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On the Bijective Half-Functors*

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O. Introduction

When a group G is regarded as a category consisting of only one object and the hom-set G, the operation $a \mapsto a^{-1}$, as we know, is a full and faithful contravariant functor: $G \to G$. Secondly, [1] showed that in a preadditive category (or more widely, in an n-preadditive category satisfying the condition (E)) the operation $\varphi : a \mapsto \bar{a}$ satisfies the following:

(1) $\varphi : (A, A') \to (A, A')$ is bijective;

(2) $\varphi(ab) = a\varphi(b) = (\varphi(a))b$ whenever the composite ab makes sense.

In this note, we shall suggest the bijective half-functors to unify both above. This finished, we can make descriptions of some systems, the φ-c-c^r and the φ^c-c-c^r categories, as is very important for S.C.T. (See[3] and [2]).

It will be mentioned that, in [3], a category is called a φ^c-c-c^r category if there exists a bijective half-functor $\varphi^c \in BH_T(\mathscr{A}, \mathscr{A}^{op})$, where $T: \mathscr{A} \to \mathscr{A}^{op}$ is a functor and $TA = A^{op}$ for each object A of $\mathscr{A}$, such that the mapping $\bar{\varphi}: (A'', A') \times (A, A') \to (A^{op}, A''^{op}): \langle a, b\rangle \mapsto a^{op}\varphi^c(b)$ is C^r, where the manifold (A^{op}, A''^{op}) has the same differential structure as the differentiable manifold (A'', A) has. In accordance to my teacher Professor Zhou Boxun's opinion, a φ^c-c-c^∞ category is called a Lie-category. Clearly, a Lie group is a Lie-category, which brings to light the contravariantness in the notion of the Lie groups. It would seem that, therefore, this supplies the study of the Lie groups with another clue—namely, to study with the aid of the category theory. Moreover, the Lie-categories extending the notion of the Lie groups, it seems to be possible to use the Lie group and Lie algebra method for reference in the research of S.C.T.

As was stated above, it is very useful to discuss the bijective half-functors. In this note, we shall show soms basic properties about them, they will be used many times in our works. §1 will introduce the concept of the bijective half-functors and show a necessary and sufficient condition of $BH_T(\mathscr{A}, \mathscr{B}) \neq \emptyset$. In §2, we shall prove that the important elementary quantities in category the-

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ory, such as limits, unions and images, are their invariants, and soms their invariances will be shown as well. §3 aims to discuss the inverses of a bijective half-functor, the main result is the proposition 3.10, which gets an answer to the problem about the structure of the inverses of a bijective half-functor. §4 is devoted to the discussion about the surjectiveness of the mapping $\psi\mapsto\psi^1_G$, —— we get a sufficient condition and a necessary condition to make a given bijective half-functor $\varphi\in BH_G(\mathscr{B}, \mathscr{A})$ an inverse of a function $\psi\in BH_T(\mathscr{A}, \mathscr{B})$.

In this note, we shall continue to use the symbols in [1]. Let $[u]=\{v|$ $\{v|\mathrm{dom}(v)=\mathrm{dom}(u)\wedge v\cong u$ and $\langle u\rangle=\{v|\mathrm{codom}(v)=\mathrm{codom}(u)\wedge v\cong u\}$ when u and v are monic. The composition of b by a is denoted as ba, $(b)a$, or $b.(a)$.

I. Bijective half-functors

1.1. Definition. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, and let $T:\mathscr{A}\to\mathscr{B}$ be a functor (resp. contravariant functor). A function $\psi:\mathscr{A}\to\mathscr{B}$ is called a bijective half-functor (resp. bijective half-contravariant functor) for T. if it satisfies the following:

(1) $\forall A\in\mathrm{ob}\mathscr{A}:\psi(A)=TA$;

(2) $\forall A_1, A_2\in\mathrm{ob}\mathscr{A}:\psi:(A_1, A_2)\to(TA_1, TA_2)$ (resp. (TA_2, TA_1)) is bijective;

(3) $\psi(fg)=(Tf)\psi(g)=\psi(f)Tg$, (resp. $\psi(fg)=(\psi g)(Tf)=(Tg)(\psi f)$), whenever the composite fg makes sense.

We write $BH_T(\mathscr{A}, \mathscr{B})$ (resp. $CBH_T(\mathscr{A}, \mathscr{B})$) for the class of all bijective half-functors (resp. all bijective half-contravariant functors) for T. Clearly, $BH_{1_{\mathscr{A}}}(\mathscr{A}, \mathscr{A})=SBH\mathscr{A}$ (see [1]).

As $\overline{\psi}:\mathscr{A}^{op}\to\mathscr{B}$ is a function such that $\overline{\psi}A^{op}=\psi A$ and $\overline{\psi}f^{op}=\psi f$ for each morphhism f in $\mathscr{A}$ and $\psi':\mathscr{A}\to\mathscr{B}^{op}$ such that $\psi'A=(\psi(A))^{op}$ and $\psi'f=(\psi f)^{op}$, we write $\overline{\psi}\doteq\psi$ and $\psi'\doteq\psi$ respectively. Then holds $CBH_T(\mathscr{A}, \mathscr{B})\doteq BH_{T^o}(\mathscr{A}^{op}, \mathscr{B})\doteq BH_{T^{oo}}(\mathscr{A}, \mathscr{B}^{op})$, where $T^o:\mathscr{A}^{op}\to\mathscr{B}$ is a functor such that $T^oA^{op}=TA$ and $T^of^{op}=Tf$, and $T^{oo}:\mathscr{A}\to\mathscr{B}^{op}$ such that $T^{oo}A=(TA)^{op}$ and $T^{oo}f=(Tf)^{op}$. Referring to the proposition 1.4, we know that when $T:\mathscr{A}\to\mathscr{B}$ is a full and faithful contravariant functor, T^{oo} is a full and faithful functor, and hence $T^{oo}\in BH_{T^{oo}}(\mathscr{A}, \mathscr{B}^{op})$, which shows that in a group G, the operation $T^{oo}:g\mapsto(g^{-1})^{op}$ belongs to $BH_{T^{oo}}(G, G^{op})$. So that the in inverse operation T^{oo} in the group G can be regarded as a bijective half-functor. On the other hand, in a preadditive category the operation $a\mapsto(-a)$ is a bijective half-functor also. Thus, we have unified the two things adove.

I.2. Lemma If $\psi\in BH_T(\mathscr{A}, \mathscr{B})$, then $\psi(1_A)$ is an isomorphism for each $A\in\mathrm{ob}\mathscr{A}$.

Proof Since $\psi:(A, A)\to(TA, TA)$ is surjective, there is a morphism $u:A\to A$ such that $\psi(u)=1_{TA}$. So $(Tu)\psi(1_A)=\psi(u1_A)=\psi(u)=1_{TA}=\psi(1_Au)=\psi(1_A)Tu$, q.e.d.

I.3 Corollary Let $\psi\in BH_T(\mathscr{A}, \mathscr{B})$. If ψ preserves monomorphisms (resp. epimorphisms), then so does T, and vice versa. In addition, ψ always preserves isomorphisms.

1.4 Proposition $BH_T(\mathcal{A}, \mathcal{B}) \neq \emptyset$ if and only if the functor T is full and faithful.

Proof ($\Rightarrow$:) Let $\psi \in BH_T(\mathcal{A}, \mathcal{B})$. Since $\psi(f) = \psi(1)T(f)$ and $\psi(1)$ is an isomorphism, the proof is clear.

($\Leftarrow$:) As T is full ann faithful, we have $T \in BH_T(\mathcal{A}, \mathcal{B})$.

1.5 Corollary If $BH_T(\mathcal{A}, \mathcal{B}) \neq \emptyset$ and if for each $b \in \mathrm{ob}\,\mathcal{B}$ there is an object a $a \in \mathrm{ob}\,\mathcal{A}$ such that $Ta \cong b$, then there is an adjoint equivalence $\langle S, T; \eta, \varepsilon \rangle : \mathcal{B} \to \mathcal{A}$; and vice versa. (See[4, IV.4. Therem 1]).

1.6 Corollary If $\langle T, S; \eta, \varepsilon \rangle : \mathcal{A} \to \mathcal{B}$ is an adjunction, then $\langle T, S; \eta, \varepsilon \rangle$ is an adjoint equivalence if and only if $BH_T(\mathcal{A}, \mathcal{B}) \neq \emptyset$ and $BH_S(\mathcal{B}, \mathcal{A}) \neq \emptyset$. See[4, IV. 3. Theorem 1 and Theorem 1 in IV.4.]).

2. Invariants of Bijective Half-Functors

2.1 Proposition Let $\psi \in BH_T(\mathcal{A}, \mathcal{B})$. If $\psi(t)$ is a monic (resp. an epi or an isomorphism), then so is t. Therefore, ψ^{-1} preserves monomorphisms, epimorphisms, and isomorphisms. In addition, if $\psi(t)$ is split monic (resp. split epi), then so is t.

Proof Suppose $\psi(t)$ is monic and $ta = tb$, then $(Tt)\psi(a) = \psi(ta) = \psi(tb) = (Tt)\psi(b) \cdot \psi(t)$ is monic, so is Tt, for $Tt = \psi(t)\psi(1)^{-1}$. Therefore, $\psi(a) = \psi(b)$, Since ψ is injective, $a = b$, so that t is monic.

Ween $\psi(t)$ is epi, the proof can similarly be completed.

If $\psi(t)$ is split monic, then there is a split epi e such that $e\psi(t) = 1$. T being full, we can write $Tu = \psi(1)e$ for an epi u in $\mathcal{A}$. Hence $\psi(ut) = \psi(1)$, so that $ut = 1$. We have proved that t is split monic, q.e.d.

2.2 Definition A functor $T: \mathcal{A} \to \mathcal{B}$ is called epi $T\to$ surjective on objects, or epi $T\to$ for short (resp. monic $T\to$), if for any object $B \in \mathrm{ob}\,\mathcal{B}$ we have an object $A \in \mathrm{ob}\,\mathcal{A}$ and an epi $h: TA \to B$ (resp. a monic $m: TA \to B$). In addition, if h is split epi as well, then T is called split epi $T\to$.

Clearly, if T is quasifull on objects (see[2, Definition 9]), then T is split epi $T\to$ as well as split monic $T\to$. The meaning of "$\to T$" is clear.

2.3 Proposition Let $\psi \in BH_T(\mathcal{A}, \mathcal{B})$. If T is epi $T\to$ (resp. monic $\to T$), then ψ preserves monomorphisms (resp. epimorphisms).

Proof Let $t: A \to A'$ be monic. Suppose $(Tt)a = (Tt)b$, where $a, b: B \to TA$. Since ψ is surjective, there are u and v such that $\psi(u) = ah$ and $\psi(v) = bh$, where $h: TA'' \to B$ is an epi. So $\psi(tu) = (Tt)\psi(u) = (Tt)(ah) = (Tt)(bh) = (Tt)\varphi(v) = \psi(tv)$. Since ψ is injective, $tu = tv$, hence $u = v$ for t is monic. Therefore, $\psi(u) = ah = \psi(v) = bh$, so that $a = b$. This means Tt is monic. Further Corollary 1.3 shows $\psi(t)$ is monic also. Q.E.D.

2.4 Proposition Given two functors $\mathscr{A} \rightleftarrows \mathscr{B}$, suppose $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and $\langle B, r: A \to GB \rangle$ is a universal arrow from $A \in ob\mathscr{A}$ to G. If G is split monic $\to G$ and r is epi, or G is full and split monic $\to G$, then $\langle GB, \psi(r): TA \to TGB \rangle$ is a universal arrow from TA to T.

Proop Suppose G is full and split monic $\to G$. Given $A' \in ob\mathscr{A}$ and $s: TA \to TA'$ because ψ is bijective, there is a morphism $u: A \to A'$ such that $\psi(u) = s$. In addition, since G is split monic $\to G$ there is a monic $h: A' \to GB'$ and $th = 1_{A'}$ for an epi t. Since $\langle B, r \rangle$ is a universal arrow, there is a unique $m: B \to B'$ such that $(Gm)r = hu$. Hence $u = (th)u = t(Gm)r$, then $\psi(t(Gm)r) = T(t(Gm))\psi(r) = \psi(u) = s$, where $tGm: GB \to A'$. On the other hand, assume that there is a morphism $n: GB \to A'$ such that $(Tn)\psi(r) = s$. Then $\psi(nr) = (Tn)\psi(r) = s = T(t(Gm))\psi(r) = \psi(t(Gm)r)$. Hence $nr = t(Gm)r$ for ψ is injective. (If r is epi, from the fact that $t(Gm)r = nr$ we know $t(Gm) = n$, so that the universal property of $\langle GB, \psi(r) \rangle$ is showh). Since G is full, there are morphisms $b, b': B \to B'$ such that $Gb = hn$ and $Gb' = htGm$. Hence $(Gb)r = (hn)r = (htGm)r = (Gb')r$. Because r is a universal arrow, holds $b = b'$, so $hn = htGm$, and hence $n = tGm$, so that the universal property of $\langle GB, \psi(r) \rangle$ is proved, the proof is completc.

2.5 Proposition Given two functors $\mathscr{A} \underset{G}{\overset{T}{\rightleftarrows}} \mathscr{B}$, suppose $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and $\langle B, r: GB \to A \rangle$ is a universal arrow from G to A. If G is split epi $G \to$ and r is monnic, or if G is full and split epi $G \to$, then $\langle GB, \psi(r): TGB \to TA \rangle$ is a universal arrow from T to TA.

2.6 Let J be a category, and let $\Delta_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}^J$ be diagonal functor.

Proposition Suppose $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and $G \in ob\mathscr{A}^J$. If T is split epi $T \to$, then that $\langle A, a = (a_j)_{j \in obJ}: \Delta_{\mathscr{A}} \dot{\to} G \rangle$ is a limit for the functor G implies that $\langle TA, \psi(a) = (\psi(a_j))_{j \in obJ}: \Delta_{\mathscr{B}} TA \dot{\to} TG \rangle$ is a limit for the functor TG.

Proof Clearly, if $(A, s_j: A \to G_j)_{j \in obJ}$ is a cone from the vertex A to the base G, then $(TA, \psi(s_j): TA \to TGj)_j$ is a cone from the vertx TA to the base TG. Now suppos there is a cone $(B, r_j: B \to TGj)_j$ from the vertex $B \in ob\mathscr{B}$ to the base TG. Since T is split epi $T \to$, there is an epi $e: TA' \to B$, and $em = 1_B$ for a monnic $m: B \to TA'$. So for each $j \in obJ$ there is a morphism $b_j: A' \to Gj$ with $\psi(b_j) = r_j e$. Given $j_1, j_2 \in obJ$ and $f: j_1 \to j_2$, because (B, r_j) is a cone, we have $(TGf)r_{j_1} = r_{j_2}$. Hence $\psi((Gf)b_{j_1}) = (TGf)\psi(b_{j_1}) = (TGf)r_{j_1}e = r_{j_2}e = \psi(b_{j_2})$, so $(Gf)b_{j_1} = b_{j_2}$. So that (A', b_j) is a cone from the vertex A' to the base G. Since $\langle A, a: \Delta_{\mathscr{A}} A \dot{\to} G \rangle = (A, a_j: A \to Gj)_j$ is a universal cone, there is a unique $d: A' \to A$ such that $a_j d = b_j$. Therefore, $\psi(a_j)Td = \psi(b_j) = r_j e$, so that $\psi(a_j)((Td)m) = r_j$. On the other hand, suppose there is a morphism $t: B \to TA$ such that $\psi(a_j)t = r_j$. We are going to prove that $t = (Td)m$. In fact, there is a morphism $d': A' \to A$ such that $T(d') = te$ for T is surjective (see Proposition1.4). So $\psi(a_j d') = (\psi(a_j))(te) = r_j e = \psi(b_j)$, hence

$a_j d' = b_j$. As was stated above, there is a unique d such that $a_j d = b_j$, hence $d' = d$. So that $t = (Td')m = (Td)m$. The proof is compete .

2.7 Proposition Suppose $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and $G \in ob\,\mathscr{A}^J$. If T is split monic $\dashrightarrow T$ than that $\langle A, a = (a_j)_{j \in obJ}: G \dot{\to} \Delta_{\mathscr{A}} A\rangle$ is a colimit for the functor G implies that $\langle TA$ $\psi(a) = (\psi(a_j))_j : TG \dot{\to} \Delta_{\mathscr{B}} TA\rangle$ is a colimit for the functor TG .

2.8 Corollary Suppose $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and T is split epi $T \dashrightarrow$ (that is, split m monic $\to T$), then the following hold:

(1) If (A, a_j) is a product diagram (resp . coproduct diagram) in $\mathscr{A}$. then so are both $(TA, \psi(a_j))$ and (TA, Ta_j) in $\mathscr{B}$.

(2) If $\langle d\rangle = \mathrm{Equ}(a, b)$ in $\mathscr{A}$, then $\langle Td\rangle = \langle \psi(d)\rangle = \mathrm{Equ}(Ta, Tb) = \mathrm{Equ}(\psi(a), \psi(b))$ in $\mathscr{B}$. If $[d] = \mathrm{Coequ}(a, b)$, then $[Td] = [\psi(d)] = \mathrm{Coequ}(Ta, Tb) = \mathrm{Coequ}(\psi(a), \psi(b))$.

(3) If $\langle u\rangle = \bigcap_j u_j$, then $\langle Tu\rangle = \langle \psi(u)\rangle = \bigcap_j Tu_j = \bigcap_j \psi(u_j)$.

(4) If $\mathscr{A}$ is complete (resp. cocomplete), then so is $\mathscr{B}$.

2.9 Proposition Let $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and T be quasifull on objects. If $\langle u\rangle = \bigcup_i u_i$ in the category $\mathscr{A}$, then $\langle Tu\rangle = \langle \psi(u)\rangle = \bigcup_i Tu_i = \bigcup_i \psi(u_i)$ in the category $\mathscr{B}$.

Proof Suppose $u_i : A_i \rightarrowtail A$ and $\langle u\rangle = \bigcup u_i$. By Proposition 2.3, $\psi(u_i)$ and $\psi(u)$ are monic . If each $\psi(u_i)$ is carried into a monic $m: B \rightarrowtail B'$ by a morphism $f: TA \to B'$, that is, for each $\psi(u_i)$ there is a morphism l_i such that $f\psi(u_i) = ml_i$. Because T is quasifull on objects, we can write $Tk = sf$, where $k: A \to A'$, $s: B' \rightarrowtail TA'$, and $ts = 1_{B'}$; $Tw_i = nl_i$, where $w_i: A_i \to A''$, $n: B \rightarrowtail TA''$, $hn = 1_B$ and $nh = 1_{TA''}$; and $\psi(x) = smh$, where $x: A'' \to A'$. By Propositton 2.1, x is monic . Since $\psi(xw_i) = \psi(x)Tw_i = smh \cdot nl_i = sml_i = sf \cdot \psi(u_i) = (Tk)\psi(u_i) = \psi(ku_i)$, we have $xw_i = ku_i$. So there is a morphism β such that $x\beta = ku$. Therefore, $\psi(x)T\beta = (smh)T\beta = (Tk)\psi(u) = sf\psi(u)$, so that $m(hT\beta) = f\psi(u)$, that is, $\psi(u)$ is also carried into m by f. This means $\langle \psi(u)\rangle = \bigcup \psi(u_i)$. Since $\psi(u) = \psi(1)T(u)$ and $\psi(1)$ isan isomorphism (see Lemma 1.2), $\langle \psi(u)\rangle = \langle Tu\rangle$. the proof is complete .

2.10 Proposition Let $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and T be quasifull on objects. If $\langle h\rangle = Im(f)$ in the category $\mathscr{A}$, then $\langle Th\rangle = \langle \psi(h)\rangle = Im(\psi(f)) = Im(Tf)$ in the category $\mathscr{B}$.

Imitating the last proof, this proof can easily be completed .

2.11 Proposition If $\psi \in BH_T(\mathscr{A}, \mathscr{B})$ and T is split epi $T \to$, then the following are equivalent :

(1) $\begin{array}{ccc} C & \xrightarrow{d} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & D \end{array}$ is a pullback (resp . a pushout).

(2) $\begin{array}{ccc} TC & \xrightarrow{\psi(d)} & TB \\ \psi(u)\downarrow & & \downarrow Th \\ TA & \xrightarrow[Tf]{} & TD \end{array}$ is a pullback (resp. a pushout).

(3) $\begin{array}{ccc} TC & \xrightarrow{Td} & TB \\ Tu\downarrow & & \downarrow \psi(h) \\ TA & \xrightarrow[\psi(f)]{} & TD \end{array}$ is a pullbk (resp. a pushout).

(4) $\begin{array}{ccc} TC & \xrightarrow{Td} & TB \\ \psi(u)\downarrow & & \downarrow \psi(h) \\ TA & \xrightarrow[Tf]{} & TD \end{array}$ is a pullback (resp. a pushout).

(5) $\begin{array}{ccc} TC & \xrightarrow{\psi(d)} & TB \\ Tu\downarrow & & \downarrow Th \\ TA & \xrightarrow[\psi(f)]{} & TD \end{array}$ is a pullback (resp. a pushout).

Proof (1)⇒(2): The use of Proposition 2.6 (resp. Proposition 2.7). (2)⇒(1): Suppose $fa=hb$, then $(Tf)\psi(a)=(Th)\psi(b)$, so $(Tf)\cdot(\psi(a)\psi(1))=(Th)\cdot(\psi(b)\psi(1))$. Hence there is a unique $t''=\psi(t)$ such that $\psi(u)\psi(t)=\psi(a)\psi(1)$ and $\psi(d)\psi(t)=\psi(b)\psi(1)$. Then $\psi(u)\psi(t)=\psi(u)(Tt)\psi(1)$, hence $\psi(u)Tt=\psi(a)$ and $\psi(d)Tt=\psi(b)$, therefore $ut=a$ and $dt=b$. On the other hand, assumu that $ut'=a$ and $dt'=b$, then $\psi(ut')=\psi(u)Tt'=\psi(a)$ and $\psi(d)Tt'=\psi(b)$, so $\psi(u)\cdot((Tt')\psi(1))=\psi(a)\psi(1)$ and $\psi(d)\cdot((Tt')\psi(1))=\psi(b)\psi(1)$. Hence $(Tt')\psi(1)=\psi(t)$, so that $t'=t$. The required universal property is proved.

That (2)⇔(3)⇔(4)⇔(5) is a corollary of Lemma 1.2. for $\psi(h)=(Th)\psi(1)=\psi(1)Th$.

3. The Inverses of a Bijective Half-Functor

We shall use the following theorem corresponding to [4, IV.3. Theorem 1]:

Theorem 1' For an adjunction $\langle F, G; \eta, \varepsilon\rangle : \mathscr{X}\to\mathscr{A}$: (i) F is faithfull if and only if every component η_x of the unit η is monic, (ii) F is full if and only if every η_x is split epi. Hence F is full and faithful if and only if each η_x is an isomorphism $x\cong GFx$.

By imitating what Prof. S. Mac Lane did in [4, IV.3], the proof can be completed. That is, we must prove the following lemma:

Lemma A Let $f_*=(f_{*c})_{c\in \mathrm{ob}\mathscr{A}}:(-, a)\xrightarrow{\bullet}(-, b)$ be the natural transfomation induced by a morphism $f: a\to b$ of $\mathscr{A}$. Then for each $c\in\mathrm{ob}\mathscr{A}$ f_{*c} is monic if and only if f is monic, while f_{*c} is epi if and only if f is split epi (i.e., if and only if f has a right inverse).

3.1 Definition Given two functors: $\mathscr{A}\underset{G}{\overset{T}{\rightleftarrows}}\mathscr{B}$, suppose $\psi\in \mathrm{BH}_T(\mathscr{A}, \mathscr{B})$. A function $\varphi\in\mathrm{BH}_G(\mathscr{B}, \mathscr{A})$ is called a G-inverse of ψ, if $\varphi(g)=\psi^{-1}(TGg): GB_1\to GB_2$ for every morphism $g: B_1\to B_2$ of $\mathscr{B}$.

Clearly, if there is a G-inverse of ψ, then it must be unique. We write $\bar{\psi}_G^1$ for it. If $\mathscr{A}=\mathscr{B}$ and $T=G=I_{\mathscr{A}}$, then $\bar{\psi}_{I_{\mathscr{A}}}^1=\psi^{-1}$, the inverse of the self-bijective half-functor ψ. Which aroused the definition.

It should be remarked that in general $\psi^{-1}(TGg)$ is not unique, but in the present case, $\psi^{-1}(TGg):GB_1\to GB_2$ is indeed unique. In this paper, we always write $\psi^{-1}(TGg)$ for $\psi^{-1}(TGg):GB_1\to GB_2$; there is no danger of confusion.

3.2 Remark Suppose $\psi\in \mathrm{BH}_T(\mathscr{A},\mathscr{B})$. If $\bar{\psi}_G^1$ and $\bar{\psi}_S^1$ exist, then $\bar{\psi}_G^1=\bar{\psi}_S^1$ if and only if $G=S$.

Observing T is faithfull the proof can be easily comleted.

3.3 Proposition Suppose $G:\mathscr{B}\to\mathscr{A}$ is a functor, then $\psi\in\mathrm{BH}_T(\mathscr{A},\mathscr{B})$ has the G-inverse if and only if the functor TG is both full and faithful.

Proof $(\Rightarrow:)$ Proposition 1.4.

$(\Leftarrow:)$ We define a function $\varphi:\mathscr{B}\to\mathscr{A}$ as follows:

(1) $\forall B\in\mathrm{ob}\,\mathscr{B}:\varphi(B)=GB$.

(2) $\forall g:B_1\to B_2:\varphi(g)=\psi^{-1}(TGg)$.

The ordinary composite of φ and ψ is denoted by $\psi\varphi$. Suppose $B\in\mathrm{ob}\,\mathscr{B}$, then $(\psi\varphi)(B)=\psi(GB)=TGB$. On the other hand, assume that $g:B_1\to B_2$ is a morphism of $\mathscr{B}$, we have $(\psi\varphi)(g)=\psi(\psi^{-1}(TGg))=TGg$. Therefore, $\psi\varphi=TG:\mathscr{B}\to\mathscr{B}$. Since TG is full and faithful, $\psi\varphi:[B_1,B_2]\to[TGB_1,TGB_2]$ is bijective $\psi:[GB_1,GB_2]\to[TGB_1,TGB_2]$ is bijective, so is $\varphi:[B_1,B_2]\to[GB_1,GB_2]$.

Finally, given two morphisms $p:B_1\to B_2$ and $h:B_2\to B_3$ in $\mathscr{B}$, we have $\varphi(hp)=\psi^{-1}(TG(hp))=\psi^{-1}((TGh)TGp)$. Since ψ is surjective, there is a morphism $s:GB_2\to GB_3$ such that $\psi(s)=TGh$. So $\varphi(hp)=\psi^{-1}(\psi(s)TGp)=\psi^{-1}(\psi(sGp))=sGp=\psi^{-1}(TGh)Gp=\varphi(h)Gp$. In the same way, we have $\varphi(hp)=(Gh)\varphi(p)$. Therefore $\varphi\in\mathrm{BH}_G(\mathscr{B},\mathscr{A})$, so that $\varphi=\bar{\psi}_G^1$, q.e.d.

3.4 Corollary If there is an adjunction $\langle T,G;\eta,\varepsilon\rangle:\mathscr{A}\to\mathscr{B}$, and if $\psi\in\mathrm{BH}_T(\mathscr{A},\mathscr{B})$, then ψ has the G-inverse iff $\langle T,G;\eta,\varepsilon\rangle:\mathscr{A}\to\mathscr{B}$ is an adjoint equivalence.

Proof $(\Rightarrow:)$ T is full and faithful by Proposition 1.4. So from [4, IV.3. Theorem 1′] we know that $\eta:I_{\mathscr{A}}\overset{\bullet}{\to}GT$ is a natural isomorphism. In addition, because ψ has the G-inverse $\bar{\psi}_G^1$, we know $\mathrm{BH}_G(\mathscr{B},\mathscr{A})\neq\varnothing$. Hence G is also full and faithful. By [4, IV.3. Theorem 1], $\varepsilon:TG\to I_{\mathscr{B}}$ is also a natural isomorphism, so $\langle T,G;\eta,\varepsilon\rangle:\mathscr{A}\to\mathscr{B}$ is an adjoint equivalence.

$(\Leftarrow:)$ By [4, IV.4. Theorem 1] G is full and faithful, so is TG. Then Proposition 3.3 shows that the G-inverse $\bar{\psi}_G^1$ exists.

3.5 Lemma If there are two functors $\mathscr{A}\underset{G}{\overset{T}{\leftrightarrows}}\mathscr{B}$ and $\varepsilon=(\varepsilon_B)_{B\in\mathrm{ob}\,\mathscr{B}}:TG\overset{\bullet}{\to}I_{\mathscr{B}}$ is a natural transformation with every component ε_B epi, then G is faithful.

Proof Given two morphisms $g_1g_2\in(B_1,B_2)$, we have the following two diagrams:

$$\begin{array}{ccc} TGB_1 & \xrightarrow{\varepsilon_{B_1}} & B_1 \\ TGg_i \downarrow & & \downarrow g_i \\ TGB_2 & \xrightarrow[\varepsilon_{B_2}]{} & B_2 \end{array} \qquad (i=1,2),$$

which commute. Assume $Gg_1 = Gg_2$, then $g_1\varepsilon_{B_1} = g_2\varepsilon_{B_1}$. Since ε_{B_1} is epi, $g_1 = g_2$. So that G is faithful.

3.6 Corollary Given two functors $\mathscr{A} \underset{G}{\overset{T}{\rightleftarrows}} \mathscr{B}$, assume $\psi \in \mathrm{BH}_T(\mathscr{A}, \mathscr{B})$. If $\varepsilon = (\varepsilon_B): TG \dot{\to} I_{\mathscr{B}}$ is a natural transformation with every component ε_B eqi and if G is full, then ψ has the G-inverse.

3.7 Definition Given two functors $T, S: \mathscr{A} \to \mathscr{B}$, suppose that $\psi \in \mathrm{BH}_T(\mathscr{A}, \mathscr{B})$ a and $\varphi \in \mathrm{BH}_S(\mathscr{A}, \mathscr{B})$. a natural transformation $n: \psi \dot{\to} \varphi$ is a function which assigns to each object A of $\mathscr{A}$ a morphism $n_A: \psi(A) \to \varphi(A)$ of $\mathscr{B}$ in such a way that every morphism $f: A \to A'$ in $\mathscr{A}$ yields a diagram

$$\begin{array}{ccc} \psi(A) & \xrightarrow{n_A} & \varphi(A) \\ \psi(f) \downarrow & & \downarrow \varphi(f) \\ \psi(A') & \xrightarrow[n_{A'}]{} & \varphi(A') \end{array}$$

which commutes.

Let $\mathscr{A}$ be a category, the class of all objects of $\mathscr{A}$ is denoted by $O_{\mathscr{A}}$.

3.8 Definition Given two categories $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{A}$ is said to be equal on morphisms to $\mathscr{B}$, if there exists a bijective map $n: O_{\mathscr{A}} \to O_{\mathscr{B}}$ such that $(A_1\ A_2) = (nA_1, nA_2)$ and if the two compositions of morphisms are the same. When this holds, we write $\mathscr{A} \overset{M}{\cong} \mathscr{B}$.

3.9 Proposition Given two functors $G, S: \mathscr{B} \to \mathscr{A}$, suppose that $\psi \in \mathrm{BH}_T(\mathscr{A}, \mathscr{B})$ and both $\bar{\psi}_G^1$ and $\bar{\psi}_S^1$ exist, then $n = (n_B)_{B \in \mathrm{ob}\mathscr{B}}: \bar{\psi}_G^1 \dot{\to} \bar{\psi}_S^1$ is a natural transformation if and only if $n = (n_B)_{B \in \mathrm{od}\mathscr{B}}: G \dot{\to} S$ is a natural transformation.

Proof $(\Leftarrow:)$ Assume that $n = (n_B): G \dot{\to} S$ is a natural tranformation. Given B, $B' \in \mathrm{ob}\mathscr{B}$ and $f: B \to B' \in$ we have $(Sf)n_B = (n_{B'})Gf$. So $(TSf)Tn_B = (Tn_{B'})TGf$. Since $TSf = \psi(\bar{\psi}_S^1(f))$ and $TGf = \psi(\bar{\psi}_G^1(f))$, holds that $\psi(\bar{\psi}_S^1(f))Tn_B = (Tn_{B'})\psi(\bar{\psi}_G^1(f))$, that is, $\psi(\bar{\psi}_S^1(f)n_B) = \psi(n_{B'}\bar{\psi}_G^1(f))$. So $\bar{\psi}_S^1(f)n_B = n_{B'}\bar{\psi}_G^1(f)$. This means that $n = (n_B): \bar{\psi}_G^1 \dot{\to} \bar{\psi}_S^1$ is a natural transformation.

$(\Rightarrow:)$ Reversing the above discussion and remarking that the functor T is fa faithful, we complete the proof.

To be explicit, there is a category consisting of (1) objects, all the invers ses of ψ, (2) morphisms, all natural transformations between two objects, and (3) the vertical composition of two natural transformations. The category is s denoted by $\bar{\psi}^1$. By Proposition 3.3 and Remark 3.2, there is a bijection from the class of all inverses of ψ to the class $\{s \mid s: \mathscr{B} \to \mathscr{A}$ is both full and faithful$\}$, $\bar{\psi}_S^1 \mapsto S$. In addition, by Proposition 3.9 we have $\bar{\psi}^1(\bar{\psi}_S^1, \bar{\psi}_E^1) = \{n \mid n: S \dot{\to} G$ is a natu ral tranformation $\} = \mathscr{A}^{\mathscr{B}}(S,G)$, so $\bar{\psi}^1$ is equal on morphisms to a category which is a full subcategory of $\mathscr{A}^{\mathscr{B}}$ and possesses the objects, all full and faithful functors: $\mathscr{B}\to\mathscr{A}$. We denote the full subcategory of $\mathscr{A}^{\mathscr{B}}$ by $FF\mathscr{A}^{\mathscr{B}}$. Sincc ψ is an arbitrary bijective half functor in $BH_T(\mathscr{A}, \mathscr{B})$ and the above statement holds for any functor $T:\mathscr{A}\to\mathscr{B}$, we obtain the following proposition.

3.10 Proposition For every functor functor $T:\mathscr{A}\to\mathscr{B}$ which is full and faithful, if $\psi\in BH_T(\mathscr{A}, \mathscr{B})$, then $\bar{\psi}^1 \overset{M}{\cong} FF\mathscr{A}^{\mathscr{B}}$.

4. Surjectiveness of the Mapping $\psi\mapsto\bar{\psi}^1_G$

This paragraph deals with the following question:

For each bijective half functor $\varphi\in BH_G(\mathscr{B}, \mathscr{A})$, is there a bijective half functor $\psi\in BH_T(\mathscr{A}, \mathscr{B})$ such that $\varphi=\bar{\psi}^1_G$?

4.1 Proposition Let $\psi\in BH_T(\mathscr{A}, \mathscr{B})$. If $\varphi=\bar{\psi}^1_G$ then φ satisfies the following condition: If $GB=GB'$ then $\varphi(1_B)=\varphi(1_{B'})$.

Proof Let $GB=GB'=A$, we have $\varphi(1_B)=\psi^{-1}(TG1_B)=\psi^{-1}(T1_{GB})=\psi^{-1}(1_{TA}):A\to A$; $\varphi(1_{B'})=\psi^{-1}(TG1_{B'})=\psi^{-1}(1_{TA}):A\to A$, so that $\varphi(1_B)=\varphi(1_{B'})$, q.e.d.

4.2 Proposition Given a functor $G:\mathscr{B}\to\mathscr{A}$ which is quasifull on objects, suppose $\varphi\in BH_G(\mathscr{B}, \mathscr{A})$, then for each full and faithful functor $T:\mathscr{A}\to\mathscr{B}$ there is a unique function $\psi\in BH_T(\mathscr{A}, \mathscr{B})$ such that $\varphi=\bar{\psi}^1_G$.

Proof First, let us define an appropriate function $\psi\in BH_T(\mathscr{A}, \mathscr{B})$. Since G is quasifull on objects, for each object $A\in ob\mathscr{A}$ there is a nonempty class $C_A=\{B|B\in ob\mathscr{B}$ and exists an isomorphism $a:GB\to A\}$. Hence by the axion of choice there is a mapping $C:ob\mathscr{A}\to\bigcup_{A\in ob\mathscr{A}} C_A$ such that $C(A)\in C_A$. We define a function $\psi:\mathscr{A}\to\mathscr{B}$ as follows:

(1) $\forall A\in ob\mathscr{A}: \psi(A)=TA$,

(2) Given $A_1, A_2\in ob\mathscr{A}$ and $f:A_1\to A_2$, we denote $C(A_i)$ by B_i, $i=1,2$, then there is an isomorphism a_i with $a_i\beta_i=1$ and $\beta_i a_i=1$. Since $\varphi:\mathscr{B}(B_1, B_2)\to\mathscr{A}(GB_1, GB_2)$ is bijective, there is a unique $g\in\mathscr{B}(B_1, B_2)$ shch that $\varphi(g)=\beta_2 f a_1$. We define $\psi(f)=T(a_2(Gg)\beta_1):TA_1\to TA_2$. Then $\psi(\beta_2 f a_1)=(T\beta_2)(\psi f)(Ta_1)=TGg$.

Next, we are going to prove that $\psi\in BH_T(\mathscr{A}, \mathscr{B})$.

1. First, we are going to show that for any $A_1, A_2\in ob\mathscr{A}$, $\psi:\mathscr{A}(A_1, A_2)\to\mathscr{B}(TA_1, TA_2)$ is bijective. From now on, we always write B_i fof $C(A_i)$. Suppose $f_1, f_2\in\mathscr{A}(A_1, A_2)$ and $\psi(f_1)=\psi(f_2)$, then $(Ta_2)(TGg_1)(T\beta_1)=(Ta_2)(TGg_2)(T\beta_1)$, where $g_i\in\mathscr{B}(B_1, B_2)$ and $\varphi(g_i)=\beta_2 f_i a_1$, $i=1,2$. Since T is faithful, we have $a_2(Gg_1)\beta_1=a_2(Gg_2)\beta_1\cdot\beta_1$ being epi and a_2 being monic, we have $Gg_1=Gg_2$. Hence $g_1=g_2$ for G is faithful. Therefore, $f_1=f_2$, so that ψ is injective. On the other hand, assume $h\in\mathscr{B}(TA_1, TA_2)$, then there is a unique $g:B_1\to B_2$ such that $TGg=$

$(T\beta_2)h(Ta_1)$, for TG is full and faithful. Then $(Ta_2)(TGg)(T\beta_1)=h$, and hence $\psi(f)=(Ta_2)(TGg)(T\beta_1)=h$, where $f=a_2\varphi(g)\beta_1:A_1\to A_2$, so that ψ is surjective.

2. Given $h\in\mathscr{A}(A_1,A_2)$ and $s\in\mathscr{A}(A_2,A_3)$, we denote $C(A_i)$ by B_i, $i=1,2,3$. Then we have both $g\in\mathscr{B}(B_1,B_2)$ and $r\in\mathscr{B}(B_2,B_3)$ such that $\varphi(g)=\beta_2ha_1$ and $Gr=\beta_3sa_2$, hence $\psi(sh)=\psi(a_3(Gr)\beta_2a_2\varphi(g)\beta_1)=\psi(a_3\varphi(rg)\beta_1)=T(a_3(Grg)\beta_1)=T(a_3(Gr)\beta_2a_2(Gg)\beta_1)=T(a_3(Gr)\beta_2)T(a_2(Gg)\beta_1)=(Ts)(\psi(h))$.

Similarly, we can prove $\psi(sh)=\psi(s)Th$, so that $\psi\in\mathrm{BH}_T(\mathscr{A},\mathscr{B})$.

Further, we are going to prove $\varphi=\bar\psi_G^1$. Given $B^1,B^2\in\mathrm{ob}\mathscr{B}$ and $g\in\mathscr{B}(B^1,B^2)$, let $B_i=C(GB^i)$, $i=1,2$, then there are four isomorphisms $a_i:GB_i\to GB^i$ and $\beta_i:GB^i\to GB_i$ such that $a_i\beta_i=1$ and $\beta_ia_i=1$. As G is full and faithful, there are four morphisms $a_i:B_i\to B^i$ and $b_i:B^i\to B_i$ such that $Ga_i=a_i$ and $Gb_i=\beta_i$. From Proposition 2.1 we know that a_i and b_i are split. G being faithful, hold $a_ib_i=1$ and $b_ia_i=1$. Let $g'=b_2ga_1:B_1\to B_2$, then $\varphi(g')=(Gb_2)(\varphi(g))(Ga_1)=\beta_2(\varphi(g))a_1$, hence $\psi(\varphi(g))=\psi(a_2\varphi(g')\beta_1)=T(a_2(Gg')\beta_1)=T(G(a_2g'b_1))=TGg$, so that $\varphi(g)=\psi^{-1}(TGg):GB^1\to GB^2$.

Finally, we arr going to prove that such a function $\psi\in\mathrm{BH}_T(\mathscr{A},\mathscr{B})$ is unique. Given A_1, $A_2\in\mathrm{ob}\mathscr{A}$, let $B_i=C(A_i)$, $i=1,2$, there are two split epic morphisms $a_1:GB_1\to A_1$ and $a_2:GB_2\to A_2$ such that $a_i\beta_i=1$. Given a morphism $f:A_1\to A_2$, let $t=\psi(\beta_2fa_1)$, there is a morphism $g:B_1\to B_2$ such that $TGg=t$, for TG is full. If there is a function $\phi\in\mathrm{BH}_T(\mathscr{A},\mathscr{B})$ such that $\bar\phi_G^1=\bar\psi_G^1$, then that $\bar\phi_G^1(g)=\bar\psi_G^1(g)$ implies that $\phi^{-1}(TGg)=\psi^{-1}(TGg)=\psi^{-1}(t)=\beta_2fa_1$. Therefore, holds $\phi(\beta_2fa_1)=TGg=t=\psi(\beta_2fa_1)$. That is, $(T\beta_2)(\phi(f))Ta_1=(T\beta_2)(\psi(f))Ta_1$, since $T\beta_2$ is monic and Ta_1 is epi, $\phi(f)=\psi(f)$, so that $\phi=\psi$, q.e.d.

4.3 Corollary If there is a full and faithful functor $G:\mathscr{B}\to\mathscr{A}$ being quasiful on objects and if $\mathrm{BH}_T(\mathscr{A},\mathscr{B})\neq\varnothing$, then each function $\varphi\in\mathrm{BH}_G(\mathscr{B},\mathscr{A})$ satisfies the following condition

$$\text{If } GB=GB' \quad \text{then } \varphi(1_B)=\varphi(1_{B'}).$$

Pemark To be explicit, if $\psi\in\mathrm{BH}_T(\mathscr{A},\mathscr{B})$ and if $\varphi\in\mathrm{BH}_R(\mathscr{B},\mathscr{C})$, then $\varphi\psi\in\mathrm{BH}_{RT}(\mathscr{A},\mathscr{C})$. We have the following proposition:

Proposition Suppose $\psi\in\mathrm{BH}_T(\mathscr{A},\mathscr{B})$ and $\bar\psi_G^1$ exists, then

(1) $\bar\psi_G^1(\overline{\bar\psi_G^1})_T^1\cong I_{\mathscr{A}}$ is a natural isomorphism between two bijective half-functors if and only if $\langle T,G;\eta,\varepsilon\rangle:\mathscr{A}\to\mathscr{B}$ is an adjoint equivalence;

(2) $\bar\psi^1(\overline{\bar\psi_G^1})_T^1\cong I_{\mathscr{A}}$ if and only if $\psi\bar\psi_G^1=I_{\mathscr{B}}$.

(1): ($\Rightarrow$:): By the definition of the inverses we have $\bar\psi_G^1(\overline{\bar\psi_G^1})_T^1=GT$. So $GT\cong I_{\mathscr{A}}:\mathscr{A}\to\mathscr{A}$, hence for each object $A\in\mathrm{ob}\mathscr{A}$ holds $A\cong GB$, where $B=TA$. In addition, G is full and faithful, from [4, IV.4, Th.1 and IV.1. Th. 2(ii)] $\langle T,G;\eta,\varepsilon\rangle:\mathscr{A}\to\mathscr{B}$ is an adjoint equivalence.

($\Leftarrow$:) Observing $\bar\psi_G^1(\overline{\bar\psi_G^1})_T^1=GT$ and the definilion of the adjoint equivalence, the proof is clear .

(2) : ($\Rightarrow$:) Since $\psi_G^1(\overline{\psi_G^1})_T^1 = GT \cong I_{\mathscr{A}}$, $\langle T, G; \eta, \varepsilon\rangle : \mathscr{A} \to \mathscr{B}$ is an adjoint equivalence. So $TG = \psi\overline{\psi}_G^1 \cong I_{\mathscr{B}}$.

($\Leftarrow$:) Since $\psi\overline{\psi}_G^1 = TG \cong I_{\mathscr{B}}$ and T is full and faithful, [4, IV.4. Th.1] shows that $\langle G, T; \eta', \varepsilon'\rangle : \mathscr{B} \to \mathscr{A}$ is an adjoint equivalence. So $GT = \overline{\psi}_G^1(\overline{\psi_G^1})_T^1 \cong I_{\mathscr{A}}$.

Proposition If $\psi \in \mathrm{BH}_T(\mathscr{A}, \mathscr{B})$ and $S : \mathscr{B} \to \mathscr{A}$ is full and faithful, then for every morphism $f : a \to a'$ of $\mathscr{A}$ hold the following

$$[\psi(f)] = [(\overline{\overline{\psi}_S^1})_T^1(f)], \quad \langle \psi(f)\rangle = \langle (\overline{\overline{\psi}_S^1})_T^1(f)\rangle .$$

It holding that $\psi(1)$ is an isomorphism, the proof is clear.

Finally, we know that, in general, there are many bijective half-functors but the operations $a \mapsto a^{-1}$ and $a \mapsto \overline{a}$. In fact, if $u_B : B \to B$ is an isomorphism, then $\psi \in \mathrm{BH}_T(\mathscr{A}, \mathscr{B})$, where $\psi : (A, A') \to (TA, TA') :\ \mapsto u_{TA'} Tf$.

Let us recall what we discuss in the full text, isomorphic as T and ψ are, ψ has some more interesting properties than T .

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Corollary 2.5

$$\mathrm{RF}_{r \cdot c}(n; q) = \prod_{i=1}^{r} \begin{bmatrix} n+c+i-1 \\ c \end{bmatrix} \Big/ \begin{bmatrix} c+i-1 \\ c \end{bmatrix}$$

And dually

$$\mathrm{RF}_{r \cdot c}(n; q) = \prod_{i=1}^{c} \begin{bmatrix} n+r+i-1 \\ r \end{bmatrix} \Big/ \begin{bmatrix} r+j-1 \\ r \end{bmatrix}.$$

Corollary 2.6 (Stanley, 1971).

$$\mathrm{RF}_{\lambda}(\infty; q) = \prod_{(i,j) \in \lambda} \langle h_{ij}\rangle^{-1}$$