t))$ is said to be locally asymptotically stable if*

$$r_1 T - \frac{a_2 p}{b_2 m_2} < 0.$$

**Proof** The local stability of periodic solution $(0, 0, z^*(t))$ may be determined by considering the behavior of small amplitude perturbation of the solution. Define

$$x(t) = u(t),\ \ y(t) = v(t),\ \ z(t) = w(t) + z^*(t). \tag{2.4}$$

Substituting (2.4) into (1.1) gives

$$\begin{cases} \left.\begin{array}{l} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = (r_1 - \frac{a_2 z^*(t)}{b_2} u(t) \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} = (-d_1 z^*(t) - m_1) v(t) \\ \frac{\mathrm{d}w(t)}{\mathrm{d}t} = \frac{a_2 z^*(t)}{b_2} u(t) - d_2 z^*(t) v(t) - m_2 w(t) \end{array}\right\} & t \neq nT \\ \left.\begin{array}{l} \Delta u(t) = 0 \\ \Delta v(t) = 0 \\ \Delta w(t) = 0 \end{array}\right\} & t = nT \end{cases} \tag{2.5}$$

Therefore, we have

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix}, \quad 0 \leq t < T$$

where $\Phi(t)$ satisfies

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \begin{pmatrix} r_1 - \frac{a_2 z^*(t)}{b_2} & 0 & 0 \\ 0 & -d_1 z^*(t) - m_1 & 0 \\ \frac{a_2 z^*(t)}{b_2} & -d_2 z^*(t) & -m_2 \end{pmatrix} \Phi(t)$$

and $\Phi(0) = I$, the identity matrix, and

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \\ w(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \\ w(nT) \end{pmatrix}.$$

The stability of the periodic solution $(0, 0, z^*(t))$ is determined by the eigenvalues of

$$\Theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(T)$$

with absolute value less than one. Then the periodic solution $(0, 0, z^*(t))$ is locally stable. All eigenvalues of $\Theta$ are

$$u_1 = \exp\Big(\int_0^T (r_1 - \frac{a_2 z^*(t)}{b_2})\mathrm{d}t\Big), \quad u_2 = \exp\Big(\int_0^T (-d_1 z^*(t) - m_1)\mathrm{d}t\Big) < 1,$$
$$u_3 = \exp(-m_2 T) < 1,$$

respectively. According to Floquet theory of impulsive differential equation, $(0, 0, z^*(t))$ is locally asymptotically stable if $| u_1 |< 1$, that is to say

$$r_1 T - \frac{a_2 p}{b_2 m_2} < 0.$$

This completes the proof. □

**Theorem 2.2** *There exists a constant $M > 0$, such that $x(t) \le M$, $y(t) \le M$, $z(t) \le M$ for each solution $X(t) = (x(t), y(t), z(t))$ of system (1.1) with all $t$ large enough.*

**Proof** Define $V(t, X(t))$ such that

$$V(t, X(t)) = x(t) + \frac{1}{e_1} y(t) + \frac{1}{e_2} z(t).$$

Then $V \in V_0$. We calculate the upper right derivative of $V(t, X)$ along a solution of system (1.1) and get the following impulsive differential equation

$$\begin{cases} D^+V(t) + LV(t) = (r_1 + L)x(t) - \frac{r_1}{k_1} x(t)^2 + \frac{L-m_1}{e_1} y(t) + \frac{L-m_2}{e_2} z(t) - \\ \qquad\qquad \frac{d_1}{e_1} y(t)z(t) - \frac{d_2}{e_2} y(t)z(t), & t \neq n, \\ V(t^+) = V(t) + p, & t = nT. \end{cases} \tag{2.6}$$

Let $0 < L < \min\{m_1, m_2\}$. Then $D^+V(t) + LV(t)$ is bounded. Select $L_1$ and $L_2$ such that

$$\begin{cases} D^+V(t) \le -L_1 V(t) + L_2, & t \neq nT, \\ V(t^+) = V(t) + p, & t = nT, \end{cases} \tag{2.7}$$

where $L_1$, $L_2$ are two positive constants.

According to Lemma 2.2, we have

$$V(t) \le (V(0^+) - \frac{L_2}{L_1}) \exp(-L_1 t) + \frac{p(1 - \exp(-nL_1 T))}{\exp(L_1 T) - 1} \exp(L_1 T) \exp(-L_1(t - nT)) + \frac{L_2}{L_1},$$

where $t \in (nT, (n + 1)T]$. Hence

$$\lim_{t\to\infty} V(t) \le \frac{L_2}{L_1} + \frac{p \exp(L_1 T)}{\exp(L_1 T) - 1}.$$

Therefore $V(t, X(t))$ is ultimately bounded, and we know that each positive solution of system is uniformly ultimately bounded. This completes the proof. □

**Theorem 2.3** *Let $(x(t), y(t), z(t))$ be any solution of system (1.1). Then $(0, 0, z^*(t))$ is said to be globally asymptotically stable if*

$$r_1T - \frac{a_2p}{b_2m_2} < 0$$

*and*

$$p > \frac{r_1m_2T}{a_2}(b_2 + M).$$

**Proof** By Theorem 2.1, we know that $(0, 0, z^*(t))$ is locally asymptotically stable. In the following, we shall prove its global attraction. Let

$$V(t) = e_1x(t) + y(t).$$

Then we get

$$V'|_{(1.1)} = e_1r_1x(t) - \frac{r_1e_1x^2(t)}{k_1} - \frac{e_1a_1x(t)y(t)}{b_1 + x(t)} - \frac{e_1a_2x(t)z(t)}{b_2 + x(t)} + \frac{a_1e_1x(t)y(t)}{b_1 + x(t)} - d_1y(t)z(t) - m_1y(t).$$

By Theorem (2.2), there exists a constant $M > 0$, such that $x(t) \leq M$, $y(t) \leq M$, $z(t) \leq M$ for each solution $X(t) = (x(t), y(t), z(t))$ of system (1.1) with all $t$ large enough.

Thus,

$$V'|_{(1.1)} \leq e_1r_1x(t) - \frac{e_1a_2x(t)z(t)}{b_2 + M} - d_1y(t)z(t) - m_1y(t).$$

By Lemmas 2.1 and 2.3, we know that there exist $t_1 > 0$, and $\epsilon > 0$ small enough, such that $z(t) \geq z^*(t) - \epsilon$, for all $t \geq t_1$. We have

$$z(t) \geq \frac{p\exp(-m_2T)}{1 - \exp(-m_2T)} - \epsilon, \quad \gamma_1 \triangleq \frac{p\exp(-m_2T)}{1 - \exp(-m_2T)} - \epsilon.$$

Then

$$V'|_{(1.1)} \leq (e_1r_1 - \frac{e_1a_2\gamma_1}{b_2 + M})x(t) + (-d_1\gamma_1 - m_1)y(t)$$

if $e_1r_1 - \frac{e_1a_2\gamma_1}{b_2+M} < 0$, that is to say:

$$p \geq \frac{r_1m_2T}{a_2}(b_2 + M).$$

Thus, for $t \geq t_1$, we have

$$V'|_{(1.1)} \leq (e_1r_1 - \frac{e_1a_2\gamma_1}{b_2 + M})x(t) + (-d_1\gamma_1 - m_1)y(t) < 0,$$

so $V(t) \to 0$, and $x(t) \to 0, y(t) \to 0$ as $t \to \infty$. Notice that the limit system of the system (1.1) is exactly system (2.3). Together with Lemma 2.3, we know that the prey-($x$) and predator-($y$) eradication periodic solution $(0, 0, z^*(t))$ is global attractor. The proof is completed. □

Then we investigate the permanence of the system (1.1).

**Theorem 2.4** *The system (1.1) is permanent if*

$$r_1T - \frac{a_2p}{b_2m_2} > 0$$

*and*

$$p \leq \frac{r_1 m_2 T}{a_2}(b_2 + M).$$

**Proof** Suppose $X(t) = (x(t), y(t), z(t))$ is any solution of the system (1.1) with $X(0) > 0$. From Theorem 2.2, we assume that $x(t) \leq M$, $y(t) \leq M$ and $z(t) \leq M$ with $t \geq 0$. From (2.3), we have $z(t) > z^*(t) - \epsilon$ for all $t$ large enough and some $z(t) \geq \frac{p\exp(-m_2T)}{1-\exp(-m_2T)} - \epsilon \triangleq \zeta_1$ for $t$ large enough. Thus we only need to find $\zeta_2$ and $\zeta_3$ such that $x(t) \geq \zeta_2$ and $y(t) \geq \zeta_3$ for $t$ large enough.

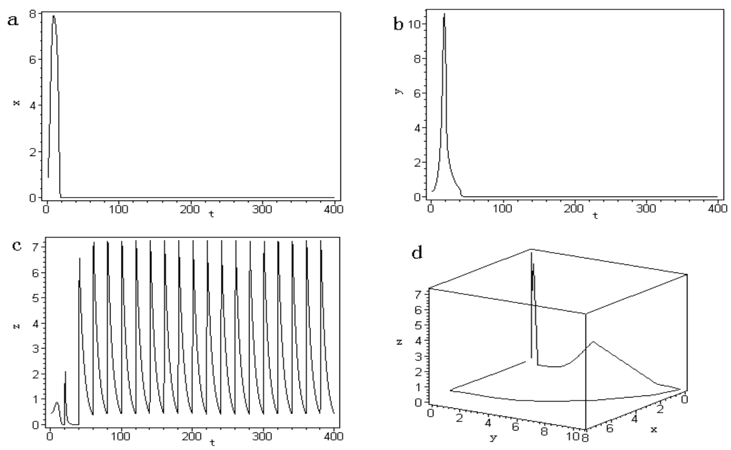

Figure 1 The dynamics of system (1.1) when $p >= 7.805052$, $(0, 0, z^*(t))$ is locally asymptotically stable, that is to say, the prey-($x$) and predator-($y$) go extinction: (a) time series of prey ($x$ population, $x(t) \to 0$, as $t \to \infty$ when $p = 8$, (b) time series of predator $y$ population, $y(t) \to 0$, as $t \to \infty$ when $p = 8$,(c) time series of predator $z$ population when $p = 8$,(d)phase portrait of system (1.1).

Let $\epsilon_1 > 0$ be small enough such that

$$\eta_1 \triangleq \exp\Big(\int_{nT}^{(n+1)T} \big([r_1(1 - \frac{M}{k_1}) - \frac{a_1 M}{b_1} - \frac{a_2(\upsilon_3^* + \epsilon_1)}{b_2}]\big)\mathrm{d}t\Big) > 1.$$

Now, we prove that there exists a constant $\zeta_2$, such that $x(t) \geq \zeta_2$ for $t$ large enough.

We will prove that there exists a $t_1 \in (0, \infty)$ such that $x(t) \geq \zeta_2$. Otherwise $x(t) < \zeta_2$ for all $t > 0$. From system (1.1), we can obtain that

$$\begin{cases} \frac{\mathrm{d}z(t)}{\mathrm{d}t} \leq (\frac{e_2a_2\zeta_2}{b_2} - m_2)z(t), & t \neq nT, \\ z(t^+) = z(t) + p, & t = nT, \\ z(0^+) = z_0. \end{cases} \tag{2.8}$$

Then we have $z(t) \leq \upsilon_3$ and $\upsilon_3 \to \upsilon_3^*$ $(t \to \infty)$, where $\upsilon_3$ is the solution of

$$\begin{cases} \frac{\mathrm{d}\upsilon_3(t)}{\mathrm{d}t} = (\frac{e_2a_2\zeta_2}{b_2} - m_2)\upsilon_3(t), & t \neq nT, \\ \upsilon_3(t^+) = \upsilon_3(t) + p, & t = nT, \\ \upsilon_3(0^+) = z_0 \end{cases} \tag{2.9}$$

and $\upsilon_3^* = \frac{p\exp(-(m_2 - \frac{e_2a_2\zeta_2}{b_2})(t-nT))}{1-\exp(-(m_2-\frac{e_2a_2\zeta_2}{b_2})T)}$, $t \in (nT, (n+1)T]$, $n \in N$. Therefore, there exists a $T_1 > 0$

such that

$$z(t) \le \upsilon_3(t) < \upsilon_3^* + \epsilon_1$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \ge x(t)[r_1(1-\frac{M}{k_1}) - \frac{a_1 M}{b_1} - \frac{a_2(\upsilon_3^*+\epsilon_1)}{b_2}]. \tag{2.10}$$

Let $N_1 \in N$ and $N_1 T \ge T_2 > T_1$. Integrating (2.10) on $(nT,(n+1)T), n \ge N_1$, we get

$$\begin{aligned} x((n+1)T) &\ge x(nT^+)\exp\Big(\int_{nT}^{(n+1)T}\big([r_1(1-\frac{M}{k_1}) - \frac{a_1 M}{b_1} - \frac{a_2(\upsilon_3^*+\epsilon_1)}{b_2}]\big)\mathrm{d}t\Big) \\ &= x(nT)\exp\Big(\int_{nT}^{(n+1)T}\big([r_1(1-\frac{M}{k_1}) - \frac{a_1 M}{b_1} - \frac{a_2(\upsilon_3^*+\epsilon_1)}{b_2}]\big)\mathrm{d}t\Big) \\ &= x(nT)\eta_1. \end{aligned} \tag{2.11}$$

Then $x((N_1+k)T) \ge x(N_1 T)\eta_1^k \to \infty$, $k \to \infty$, which is a contradiction to the boundedness of $x(t)$. Hence there exists a $t_1 > 0$ such as $x(t_1) \ge \zeta_2$

Secondly, if $x(t) \ge \zeta_2$ for all $t \ge t_1$, then our aim is obtained. Hence we only need to consider those solutions which leave the region $R = \{x(t) : x(t) < \zeta_2\}$ and reenter it again. Let $t^* = \inf_{t \ge t_1}\{x(t) < \zeta_2\}$. We have $x(t) \ge \zeta_2$, $t \in (t, t^*)$ and $t^* \in (n_1 T, (n_1+1)T), n_1 \in N$. It is easy to prove $x(t^*) = \zeta_2$ since $x(t)$ is continuous.

We claim that there must exist a $t_2 \in ((n_1+1)T, (n_1+1)T + \bar{T})$ such that $x(t_2) \ge \zeta_2$, otherwise $x(t) < \zeta_2, t \in ((n_1+1)T, (n_1+1)T+\bar{T}), \bar{T} = n_2 T + n_3 T$. Select $n_2, n_3 \in N$ such that

$$(n_2-1)T > \frac{\ln(\frac{\epsilon_1}{(M+p)})}{-(m_2 - \frac{e_2 a_2 \zeta_2}{b_2})}, \quad \exp(\gamma(n_2+1)T)\eta_1^{n_3} > 1.$$

Considering (2.9) with $\upsilon_3(t^{*+}) = z(t^{*+})$, we have

$$\upsilon_3(t) = (\upsilon_3((n_1+1)T^+) - \frac{p}{1-\exp(-(m_2 - \frac{e_2 a_2 \zeta_2}{b_2})T))})\exp(-(m_2 - \frac{e_2 a_2 \zeta_2}{b_2})t) + \upsilon_3^*(t)$$

for $t \in (nT, (n+1)T)$, $n_1 + 1 < n < n_1 + n_2 + n_3 + 1$, then

$$|\upsilon_3(t) - \upsilon_3^*(t)| < (M+p)\exp(-(m_2 - \frac{e_2 a_2 \zeta_2}{b_2})(t-(n_1+1)T)) < \epsilon_1$$

and

$$z(t) \le \upsilon_3(t) \le \upsilon_3^*(t) + \epsilon_1,$$

$(n_1+n_2+1)T \le t \le (n_1+1)T + \bar{T}$, which implies that (2.10) holds for $(n_1+n_2+1)T \le t \le (n_1+1)T+\bar{T}$. Integrating (2.10) on $((n_1+n_2+1)T, (n_1+1)T+\bar{T})$, we have

$$x((n_1+n_2+n_3+1)T) \ge x((n_1+n_2+1)T)\eta_1^{n_3}.$$

There are two possible cases for $t \in (t^*, (n_1+1)T)$.

**Case I** If $x(t) < \zeta_2$ for all $t \in (t^*, (n_1+1)T)$, then $x(t) < \zeta_2$ for all $t \in (t^*, (n_1+1+n_2)T)$. We have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \ge x(t)[r_1(1-\frac{\zeta_2}{k_1}) - \frac{a_1 M}{b_1}] = \gamma x(t). \tag{2.12}$$

Integrating (2.12) on $(t^*, (n_1+1+n_2)T)$ yields $x((n_1+n_2+1)T) \geq x(t^*)\exp(\gamma(n_2+1)T)$. Then $x((n_1+n_2+n_3+1)T) \geq \zeta_2 \exp(\gamma(n_2+1)T)\eta_1^{n_3} > \zeta_2$, which is a contradiction.

Let $t_3 = \inf_{t>t^*}\{x(t) \geq \zeta_2\}$. Then $x(t_3) = \zeta_2$ and (2.12) holds for $t \in [t^*, t_3)$. Then integrating (2.12) on $[t^*, t_3)$ yields

$$x(t) \geq x(t^*)\exp(\gamma(t-t^*)) \geq \zeta_2 \exp(\gamma(1+n_1+n_3)T) \triangleq \bar{\zeta_2}.$$

For $t > t_3$ the same arguments can be continued since $x(t_3) \geq \zeta_2$. Hence $x(t) \geq \bar{\zeta_2}$ for all $t > t_3$.

**Case II** There exists a $t_5 \in (t^*, (n_1+1)T]$ such that $x(t_5) \geq \zeta_2$. Let $t_4 = \inf_{t>t^*}\{x(t) \geq \zeta_2\}$. Then $x(t) < \zeta_2$ for $t \in [t^*, t_4)$ and $x(t_4) = \zeta_2$. For $t \in [t^*, t_4)$, (2.12) holds. Integrating (2.12) on $[t^*, t_4)$, we have $x(t) \geq x(t^*)\exp(\gamma(t-t^*)) > \bar{\zeta_2}$. This process can be continued since $x(t_4) \geq \zeta_2$. And we have $x(t) \geq \bar{\zeta_2}$for $t > t_4$. Thus in both cases, we conclude $x(t) \geq \bar{\zeta_2}$ for all $t \geq t_1$.

Similarly, we can prove $y(t) \geq \bar{\zeta_3}$ for all $t \geq t_2$.

Set $\Omega = \{(x, y, x) : x \geq \zeta_2, y \geq \zeta_3, z \geq \zeta_1, x+y+z \leq 3M\}$. Obviously, we know that the set $\Omega \in \text{int } R_+^3$ is global attractors. Every solution of system (1.1) will eventually enter and remain in region $\Omega$. Therefore, system (1.1) is permanent. The proof is completed. □

**Corollary 1** *The prey-x and predator-z of system (1.1) can coexist and the predator-y extinct if* $r_1T - \frac{a_2p}{b_2m_2} > 0$ *and* $p > \frac{r_1m_2T}{a_2}(b_2+M)$.

## 3. Numerical analysis

To study the dynamics of an impulsively controlled one-prey two-predator system. The solution of system (1.1) with initial conditions $(x_0 = 0.5, y_0 = 0.3, z_0 = 0.5)$ is obtained numerically for biologically feasible range of parametric value $r_1 = 0.8$, $k_1 = 10$, $m_1 = 0.1, m_2 = 0.15$, $d_1 = 0.15, d_2 = 0.2$, $a_1 = 0.3, a_2 = 0.24$, $e_1 = 0.8, e_2 = 0.7$, $b_1 = 0.5$, $b_2 = 1.5$,$T = 20$.

From Theorem 2.1 we know that the prey-$x$-eradication and predator-$y$-eradication periodic solution $(0, 0, z^*(t))$ is locally asymptotically stable provided that $p >= 7.805052$. A type of prey-$x$-eradication and predator-$y$-eradication periodic solution of system (1.1) is shown in Fig.1, where we may observe how the variable $z(t)$ oscillates in a stable cycle. In contrast, the the prey $x$ and predator $y$ rapidly decrease to zero when $p >= 7.805052$. When the value of $p$ is smaller than 7.805052, the prey-$x$-eradication and predator-$y$-eradication periodic solution will become unstable. It is possible that the prey and two predators can coexist on a stable positive periodic solution. When $p = 2.5$, the prey $x$ and the predator $z$ can coexist, but the predator $y$ finally becomes extinct, as shown in Figure 2. This result agrees well with Corollary 1. When $p = 0.5$ , three species can coexist. Figure 3 shows this result. From Figure 3, we can know that the prey and two predators finally coexist in a stable limit circle. In a word, the impulsive perturbation can control the dynamical behavior of this ecological system and the impulsive control method is very effective.

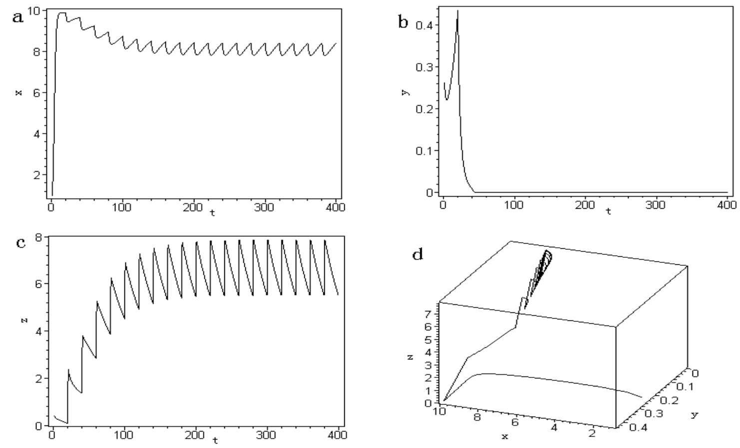

Figure 2 The dynamics of system (1.1) when $p = 2.5$: (a) time series of prey $x$ population $x(t)$, (b) time series of predator $y$ population, $y(t) \to 0$, as $t \to \infty$, (c) time series of predator $z$ population, (d) phase portrait of system (1.1).

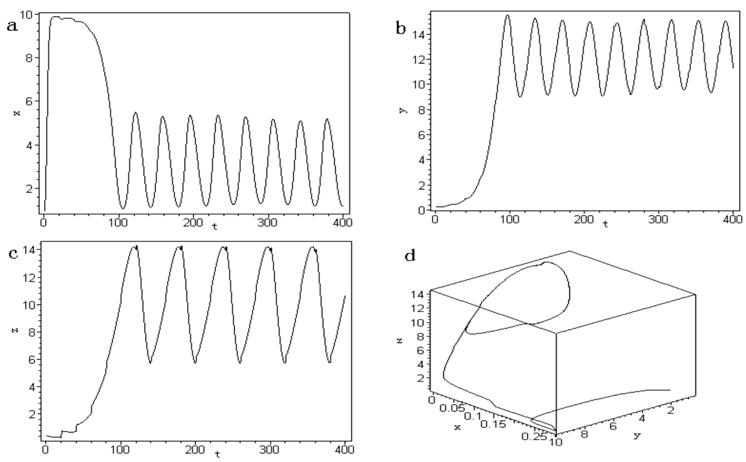

Figure 3 The dynamics of system (1.1) when $p = 0.5$: (a) time series of prey $x$ population, (b) time series of predator $y$ population, (c) time series of predator $z$ population, (d) phase portrait of system (1.1).

## 4. Conclusions and remarks

In this paper, the dynamic complexities of an ecological model with an impulsive control strategy are studied numerically and analytically. By using Floquet theorem and small amplitude perturbation skills, we have proved that the periodic solution $(0, 0, z^*(t))$ is globally asymptotically stable if $r_1T - \frac{a_2p}{b_2m_2} < 0$ and $p > \frac{r_1m_2T}{a_2}(b_2 + M)$ and the system (1.1) is permanent if $r_1T - \frac{a_2p}{b_2m_2} > 0$ and $p \leq \frac{r_1m_2T}{a_2}(b_2 + M)$. By choosing impulsive perturbation $p$ as a parameter, we have obtained time series diagrams. Time series diagrams have shown dynamical complexity for system (1.1). All these results show that the dynamical behaviors of system (1.1) are more complex under periodically impulsive perturbations.

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