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Inverse Semigroups of Matrices

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Abstract We discuss some fundamental properties of inverse semigroups of matrices, and prove that the idempotents of such a semigroup constitute a subsemilattice of a finite Boolean lattice, and that the inverse semigroups of some matrices with the same rank are groups. At last, we determine completely the construction of the inverse semigroups of some 2×2 matrices: such a semigroup is isomorphic to a linear group of dimension 2 or a null-adjoined group, or is a finite semilattice of Abelian linear groups of finite dimension, or satisfies some other properties. The necessary and sufficient conditions are given that the sets consisting of some 2×2 matrices become inverse semigroups.

Keywords matrix semigroup; inverse semigroup; Green's relation; Clifford semigroup; semilattice.

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1. Introduction

Throughout the paper, F is an arbitrary field. Let $M_n(F)$ and $\mathrm{GL}(n, F)$ be the sets of all matrices and all invertible matrices of $n \times n$ over F, respectively. A matrix semigroup is defined as a semigroup whose set is a subset of $M_n(F)$ and whose composition is the usual multiplication of matrices. If a matrix semigroup is a group, we call it a matrix group. For example, $M_n(F)$ and $\mathrm{GL}(n, F)$ are both matrix semigroups, and the latter is even a matrix group. Evidently, any matrix semigroup must be a subsemigroup of $M_n(F)$. By a linear group, we mean a subgroup of $\mathrm{GL}(n, F)$. Note that in general a matrix group is different from a linear group. We will prove later that a matrix group must be isomorphic to some linear group. For example, $\{\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} : u \in F\}$ is a group of matrices, which is not a linear group of dimension 2, but is isomorphic to $\mathrm{GL}(1, F)$, the linear group of dimension 1.

The domestic scholars have obtained many good results in regular semigroups[1−3], but in China, few people study the matrix semigroups. Although the international research on the matrix semigroups is very active[4−6], but it is not so on the inverse semigroups of matrices. In fact, many researches were about some other special classes of matrix semigroups. For example, [6] studied the nonnegative matrix semigroups. We have determined the structure of completely

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simple matrix semigroups in [7], and made more thorough research on the compact abelian semigroups of matrices in [8]. It is well known that the inverse semigroups are an important class of semigroups and have been studied by many researchers[9–11]. A very natural and important problem is as follows: when does a matrix semigroup become an inverse semigroup? We will reply this question in this paper. If a matrix semigroup is an inverse semigroup, then we call it an inverse semigroup of matrices. So the subject of this paper is the inverse semigroups of matrices.

As we know, dealing with the problems of matrices is a complex thing, so sometimes we first need to treat the case of smaller dimensions. For example, in [12], the homomorphisms were studied of linear groups of dimension 2. In this paper, after discussing some basic properties of matrix semigroups and of inverse semigroups of matrices, respectively, we further determine the structures of the inverse semigroups of matrices.

For the general knowledge of semigroup theory, one can see [9–11], [13]; for the theory of matrix semigroups, we refer to [4] and [5]; and the knowledge of linear algebra and of matrix theory used in this paper is basic and so we do not refer to any references.

2. Some fundamental properties of matrix semigroups

We say that a subset M of a matrix semigroup S is monoranked if all matrices of M have the same rank. Let $T = \mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$ stand for Green's relations as in [9] and [13]. By T_a, we denote the T-class of an element a of S. The notions of bisimple semigroups and 0-bisimple semigroups can be found in [9, I.6.5]. The rank of a matrix a is denoted by $r(a)$. Now we shall present some fundamental properties of matrix semigroups without proofs.

Lemma 2.1 *Let a and b be two elements of a matrix semigroup S such that $aba = a$. If there exist invertible matrix P and Q such that $a = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q$, with I_r the identity matrix of $r \times r$, then there exist three matrices C, D, F such that $b = Q^{-1}\begin{pmatrix} I_r & C \\ D & F \end{pmatrix}P^{-1}$.*

Corollary 2.2 *Let a and b be two elements of a matrix semigroup S such that $aba = a$ and $bab = b$. If there exist two invertible matrices P and Q such that $a = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q$, with I_r the identity matrix of $r \times r$, then there exist matrices C and D such that $b = Q^{-1}\begin{pmatrix} I_r & C \\ D & DC \end{pmatrix}P^{-1}$. In particular, $r(a) = n$ if and only if b is the usual inverse of matrix a.*

Lemma 2.3 *Suppose that $S \subseteq M_n(F)$ and e is an idempotent of S and $r(e) = r$. Then there exists $P \in GL(n, F)$ such that $P^{-1}eP = a = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ with I_r the identity matrix of $r \times r$.*

Lemma 2.4 *A matrix group is monoranked and isomorphic to a linear group with dimension equal to the rank of an arbitrary matrix of this matrix group.*

Lemma 2.5 *An $\mathcal{H}$-class of an idempotent of a matrix semigroup is isomorphic to a linear group with dimension equal to the rank of this idempotent.*

Lemma 2.6 *Suppose S is a matrix semigroup and $a, b \in S^1$. Then the following statements are valid: (1) $a\mathcal{L}b$ implies that the row vectors of a are equivalent to those of b; (2) $a\mathcal{R}b$ implies that the column vectors of a are equivalent to those of b; (3) $a\mathcal{H}b$ implies that the row vectors of a are equivalent to those of b, and the column vectors of a are also equivalent to those of b.*

Corollary 2.7 *Let $T = \mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}, \mathcal{H}$. Each T-class of a matrix semigroup is all monoranked.*

Corollary 2.8 *Any simple [bisimple] matrix semigroup is monoranked; any 0-simple [0-bisimple] matrix semigroup except 0 is also monoranked.*

3. Some fundamental properties of inverse semigroups of matrices

In this section, we discuss some fundamental properties of inverse semigroups of $n \times n$ matrices with n an arbitrary natural number, which is the basis of analyzing the structure of inverse semigroups of 2×2 matrices in the next section. Theorem 3.1 is a key to studying the inverse semigroups of matrices. It states that all idempotents of an inverse semigroups of matrices constitute a subsemilattice of a finite Boolean lattice. A Boolean lattice means a contributive lattice with complements. For a given finite set $A = \{a_1, \ldots, a_n\}$, let $\rho(A)$ denote the set of all subsets of A ordered by $\subseteq$, the usual inclusion relation. Then $(\rho(A), \subseteq)$ is a finite Boolean lattice. As we know, any finite Boolean lattice with n atoms is isomorphic to $(\rho(A), \subseteq)$. If $X \in \rho(A)$, we may assume $X = \{a_{i_1}, \ldots, a_{i_r}\}$ with $1 \leq i_1 \leq \cdots \leq i_r \leq n$. Thus we obtain an $n-$digit binary number $\delta = \delta_1 \ldots \delta_n$, where $\delta_j = \begin{cases} 1 & \text{if } j = i_k (\exists k) \\ 0 & \text{otherwise} \end{cases}$. We may identify δ with X.

Theorem 3.1 *All idempotents of an inverse semigroup of some $n \times n$ matrices constitute a subsemilattice of the finite Boolean lattice $\rho(A)$ with n atoms.*

Proof Suppose S is an inverse semigroup of $n \times n$ matrices, and E is the set of all idempotents of S. According to [9, II.1.2], E is a semilattice. For all $e \in E$, by Lemma 2.3, e is diagonalizable. Furthermore, in light of the knowledge of the linear algebra, all elements of E can be simultaneously diagonalizable, i.e., there exists $P \in \mathrm{GL}(n, F)$ such that $\forall e \in E$, $P^{-1}eP$ is a diagonal matrix and each number on the diagonal is 0 or 1. Assume $P^{-1}eP = \mathrm{diag}(\delta_1 \cdots \delta_n)$ and $\delta = \delta_1 \cdots \delta_n$. Then e can be identified with the binary number δ, that is, e can be viewed as an element of $\rho(A)$, the finite Boolean lattice with n atoms. Since E is closed under the multiplication of the matrices, one may view it as a subsemilattice of $\rho(A)$, and the proof is completed. □

Corollary 3.2 *A monoranked inverse semigroup of matrices is a group.*

Proof Let S be a monoranked inverse semigroup of $n \times n$ matrices with the idempotents E. If $|E| \geq 2$, then there exist $e, f \in E$ such that $e \neq f$. In light of the proof of Theorem 3.1, e and f are in correspondence with two distinguished elements δ and γ of $\rho(A)$, respectively. If

$\delta < \gamma$, then the number of 1 contained in γ is more than that in δ, thus $r(e) < r(f)$, which is a contradiction to the assumption that S is monoranked. Analogously, the contradiction may be derived from the assumption that $\gamma < \delta$. So δ, γ are not comparable in $\rho(A)$. Let $\gamma\delta = \eta$ and $ef = g$. Then $\eta < \delta$, $g \in E$, and g is in the correspondence with η. Since $\eta < \delta$, we get $r(g) < r(e)$, a contradiction again. So we conclude that $|E| = 1$. Now using [9, II.2.10], we obtain that S is a group as desired. □

Lemma 3.3 *If S is an inverse semigroup of matrices and $a, b \in S$, then the following statements are true: (1)$r(a) = r(a^{-1}) = r(aa^{-1}) = r(a^{-1}a)$; (2) $a\mathcal{L}b \Leftrightarrow a^{-1}\mathcal{R}b^{-1}$; (3) $a\mathcal{R}b \Leftrightarrow a^{-1}\mathcal{L}b^{-1}$.*

Lemma 3.4 *An inverse semigroup of matrices is a finite chain of linear groups if its idempotent set is a chain.*

Proof Let S be an inverse semigroup of $n \times n$ matrices with idempotents being a chain. Then by Theorem 3.1, E is a finite chain. In view of [9, XI.5.1], S is a finite chain of matrix groups. But by Lemma 2.4, these matrix groups are all linear groups and the proof is completed. □

Of course, the matrix semigroup in Lemma 3.4 is actually a Clifford semigroup, that is, it is a semilattice of groups[9]. We shall discuss further the Clifford semigroup of matrices below.

Lemma 3.5 *If S is a Clifford semigroup of some $n \times n$ matrices with a matrix of rank r, then there exists an invertible matrix P so that for all $a \in S$, there exist an $r \times r$ matrix U and an $(n-r) \times (n-r)$ matrix Z such that $P^{-1}aP = a = \begin{pmatrix} U & 0 \\ 0 & Z \end{pmatrix}$.*

Proof Suppose that S is a Clifford semigroup and $S \subseteq M_n(F)$. Then S is an inverse semigroup. We denote its idempotents by E. If S contains a matrix b of rank r, then $bb^{-1} = e \in E$. By Lemma 3.3, $r(e) = r(bb^{-1}) = r(b) = r$. According to Lemma 2.3, there exists $P \in \mathrm{GL}(n, F)$ so that $P^{-1}eP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. For any $a \in S$, set $P^{-1}aP = c$. Since $P^{-1}SP$ is isomorphic to S and S is a Clifford semigroup, $P^{-1}SP$ is also a Clifford semigroup. In light of [9, II.2.6], in a Clifford semigroup, any element and any idempotent are commutative, so $ce = ec$. Assume $c = \begin{pmatrix} U & X \\ Y & Z \end{pmatrix}$ with $U \in M_r(F)$. Then $\begin{pmatrix} U & X \\ Y & Z \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} U & X \\ Y & Z \end{pmatrix}$, that is, $\begin{pmatrix} U & 0 \\ Y & 0 \end{pmatrix} = \begin{pmatrix} U & X \\ 0 & 0 \end{pmatrix}$. It follows that $X = 0$ and $Y = 0$. So $P^{-1}aP = c = \begin{pmatrix} U & 0 \\ 0 & Z \end{pmatrix}$ as desired. □

4. Structures of inverse semigroups of 2×2 matrices

Having a series of preparations as above, now we are ready to analyze the structure of inverse semigroups of 2×2 matrices. From Lemma 3.5, we can show the following Lemma 4.1, which is the basis of determination of the structure of an inverse semigroups of 2×2 matrices.

Lemma 4.1 *A Clifford semigroup of 2×2 matrices over a field F is commutative if it contains* *a matrix of rank 1.*

Proof Let S be a Clifford semigroup of 2×2 matrices containing a matrix of rank 1. According to Lemma 3.5, there exists an invertible matrix P such that for any $a\in S$, there exist two square matrices U and Z such that $P^{-1}aP=\begin{pmatrix} U & 0 \\ 0 & Z \end{pmatrix}$. Let $U=(u)$ and $Z=(z)$ with $u,z\in F$. Then $P^{-1}aP=\begin{pmatrix} u & 0 \\ 0 & z \end{pmatrix}$. Since all matrices of S are simultaneously diagonalizable, S is commutative semigroup and the proof is completed. □

The following Theorems 4.2, 4.4 and 4.5 are our main results, according to which we can actually determine completely the structure of inverse semigroups of 2×2 matrices

Theorem 4.2 *An inverse semigroup S of 2×2 matrices either is isomorphic to a linear group of dimension 2 or a zero-adjoined group , or is a finite semilattice of commutative linear groups, or satisfies the following conditions: (1) There are precisely 2 or 3 $\mathcal{H}$-classes containing idempotents, each of which is isomorphic to a linear group; (2) There are precisely 2 $\mathcal{H}$-classes without idempotents, which are mutual invertible and denoted by H_a and $H_{a^{-1}}$, respectively; (3) There exists an invertible matrix P such that each matrix of $P^{-1}H_aP$ is of the form $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, while each matrix of $P^{-1}H_{a^{-1}}P$ is of the form $\begin{pmatrix} 0 & 0 \\ v^{-1} & 0 \end{pmatrix}$, where $v\neq 0$; (4) When S contains an invertible matrix, S must contain the identity matrix $1=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and each matrix of $P^{-1}H_1P$ is of the form $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ or $\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$ with $uv\neq 0$.*

Proof Denote the idempotents of S by E. In view of Theorem 3.1, there exists $P\in \mathrm{GL}(n,F)$ such that $P^{-1}EP$ is a subsemilattice of the Boolean lattice $B_2=\{0,e,f,1\}$ with 2 atoms, where $0,e,f,1$ represent $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Let $T=P^{-1}SP$ and $E_1=P^{-1}EP$. Then $T\cong S$, $E\cong E_1$, E_1 is the idempotents of T, and T is also an inverse semigroup of matrices.

If E_1 is a chain of B_2, then by Lemma 3.4, S is a finite chain of linear groups. If $E_1=\{0\}$, then $T=0$; if $E_1=\{0,1\}$, then $T=H_1\cup\{0\}$ with H_1 the unit group of T, which is isomorphic to a linear group of dimension 2; if E_1 contains e or f, then T contains a matrix of rank 1 and by Lemma 4.1, T is a commutative semigroup. Furthermore, T is a finite chain of commutative linear groups.

If E_1 is not a chain of B_2, then $E_1=\{0,e,f\}$ or $E_1=\{0,e,f,1\}$. From the definition of Green's relations, it follows that $L_0=R_0=D_0=H_0=\{0\}$. We will discuss two cases below.

The first case is $E_1=\{0,e,f\}$. Since T is an inverse semigroup, by [9, II.1.2], each $\mathcal{L}$-class and each $\mathcal{R}$-class of T contains precisely one idempotent, respectively. Thus T has a partition $\{L_0,L_e,L_f\}$ and a partition $\{R_0,R_e,R_f\}$. Hence there are at most 5 distinguished $\mathcal{H}$-classes in

T. Clearly, H_0, H_e, H_f are all groups, which are isomorphic to some linear groups respectively. If these three classes are the whole $\mathcal{H}$-classes of T, then T is a finite semilattice of groups. Since T contains a matrix of rank 1, according to Lemma 4.1, it is a commutative Clifford semigroup. So T is a finite chain of some commutative linear groups.

If T contains at least 4 $\mathcal{H}$-classes, then the assumption $\exists a \in R_e \cap L_f$ implies that $a^{-1} \in L_e \cap R_f$ by Lemma 3.3, and also the assumption $\exists b \in L_e \cap R_f$ implies $b^{-1} \in R_e \cap L_f$. Thus both $R_e \cap L_f$ and $L_e \cap R_f$ are not empty sets. For every $a \in R_e \cap L_f$, T has a partition $\{H_0, H_e, H_f, H_a, H_{a^{-1}}\}$.

In order to determine H_a and $H_{a^{-1}}$, note that $a\mathcal{R}e$, which implies that $aS^1 = eS^1$. Thus $\exists s \in S^1$ such that $a = es$. From that $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, it follows that $a = \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}$ with some $u, v \in F$. Similarly, $a\mathcal{L}f$ implies that $a = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ with some $x, y \in F$. So we conclude that $a = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$. Dually, we have $a^{-1} = \begin{pmatrix} 0 & 0 \\ v' & 0 \end{pmatrix}$ with some $v' \in F$. Since $aa^{-1}a = a$, $aa^{-1}\mathcal{R}a$. But $a\mathcal{R}e$, so $aa^{-1} = e$. It follows that $vv' = 1$, and $v \neq 0$, $v' = v^{-1}$, $a^{-1} = \begin{pmatrix} 0 & 0 \\ v^{-1} & 0 \end{pmatrix}$.

The second case is that $E_1 = \{0, e, f, 1\}$. In light of Corollary 2.7, $L_1 = R_1 = D_1 = H_1$, which consists precisely of all invertible matrices of T. Proceeding along the line of proving the first case, we can show that either T is a finite semilattice of commutative linear groups, or both $R_e \cap L_f$ and $L_e \cap R_f$ are not empty sets. If the latter is true, then for all $a \in R_e \cap L_f$, $a^{-1} \in L_e \cap R_f$, S has a partition $\{H_0, H_e, H_f, H_a, H_{a^{-1}}, H_1\}$, each element of H_a has the form $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, and that of $H_{a^{-1}}$ is of the form $\begin{pmatrix} 0 & 0 \\ v^{-1} & 0 \end{pmatrix}$ with $v \neq 0$. For all $g \in H_1$, g is an invertible matrix, so $ge\mathcal{L}e$. Let $g = \begin{pmatrix} u & x \\ y & v \end{pmatrix}$. Then $ge = \begin{pmatrix} u & 0 \\ y & 0 \end{pmatrix}$. From $ge \in L_e = H_e \cup H_{a^{-1}}$, it follows that $u = 0$ or $y = 0$. Similarly, $gf\mathcal{L}f$ implies $x = 0$ or $v = 0$. Therefore, $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ or $g = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$. Since g is an invertible matrix, we get $uvxy \neq 0$.

Since $T = P^{-1}SP$, T is isomorphic to S. From the above discussion about T, we can derive that S satisfies all assertions in the theorem and the proof is completed. □

From Lemma 4.1 and Theorem 4.2, we can easily deduce the following corollary.

Corollary 4.3 *If S is an inverse semigroup of 2×2 matrices over a field F and there is no invertible matrix in S, then there exists $P \in \mathrm{GL}(2, F)$ such that each matrix of $P^{-1}SP$ is of the form $\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$ with $v \in F$.*

Let us turn our attention to the following problem: if S is a set of some 2×2 matrices and each element of S is of the forms appearing in Theorem 4.2(4) and Corollary 4.3, then under what conditions does S become an inverse semigroup? The next two theorems reply completely this question and give some necessary and sufficient conditions. One will be involved in subsets of the semigroup F^*, all nonzero elements of a field F with the multiplication. If X, Y are such two subsets, then we denote

$$X^{-1} = \{x^{-1} : x \in X\}, XY^{-1} = \{xy^{-1} : x \in X, y \in Y\}.$$

Theorem 4.4 *Suppose that S is a subset of $M_n(F)$ with F a field, and that every element of S is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in F$, at most one of which is not 0. Let*

$$A = \Big\{\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} \in S : v \neq 0\Big\}, \quad B = \Big\{\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in S : v \neq 0\Big\},$$

$$C = \Big\{\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in S : v \neq 0\Big\}, \quad D = \Big\{\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} \in S : v \neq 0\Big\}.$$

Assume A, B, C, D are all non-empty sets, and let

$$A_0 = \Big\{v : \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} \in A\Big\}, \quad B_0 = \Big\{v : \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in B\Big\},$$

$$C_0 = \Big\{v : \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in C\Big\}, \quad D_0 = \Big\{v : \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} \in D\Big\}.$$

Then S is an inverse semigroup if and only if S contains the zero matrix 0, $C_0 = B_0^{-1}$, and $B_0B_0^{-1} = A_0 = D_0$ is a semigroup [group].

Proof Necessity. Let $0, e, f$ represent the matrices $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Then $\{0, e, f\}$ is the set of idempotents of S. Let $u \in B_0$. Then $\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \in B$. By Theorem 4.2, we have $a^{-1} = \begin{pmatrix} 0 & 0 \\ u^{-1} & 0 \end{pmatrix} \in C$, $A = H_e$, $B = H_a = R_e \cap L_f$, $C = H_{a^{-1}} = L_e \cap R_f$, $D = H_f$. Since H_e, H_f are groups, so are A and D. Clearly, $A \cong A_0, D \cong D_0$, and thus A_0, D_0 are groups. Since $(H_a)^{-1} = H_{a^{-1}}$, we have $B^{-1} = C$. Therefore, $C_0 = B_0^{-1}$.

Sufficiency. Suppose S contains 0, $C_0 = B_0^{-1}$ and $B_0B_0^{-1} = A_0 = D_0$ are semigroups. First note that $0, e, f \in S$, and the idempotents of S is $E = \{0, e, f\}$. It is obvious that E is commutative. Secondly, since $S = \{0\} \cup A \cup B \cup C \cup D$, we can take direct calculations to verify that S is closed under the multiplication and hence S is a semigroup. At last, in order to show that S is an inverse semigroup, it remains to prove that S is regular. Since $B_0B_0^{-1} = A_0 = D_0$, for all $b_1, b_2 \in B$, we have $1 = b_1/b_1 \in A_0$, $b_1/b_2, b_2/b_1 \in A$ and $(b_1/b_2)(b_2/b_1) = 1$. Combining with that A_0 is a semigroup, we derive that A_0 and then D_0 are both groups. Furthermore, we know the regularity of the elements of A and D. Now for all $b \in \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in B$, we have $v \in B_0$. From $C_0 = B_0^{-1}$, it follows that $v^{-1} \in C_0$. Let $c = \begin{pmatrix} 0 & 0 \\ v^{-1} & 0 \end{pmatrix} \in C$. Then it is readily to verify that $bcb = b$. Thus, b is regular, and so every element of B is regular. Similarly, one can show that every element of C is also regular. It is obvious that the zero matrix 0 is regular. So we conclude that S is regular, and the proof is completed. □

In addition, if a subset of 2×2 matrices is allowed to contain an invertible matrix, then we have the following theorem to give the necessary and sufficient conditions so that the subset becomes an inverse semigroup of matrices.

Theorem 4.5 *Suppose that S is a subset of $M_n(F)$ with F a field, and that every element of S is of the form $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ or $\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$ with $u, v \in F$. Denote $U = \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in S : uv \neq 0\}$ and $V = \{\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \in S : uv \neq 0\}$. Let $A, B, C, D, A_0, B_0, C_0$ and D_0 be as in Theorem 4.4. Assume both U and V are nonempty sets, and set*

$$A_1 = \Big\{u : \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in U\Big\}, \quad B_1 = \Big\{u : \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \in V\Big\},$$

$$C_1 = \Big\{v : \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \in V\Big\}, \quad D_1 = \Big\{v : \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in U\Big\}.$$

Then S is an inverse semigroup if and only if the following conditions are all true: (1) S contains the zero matrix 0; (2) $U^{-1} \subseteq U$, $V^{-1} \subseteq V$; (3) $B_1B_1^{-1} = A_1 = D_1$ is a semigroup, and $C_1 = B_1^{-1}$; (4) $B_0B_0^{-1} = A_0 = D_0$ is a semigroup, and $C_0 = B_0^{-1}$; (5) $A_1 \subseteq A_0$, $B_1 \subseteq B_0$, $C_1 \subseteq C_0$ and $D_1 \subseteq D_0$.

Proof First note that $S = \{0\} \cup U \cup V \cup A \cup B \cup C \cup D$, thus we can prove the sufficiency of this theorem by the same method as in proving that of Theorem 4.4. Second, we prove the necessity of this theorem as follows.

Let S be an inverse semigroup of matrices as in the theorem, and $0, e, f, 1$ represent $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Then $\{0, e, f, 1\}$ is the semilattice of idempotents of S. Let $u \in B_0$. Then $a = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \in B$. According to Theorem 4.2, $a^{-1} = \begin{pmatrix} 0 & 0 \\ u^{-1} & 0 \end{pmatrix} \in C$, $A = H_e$, $B = H_a = R_e \cap L_f$, $C = H_{a^{-1}} = R_e \cap L_f$, $D = H_f$. By 4.4, A_0, D_0 are groups and $C_0 = B_0^{-1}$.

Assume $v = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in V$. Then $xy \neq 0$, and by Corollary 2.2, $v^{-1} = \begin{pmatrix} 0 & y^{-1} \\ x^{-1} & 0 \end{pmatrix} \in V$, which implies that $B_1^{-1} \subseteq C_1$, $C_1^{-1} \subseteq B_1$ and thus $B_1 = C_1^{-1}$. From $U^2 \subseteq U$, $A_1^2 \subseteq A_1$ and $D_1^2 \subseteq D_1$, it follows that both A_1 and D_1 are semigroups. Let $u = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in U$. Then $xy \neq 0$, and by Corollary 2.2, we have $u^{-1} = \begin{pmatrix} x^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} \in U$. Thus $U^{-1} \subseteq U$, $V^{-1} \subseteq V$, $A_1^{-1} \subseteq A_1$

and $D_1^{-1} \subseteq D_1$. But from $1 = uu^{-1} \in U$, it follows that $1 \in A_1$ and $1 \in D_1$. So A_1, D_1 are both groups. Obviously, $UV \subseteq V$ implies that $A_1B_1 \subseteq B_1$ and $C_1D_1 \subseteq C_1$. Hence $A_1 \subseteq B_1B_1^{-1}$, $D_1 \subseteq C_1C_1^{-1}$. From $V^2 \subseteq U$, it follows that $B_1C_1 \subseteq A_1$ and $B_1C_1 \subset D_1$. Thus $B_1B_1^{-1} \subseteq A_1$ and $C_1C_1^{-1} \subseteq D_1$. Therefore, $A_1 = B_1B_1^{-1} = B_1C_1 = C_1B_1 = C_1C_1^{-1} = D_1$.

$UB \subseteq B$ implies $B_0A_1 \subseteq B_0$, and so $A_1 \subseteq B_0B_0^{-1} = A_0$. Similarly, $UC \subseteq C$ implies $D_1 \subseteq D_0$ and $VD \subseteq B$ implies $D_0B_1 \subseteq B_0$. But since D_0 is a group, $1 \in D_0$. Thus $B_1 \subseteq D_0B_1 \subseteq B_0$. Analogously, since $VA \subseteq C$, we can derive that $C_1 \subseteq C_0$. This completes the proof. □

Let us conclude this paper with an example, which shows the significance of Theorems 4.4 and 4.5.

Example 4.6 All integers are denoted by Z. Let

$$U = \left\{ \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix} : n \in Z \right\}, \quad V = \left\{ \begin{pmatrix} 0 & 2^n \\ 2^{-n} & 0 \end{pmatrix} : n \in Z, n > 0 \right\},$$

$$A = \left\{ \begin{pmatrix} 2^n & 0 \\ 0 & 0 \end{pmatrix} : n \in Z \right\}, \quad B = \left\{ \begin{pmatrix} 0 & 2^n \\ 0 & 0 \end{pmatrix} : n \in Z, n > 0 \right\},$$

$$C = \left\{ \begin{pmatrix} 0 & 0 \\ 2^n & 0 \end{pmatrix} : n \in Z, n < 0 \right\}, \quad D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 2^n \end{pmatrix} : n \in Z \right\}.$$

Then by Theorem 4.4, $S_1 = \{0\} \cup A \cup B \cup C \cup D$ becomes an inverse semigroup of 2×2 matrices; and so does $S_2 = \{0\} \cup U \cup V \cup A \cup B \cup C \cup D$ by Theorem 4.5.

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