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Sewing Connection of Step-Step Solution for Singularly Perturbed Problems

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Abstract In view of singularly perturbed problems with complex inner layer phenomenon, including contrast structures (step-step solution and spike-type solution), corner layer behavior and right-hand side discontinuity, we carry out the process with sewing connection. The presented method of sewing connection for singularly perturbed equations is based on the two points singularly perturbed simple boundary problems. By means of sewing orbit smoothness, we get the uniformly valid solution in the whole interval. It is easy to prove the existence of solutions and deal with the high dimensional singularly perturbed problems.

Keywords sewing connection; internal layer phenomenon; asymptotic expansion.

MR(2010) Subject Classification 34E20; 34B15

1. Introduction

The research for nonlinear singularly perturbed problems arises in many efficient methods such as boundary layer function theory [1, 5], differential inequality [2], asymptotic matching principle [3], etc. These methods have severe mathematics theory basis, in which not only the asymptotic solutions are constructed but also the estimates of residual terms are given. After entering the 21st century, the internal layer phenomenon of singular perturbation problems has become the focus of attention, including contrast structures (step-step solution and spike-type solution), corner layer behavior and right-hand side of discontinuity, etc. To solve these problems, even in the scalar case with differential inequality, it is very complex to prove the existence of solutions. What is more, if we deal with high-dimensional non-linear singularly perturbed problems, these methods become powerless.

The method of sewing connection for singularly perturbed equations presented in this paper is based on the two points singularly perturbed simple boundary problems, which are in allusion

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to complicated interior layers. By means of sewing orbit smoothness, we get the uniformly valid solution in the whole interval. Our treatments have at least two advantages: first, it is easy to prove the existence of solutions and carry out the estimate of residual terms; second, the sewing connection can deal with the high dimensional singularly perturbed problems.

The theoretical basis of sewing connection is the following Vasili'eva theorem.

Lemma 1 (Vasili'eva Theorem) *Consider Tikhnov system with $z \in R^M$, $y \in R^m$*

$$\mu\frac{\mathrm{d}z}{\mathrm{d}t} = F(z,y,t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = f(z,y,t), \quad 0 \le t \le 1, \tag{1}$$

$$az(0,\mu) = az^0, \quad bz(1,\mu) = bz^0, \quad y(0,\mu) = y^0, \tag{2}$$

where $a = \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 \\ 0 & E_{M-k} \end{pmatrix}$, $1 < k < M$, *E_k and E_{M-k} are unit matrices. Assume that all the following conditions are satisfied:*

[H_1] Vector functions F and f are smooth sufficiently on the domain;

[H_2] The eigenvalue $\bar\lambda_t(t)$ $(0 \le t \le 1)$ of matrix $\bar F_z(t) \equiv F_z(\bar z(t), \bar y(t), t)$ satisfies

$$\operatorname{Re}\bar\lambda_i(t) < 0, \quad i = 1, \ldots, k < M,$$

$$\operatorname{Re}\bar\lambda_i(t) > 0, \quad i = k+1, \ldots, M.$$

Then the solution $x(t,\mu) = (z(t,\mu), y(t,\mu))^{\mathrm{T}}$ of (1) and (2) exists, and has the residual terms estimation

$$\|x(t,\mu) - X_n(t,\mu)\| \le C\mu^{n+1},$$

where $X_n(t,\mu) = \sum_{k=0}^{n} \mu^k(\bar x_k(t) + \Pi_k x(\tau_0) + R_k x(\tau_1))$ is the n-order part sum with $\Pi_k x(\tau_0)$ and $R_k x(\tau_1)$, which are exponential decay as $\tau_0 \to \infty (\tau_1 \to -\infty)$.

In allusion to step-step solution, in this paper, we will introduce how to construct uniformly valid asymptotic solutions by using sewing connection.

We discuss the semi-linear boundary value problem as follows:

$$\mu^2 y'' = F(y,t), \quad 0 \le t \le 1, \quad 0 < \mu \le 1, \tag{3}$$

$$y(0,\mu) = y^0, \quad y(1,\mu) = y^1. \tag{4}$$

First we give several hypotheses:

[A_1] (smooth condition) $F(y,t)$ is smooth sufficiently in $D = \{(y,t) : |y| \le l, 0 \le t \le 1\}$, where l is a given real number;

[A_2] (isolated-solution condition) The degenerate equation $F(\bar y(t), t) = 0$ has three disjoint real roots $\bar y(t) = \varphi_i(t)$ $(i = 1,2,3)$ satisfying $\varphi_1(t) < \varphi_2(t) < \varphi_3(t)$;

[A_3] (stable condition) $F_y(\varphi_i(t), t) > 0 (i = 1,3), F_y(\varphi_2(t), t) < 0$.

2. The form of the construction of asymptotic solutions

The solution $y(t,\mu)$ has the contrast structure. Consider the left problem and the right problem which joint smoothly at t_* (t_* is called transfer point).

The left problem $0 \le t \le t_*$:

$$\mu^2 y''^{(-)} = F(y^{(-)}, t), \quad y^{(-)}(0,\mu) = y^0, \quad y^{(-)}(t_*,\mu) = \varphi_2(t_*). \tag{5}$$

The right problem $t_* \le t \le 1$:

$$\mu^2 y''^{(+)} = F(y^{(+)}, t), \quad y^{(+)}(t_*,\mu) = \varphi_2(t_*), \quad y^{(+)}(1,\mu) = y^1, \tag{6}$$

where $0 < t_* < 1$ is a parameter. From (5)–(6), we know that $y^{(\mp)}(t,\mu)$ is continuous at $t = t_*$. In order to make $y^{(\mp)}(t,\mu)$ joint smoothly at $t = t_*$, the following equation must be satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t} y^{(-)}(t_*,\mu) = \frac{\mathrm{d}}{\mathrm{d}t} y^{(+)}(t_*,\mu). \tag{7}$$

Suppose that the formal asymptotic solutions of (3) and (4) having the contrast structures can be represented:

$$y(t,\mu) = \begin{cases} \sum\limits_{k=0}^{\infty} \mu^k (\bar{y}_k^{(-)}(t) + \Pi_k y(\tau_0) + Q_k^{(-)} y(\tau)), & 0 \le t \le t_*, \\ \sum\limits_{k=0}^{\infty} \mu^k (\bar{y}_k^{(+)}(t) + Q_k^{(+)} y(\tau) + R_k y(\tau_1)), & t_* \le t \le 1, \end{cases} \tag{8}$$

where $\tau_0 = \mu^{-1} t > 0$, $\tau = \mu^{-1}(t - t_*)$, $\tau_1 = \mu^{-1}(t-1) < 0$, $\bar{y}_k^{(\mp)}(t)$ $(k \ge 0)$ are the coefficients of regular series, $\Pi_k y(\tau_0)$ $(k \ge 0)$ are the coefficients of boundary layer series at $t = 0$, $Q_k^{(\mp)} y(\tau)$ $(k \ge 0)$ are the coefficients of interior layer series at $t = t_*$, and $R_k y(\tau_1)$ $(k \ge 0)$ are the coefficients of boundary layer series at $t = 1$.

Partially, t_* can be expanded in the power series of μ, that is,

$$t_* = t_0 + \mu t_1 + \cdots + \mu^k t_k + \cdots. \tag{9}$$

According to boundary layer function theory, we put formal asymptotic solution (8) into (5) and (6), and separate equations by measures t, τ_0, τ, τ_1, then compare the same order of μ, thus each term's coefficients of equation and boundary value are obtained. At the same time in accordance with requirements of sewing connection, we substitute (8) into (7), and get a series of relation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Q_0^{(-)} y(0) = \frac{\mathrm{d}}{\mathrm{d}\tau} Q_0^{(+)} y(0), \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Q_1^{(-)} y(0) + \varphi_1'(t_0) = \frac{\mathrm{d}}{\mathrm{d}\tau} Q_1^{(+)} y(0) + \varphi_3'(t_0). \tag{11}$$

Step 1. Write the equation for determining $\bar{y}_0(t)$:

$$F(\bar{y}_0(t), t) = 0. \tag{12}$$

Eq.(10) is the degenerate equation. In view of the properties of step-step solution and condition $[A_2]$, we have $\bar{y}_0 = \varphi_1(t)$ $(0 \le t \le t^0)$, $\bar{y}_0 = \varphi_3(t)$ $(t_0 \le t \le 1)$. Here we only discuss the step-step solution from $\varphi_1(t)$ to $\varphi_3(t)$. Similarly, we can consider the step-step solution from $\varphi_3(t)$ to $\varphi_1(t)$.

Step 2. Write the equation for determining $\bar{y}_k(t)$ $(k \ge 1)$:

$$F_y(\bar{y}_0(t), t)\bar{y}_k = g_{k-1}, \quad k \ge 1, \tag{13}$$

where $g_{k-1}(k \geq 1)$ are the known functions which depend only on $\bar{y}_j$ $(0 \leq j \leq k-1)$. By virtue of [A$_3$], $\bar{F}_y^{-1}$ are known to exist, and we get $\bar{y}_k = \bar{F}_y^{-1} g_{k-1}$.

Step 3. Write the equation and boundary value to determine $Q_0^{(\mp)}y(\tau)$:

$$\frac{\mathrm{d}^2 Q_0^{(\mp)}y}{\mathrm{d}\tau^2} = F(\varphi_{1,3}(t_0) + Q_0^{(\mp)}y, t_0), \tag{14}$$

$$Q_0^{(\mp)}y(0) = \varphi_2(t_0) - \varphi_{1,3}(t_0), \quad Q_0^{\mp)}y(\mp\infty) = 0. \tag{15}$$

The equations (14) and (15) are equivalent to the following problem:

$$\frac{\mathrm{d}z^{(\mp)}}{\mathrm{d}\tau} = F(\tilde{y}^{(\mp)}, t_0), \quad \frac{\mathrm{d}\tilde{y}^{(\mp)}}{\mathrm{d}\tau} = z^{(\mp)}, \tag{16}$$

$$\tilde{y}^{(\mp)}(0) = \varphi_2(t_0), \tilde{y}^{(-)}(-\infty) = \varphi_1(t_0), \tilde{y}^{(+)}(+\infty) = \varphi_3(t_0), \tag{17}$$

where $\tilde{y}^{(\mp)} = \varphi_{1,3}(t_0) + Q_0^{(\mp)}y$. Integrating (16), we get a first integral passing through $M_{1,3}$ as

$$[z^{(\mp)}(\tau)]^2 = 2\int_{\varphi_{1,3}(t_0)}^{\tilde{y}^{(\mp)}(\tau)} F(y, t_0)\mathrm{d}y. \tag{18}$$

Hence, $Q_0^{(-)}y(\tau)$ and $Q_0^{(+)}y(\tau)$ are the solutions of the following Cauchy problems

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}Q_0^{(-)}y = \Big[2\displaystyle\int_{\varphi_{1,3}(t_0)}^{\varphi_{1,3}(t_0)+Q_0^{(-)}y(\tau)} F(y, t_0)\mathrm{d}y\Big]^{\frac{1}{2}} = G(Q_0^{(-)}y, t_0), \\ Q_0^{(-)}y(0) = \varphi_2(t_0) - \varphi_1(t_0), \quad Q_0^{(-)}y(-\infty) = 0, \end{cases} \tag{19}$$

and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}Q_0^{(+)}y = -G(Q_0^{(+)}y, t_0), \\ Q_0^{(+)}y(0) = \varphi_2(t_0) - \varphi_3(t_0), \quad Q_0^{(+)}y(\infty) = 0, \end{cases} \tag{20}$$

respectively. Substituting (18) into sewing joint condition (10) gives

$$\int_{\varphi_1(t_0)}^{\varphi_2(t_0)} F(y, t_0)\mathrm{d}y = \int_{\varphi_3(t_0)}^{\varphi_2(t_0)} F(y, t_0)\mathrm{d}y,$$

that is,

$$H(t_0) \equiv \int_{\varphi_1(t_0)}^{\varphi_3(t_0)} F(y, t_0)\mathrm{d}y = 0, \tag{21}$$

which is the equation for finding t_0.

[A$_4$] Suppose that Eq.(21) is solvable for t_0 $(0 < t_0 < 1)$, and $\frac{\mathrm{d}}{\mathrm{d}t}H(t_0) \neq 0$.

After obtaining t_0, we consider the initial values of (19) and (20), which intersect the heteroclinic orbits, therefore, $Q_0^{(\mp)}y(\tau)$ exist. By using [A$_3$] and L'Hospital rule, we can prove that $Q_0^{(\mp)}y(\tau)$ are exponential decay, that is,

$$|Q_0^{(\mp)}y(\tau)| \leq Ce^{-\kappa_0|\tau|}, \quad \tau \in R.$$

Step 4. Write the equation and boundary value for determining $Q_k^{(\mp)}y(\tau)(k \geq 1)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}Q_k^{(\mp)}y = \tilde{F}_y Q_k^{(\mp)}y + \tilde{h}_k(\tau), \tag{22}$$

$$Q_k^{(\mp)}y(0) = \bar{y}_{0t}^{(\mp)}(t_0)t_k + q_k^{(\mp)}(t_0, \ldots, t_{k-1}), \quad Q_k^{(\mp)}y(\mp\infty) = 0, \tag{23}$$

where $\tilde{F}_y = F_y(\varphi_{1,3}(t_0) + Q_0^{(\mp)}y, t_0)$, $\tilde{h}_k$ is a known function depending only on $Q_j^{(\mp)}y, t_j$ $(0 \leq j \leq k-1)$.

The solutions of linear boundary value problems (22) and (23) can be expressed by square formula:

$$Q_k^{(\mp)}y(\tau) = (\bar{y}_{0t}^{(\mp)}(t_0)t_k + q^{(\mp)})\frac{\varphi(\tau)}{\varphi(0)} + \varphi(\tau)\int_0^{\tau} \varphi^{-2}(\eta)\int_{\mp\infty}^{\eta} \varphi(\sigma)\tilde{h}_k(\sigma)\mathrm{d}\sigma\mathrm{d}\eta,$$

where $\varphi(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}Q_0^{(\mp)}y(\tau)$. It can be proved that $Q_0^{(\mp)}y(\tau)$ are exponential decay, as $\tau \to \mp\infty$, that is,

$$|Q_k^{(\mp)}y(\tau)| \leq Ce^{-\kappa_k|\tau|}, \quad \tau \in R.$$

By employing the relationship (11) of sewing connection, the equation to determine t_k is obtained as

$$H'(t_0)t_k = P_k, \tag{24}$$

where P_k are known constants.

Step 5. Write the equation and boundary value to determine $\Pi_0 y(\tau_0)$:

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}\Pi_0 y = F(\varphi_1(0) + \Pi_0 y, 0), \\ \Pi_0 y(0) = y^0 - \varphi_1(0), \quad \Pi_0 y(\infty) = 0. \end{cases} \tag{25}$$

The equation (25) is equivalent to the following system:

$$\frac{\mathrm{d}\hat{z}}{\mathrm{d}\tau_0} = F(\hat{y}, 0), \quad \frac{\mathrm{d}\hat{y}}{\mathrm{d}\tau_0} = \hat{z}, \tag{26}$$

$$\hat{y}(0) = y^0, \quad \hat{y}(\infty) = \varphi_1(0), \tag{27}$$

where $\hat{y} = \Pi_0 y(\tau_0) + \varphi_1(0)$.

On the phase plane, $M^0(\varphi_1(0), 0)$ is a saddle. Separating the orbit passing through M^0, one can write the saddle of M^0 as

$$[\hat{z}(\tau_0)]^2 = 2\int_{\varphi_1(0)}^{\hat{y}(\tau_0)} F(y, 0)\mathrm{d}y.$$

As $\tau_0 \to \infty$, the separating orbit passing through M^0 is

$$W^u(M^0) : \hat{z} = -\Big[2\int_{\varphi_1(0)}^{\hat{y}} F(y, 0)\mathrm{d}y\Big]^{\frac{1}{2}}.$$

In order to ensure that (26) and (27) have solutions, we require that

[A$_5$] $\{\hat{y}(0) = y^0\} \cap W^u(M^0) \neq \emptyset$.

Thus, the solution $\Pi_0 y(\tau_0)$ of (25) is gained, and we have

$$|\Pi_0 y(\tau_0)| \leq Ce^{-\kappa_0\tau_0}, \quad \tau_0 \geq 0.$$

Step 6. Write the equation and boundary value to determine $\Pi_k y(\tau_0)$ $(k \geq 1)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau_0^2}\Pi_k y = \hat{F}_y \Pi_k y + \hat{h}_k(\tau_0), \tag{28}$$

$$\Pi_k y(0) = -\bar{y}_k(0), \quad \Pi_k y(\infty) = 0, \tag{29}$$

where $\hat{F}_y = F_y(\varphi_1(0) + \Pi_0 y, 0)$, $\hat{h}_k$ is a known function depending only on $\Pi_j y (0 \leq j \leq k-1)$. By utilizing the square formula, the solutions of (28) and (29) can be obtained from

$$\Pi_k y(\tau_0) = -\bar{y}_k(0)\frac{\hat{\varphi}(\tau_0)}{\hat{\varphi}(0)} + \int_0^{\tau_0} \hat{\varphi}^{-2}(\eta) \int_\infty^\eta \hat{\varphi}(\sigma)\hat{h}_k(\sigma)\mathrm{d}\sigma\mathrm{d}\eta, \tag{30}$$

where $\Pi_k y(\tau_0)$ is exponential decay, as $\tau_0 \to +\infty$, that is,

$$|\Pi_k y(\tau_0)| \leq Ce^{-\kappa_k \tau_0}, \quad \tau_0 \geq 0.$$

Step 7. Similarly, we can also discuss $R_k y(\tau_1)$ $(k \geq 0)$.

[A$_6$] Suppose that $\{\check{y}(0) = y^1\} \cap W^s(M^1) \neq \emptyset$, where $\check{y}(0) = R_0 y(0) + \varphi_3(1)$, $W^s(M^1)$ is a saddle separating orbit which passes through $M^1(\varphi_3(1), 0)$.

Under the condition $[A_6]$, $R_0 y(\tau_1)$ exists, $R_k y(\tau_1)(k \geq 1)$ have the same expressions as (30), and the estimates of exponent satisfy

$$|R_k y(\tau_1)| \leq Ce^{\kappa_k \tau_1}, \quad \tau_1 \leq 0.$$

Now, the formal asymptotic solutions of (8) have been constructed completely.

3. The existence of step-step solution

In [4], the authors have ever used differential inequality to prove the existence of step-step solution at great length. In this section, we will apply the sewing connection to determine the transfer point t_*, and complete the proof of the existence of step-step solutions. Namely, using the left problem (5), the right problem (6) and the existence of solution for any parameter $0 < t_* < 1$, we get the asymptotic solution. The step-step solutions of problems (3)-(4) are sewed smoothly, and asymptotic expansion is obtained.

We write the zeroth asymptotic expansions of the left problem (5) and the right problem (6) respectively as

$$y(t,\mu) = \begin{cases} \varphi_1(t) + \Pi_0 y(\tau_0) + Q_0^{(-)} y(\tau) + O(\mu), & 0 \leq t \leq t_*, \\ \varphi_3(t) + Q_0^{(+)} y(\tau) + R_0 y(\tau_1) + O(\mu), & t_* \leq t \leq 1, \end{cases} \tag{31}$$

and

$$z(t,\mu) = \begin{cases} \Pi_0 z(\tau_0) + Q_0^{(-)} z(\tau) + O(\mu), & 0 \leq t \leq t_*, \\ Q_0^{(+)} z(\tau) + R_0 z(\tau_1) + O(\mu), & t_* \leq t \leq 1. \end{cases} \tag{32}$$

Here, we do not expand the parameter t_*.

According to the boundary values of (3) and (4), we see

$$y^{(-)}(t_*,\mu) = y^{(+)}(t_*,\mu), \quad t_* \in (0,1), \tag{33}$$

which implies $y(t,\mu)$ is continuous at $t = t_*$. Therefore, t_* can be determined in this way provided that derivatives of $y(t,\mu)$ are equal at $t = t_*$, that is,

$$z^{(-)}(t_*,\mu) = z^{(+)}(t_*,\mu). \tag{34}$$

For this purpose, we introduce difference function $\Delta(t_*)$:

$$\Delta(t_*) = z^{(-)}(t_*,\mu) - z^{(+)}(t_*,\mu) = [Q_0^{(-)} z(0)]^2 - [Q_0^{(+)} z(0)]^2 + O(\mu), \tag{35}$$

where we have considered that $\Pi_0 z(\tau_0), R_0 z(\tau_1)$ are exponentially small in the neighborhood of the point $t = t_*$. We may consider that $O(\mu) = C\mu$, where C is a real number. The approach of solving $Q_0^{(\mp)} z(0)$ in (35) is similar to those used in Section 2. Consequently, changing t_0 of (14)–(21) into t_* yields

$$\Delta(t_*) = H(t_*) + O(\mu) = H(t_0) + \frac{\mathrm{d}}{\mathrm{d}t} H(t_0)(t_* - t_0) + O((t_* - t_0)^2) + O(\mu), \tag{36}$$

where t_0 is known by (21). Let $t_* = t_0 \pm k\mu$, and put it into (36). We obtain

$$\Delta(t_0 \pm k\mu) = \pm k\mu \frac{\mathrm{d}}{\mathrm{d}t} H(t_0) + O(\mu). \tag{37}$$

Let k in (36) be sufficiently large, and select μ sufficiently small, such that the right-hand side of (36) cannot be of the same symbol. By virtue of intermediate value theorem, there exists $\bar{t}_* \in (t_0 - k\mu, t_0 + k\mu)$ such that $\Delta(\bar{t}_*) = 0$. We can show that (34) holds, and $\bar{t}_* = t_0 + O(\mu)$.

We formulate the results in the following theorem.

Theorem 1 (transfer Limit Theorem) *Suppose that the conditions [A_1]–[A_6] are satisfied, then there exist transfer point $\bar{t}_*$ and step-step solution $y(t, \mu)$ for Eqs. (3)–(4), and the following limiting process*

$$\lim_{\mu \to 0} y(t, \mu) = \begin{cases} \varphi_1(t), & 0 \le t \le \bar{t}_*, \\ \varphi_3(t), & \bar{t}_* \le t \le 1 \end{cases}$$

holds.

If we put $t_* = t_0 + O(\mu)$ into $Q_0^{(\mp)} y(\tau)$, then the order of (31)–(32) is not $O(\mu)$. To get the uniformly valid zeroth asymptotic solution, we need to expand t_* to the first approximation, namely, $t_* = t_0 + \mu t_1 + O(\mu^2)$, where t_1 is determined by (24).

Theorem 2 *Suppose that the conditions [A_1]–[A_6] are satisfied, then Eqs. (3)–(4) have the zeroth asymptotic expression:*

$$y(t, \mu) = \begin{cases} \varphi_1(t) + \Pi_0 y(\tau_0) + Q_0^{(-)} y(\tau) + O(\mu), & 0 \le t \le \bar{t}_*, \\ \varphi_3(t) + Q_0^{(+)} y(\tau) + R_0 y(\tau_1) + O(\mu), & \bar{t}_* \le t \le 1. \end{cases}$$

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