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Structure of Quasi-Invariant Vector Spaces *

FENG Hong
(Dept. of Appl. Math., Dalian University of Technology, 116024)

Abstract: Let V be a vector space over a field F and G a group of linear transformations in V. It is proved in this note that for any subspace $U \subseteq V$, if $\dim U/(U \cap g(U)) \leq 1$, for any $g \in G$, then there is a $g \in G$ such that $U \cap g(U)$ is a G-invariant subspace, or there is an $x \in V \backslash U$ such that $U + \langle x \rangle$ is a G-invariant subspace. So a vector-space analog of Brailovsky's results on quasi-invariant sets is given.

Key words: vector space; quasi-invariant.

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The notion of set of quasistabilizers of a subset of a finite group was introduced by Füredi and Kleitman in [4]. As a generalization of this notion, L. Brailovsky introduced in [1] the notion of quasi-invariant sets as follows. A subset $A \subseteq X$ is called quasi-invariant if $|A^g \backslash A| \leq 1$ for any $g \in G$, where G is a group acting on X. Brailovsky characterized the structure of quasi-invariant sets in [1]. It was shown that a quasi-invariant subset is either an invariant subset or an invariant subset with one point added or removed (See also [2]). In this note, we present a vector-space analog of this result. Making the analog of this nature is motivated from the q-analogs in combinatorial properties of finite sets, which have been extensively studied by combinatorists (See, for example, [5] and [3]).

Let V be a vector space over a field F and G a group of linear transformations in V. For a subspace $U \subseteq V$ and $g \in G$ denote $g(U) = \{g(u) : u \in U\}$. A subspace $U \subseteq V$ is said to be G-invariant (or invariant for short) if $g(U) = U$ for any $g \in G$. Clearly, to prove that a subspace U is invariant it suffices to show that $g(U) \subseteq U$ for any $g \in G$.

Definition *A subspace U of V is said to be quasi-invariant if*

$$\dim U/(U \cap g(U)) \leq 1 \quad \text{forany} \quad g \in G, \tag{1}$$

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Biography: FENG Hong (1960-), female, born in Dalian city, Liaoning province. Ph.D, currently an associate professor Dalian University of Technology.

or equivalently

$$\dim g(U)/(U \cap g(U)) \leq 1 \ \text{ forany } \ g \in G. \tag{2}$$

Theorem *If a subspace $U \subseteq V$ is quasi-invariant, one of the following cases holds:*

(i) there is a $g \in G$ such that $U \cap g(U)$ is invariant;

(ii) there is an $x \in V \backslash U$ such that $U + \langle x \rangle$ is invariant.

Proof Put $H = \{g \in G : g(U) \subseteq U\}$. Suppose that U is not invariant, which means $H \neq G$. Then for any g_i in $G \backslash H$, there is an $a_i \in U$ such that $g_i(a_i) = x_i \notin U$. Write $U_i = U \cap g_i^{-1}(U)$. By (1) and (2) we have that

$$U = U_i + \langle a_i \rangle \tag{3}$$

and

$$g_i(U) = g_i(U_i) + \langle x_i \rangle, \tag{4}$$

where $\langle T \rangle$ denotes the subspace generated by T. In the following proof we shall use the above notations for $i = 1, 2, \ldots$.

There are two main cases to be considered in Sections 1 and 2.

1. Assume that there is a $g_1 \in G \backslash H$ such that $a_1 \in g_1(U)$, i.e., $a_1 = g_1(a_1')$ for some $a_1' \in U$.

The proof for this case will be divided into three steps.

1.1. We prove

$$g_1(U + \langle x_1 \rangle) = U + \langle x_1 \rangle. \tag{5}$$

Since both $x_1 = g_1^2(a_1')$ and $g_1(x_1) = g_1^2(a_1)$ are vectors in $g_1^2(U)$ and $x_1 \notin U$ as well, it follows from (2) that $x_1 + U \cap g_1^2(U)$ is a basis of the quotient space $g_1^2(U)/U \cap g_1^2(U)$, and $g_1(x_1) + U \cap g_1^2(U)$ can be linearly represented by $x_1 + U \cap g_1^2(U)$. This means that $g_1(x_1) \in U + \langle x_1 \rangle$, and $g_1(U + \langle x_1 \rangle) \subseteq U + \langle x_1 \rangle$.

On the other hand, since $g_1(x_1) \in U + \langle x_1 \rangle$, there are $u_1 \in U$ and $k_1 \in F$ such that $g_1(x_1) = u_1 + k_1 x_1$ which is not in $g_1(U)$. From this and $x_1 = g_1(a_1) \in g_1(U)$ it follows that $U = U \cap g_1(U) + \langle u_1 \rangle \subseteq g_1(U) + \langle g_1(x_1) \rangle = g_1(U + \langle x_1 \rangle$. So $U + \langle x_1 \rangle \subseteq g_1(U + \langle x_1 \rangle)$. Combining the above gives (5).

1.2. For any $g_2 \in G \backslash H$ with $g_2 \neq g_1$ we show that

$$U + \langle x_1 \rangle = U + \langle x_2 \rangle. \tag{6}$$

There are two subcases to be considered.

1.2.1. Suppose that $a_1 \in g_2(U)$, i.e., $a_1 = g_2(a_1'')$ for some $a_1'' \in U$.

Clearly, $\langle a_1'' \rangle \neq \langle a_2 \rangle$ since $g_2(a_1'') = a_1 \in U$ and $g_2(a_2) = x_2 \notin U$. Set $g_1 g_2 = g_3$. Since $g_3(a_1'') = g_1(a_1) = x_1 \notin U$, (2) implies that $x_1 + U \cap g_3(U)$ is a basis of $g_3(U)/U \cap g_3(U)$. Hence $g_1(x_2) = g_3(a_2) \in U + \langle x_1 \rangle = g_1(U + \langle x_1 \rangle)$ (from (5)), yielding $x_2 \in U + \langle x_1 \rangle$.

1.2.2. Suppose that $a_1 \notin g_2(U)$. Then $a_1 + U \cap g_2(U)$ is a basis of $U/U \cap g_2(U)$. So $u_1 + U \cap g_2(U)$ can be linearly represented by $a_1 + U \cap g_2(U)$, which means that $u_1 + \lambda a_1 = g_2(a_2')$ for some $\lambda \in F$ and $a_2' \in U$ (recall that $g_1(x_1) = u_1 + k_1 x_1$). Write $g_1^{-1} g_2 = g_4$. Then we have $g_4(a_2') = x_1 + k_1 a_1 + \lambda a_1' \notin U$ (recall $g_1^{-1}(a_1) = a_1' \in U$). Thus

$g_4(U) = U \cap g_4(U) + \langle g_4(a_2')\rangle \subset U + \langle g_4(a_2')\rangle = U + \langle x_1\rangle$. On the other hand, we have $g_1^{-1}(x_2) = g_4(a_2) \in g_4(U) \subset U + \langle x_1\rangle$, hence $x_2 \in g_1(U + \langle x_1\rangle) = U + \langle x_1\rangle$ (by (5)).

It has been shown that $x_2 \in U + \langle x_1\rangle$ both in subcases. Since $x_2 \notin U$, it follows that $x_1 \in U + \langle x_2\rangle$, which means that (6) holds.

1.3. We prove that $U + \langle x_1\rangle$ is invariant.

We have proved up to now that $U + \langle x_1\rangle = U + \langle x_i\rangle$ for any $g_i \in G\backslash H$, thus

$$g_i(U) = g_i(U \cap g_i^{-1}(U) + \langle a_i\rangle) \subseteq U + \langle x_i\rangle = U + \langle x_1\rangle,$$

and

$$g_i(x_1) = g_i g_1(a_1) \in g_i g_1(U) \subseteq U + \langle x_1\rangle.$$

So $g_i(U + \langle x_1\rangle) \subseteq U + \langle x_1\rangle$ for any $g_i \in G\backslash H$. On the other hand, for any $g \in H$, we have

$$g(x_1) = g g_1(a_1) \in g g_1(U) \subseteq U + \langle x_1\rangle.$$

and it is clear that $g(U + \langle x_1\rangle) \subseteq U + \langle x_1\rangle$. Thus we have proved that $U + \langle x_1\rangle$ is invariant.

2. Assume now that $a_i \notin g_i(U)$ for any $g_i \in G\backslash H$. This implies that

$$U = g_i(U_i) + \langle a_i\rangle \tag{7}$$

holds for any $g_i \in G\backslash H$.

Let us consider two possible subcases.

2.1. Suppose that there exists $g_1 \in G\backslash H$ such that $U_1 \neq g_1(U_1)$. From (3) and (7) we have that there exist $u_1 \in U_1$ and $u_1' \in U_1$ such that $g_1(u_1) = u_1' + k_2 a_1$, where $k_2 \in F\backslash\{0\}$. So $g_1^2(u_1) = g_1(u_1') + k_2 x_1 \notin U$. Let $b_1 = g_1(u_1)$. Then $b_1 \in U$, $b_1 \in g_1(U)$ and $g_1(b_1) \notin U$. Note that we have proved in Section 1 that $U + \langle g_1(b_1)\rangle$ is invariant.

2.2. Suppose that $g_i(U_i) = U_i$ holds for any $g_i \in G\backslash H$. Then two subcases will still be distinguished.

2.2.1. Assume that $U_i = U_1$ for any $g_i \in G\backslash H$. Then $g_i(U_1) = g_i(U_i) \subseteq U = U_1 + \langle a_1\rangle$.

For any $u \in U_1$ and $g \in G$, set $g(u) = u' + k_3 a_1$, where $u' \in U_1$. Then $g_1 g(u) = g_1(u') + k_3 x_1 \in U$, which yields $k_3 = 0$ (recall that $g_1(u') \in U$ and $x_1 \notin U$). This means $g(U_1) \subseteq U_1$. Hence $U_1 = U \cap g_1^{-1}(U)$ is invariant.

2.2.2. Assume that there is a $g_2 \in G\backslash H$ such that $U_1 \neq U_2$. Then

$$U = U_1 + U_2 = g_1(U_1) + g_2(U_2).$$

Let $a_1 = c_1 + c_2$ where $c_i \in g_i(U_i) = U_i \subseteq U, (i = 1, 2)$. Then $g_1(a_1) = g_1(c_1) + g_1(c_2) \notin U$, which implies that $g_1(c_1) \notin U$ or $g_1(c_2) \notin U$.

If $g_1(c_1) \notin U$, then we have $c_1 \in U$, $c_1 \in g_1(U)$ but $g_1(c_1) \notin U$. From Section 1 we have that $U + \langle g_1(c_1)\rangle$ is invariant.

Now we suppose $g_1(c_1) \in U$ but $g_1(c_2) \notin U$. If $c_2 \in g_1(U)$, then the situation is the same as that in the above. So we can suppose that $c_2 \notin g_1(U)$. Thus we have

$$U = U_1 + \langle a_1\rangle = U_1 + \langle c_2\rangle.$$

Let $c_2' \in U$ such that $c_2 = g_2(c_2')$. Then $g_1(c_2) + U \cap g_1g_2(U) = g_1g_2(c_2') + U \cap g_1g_2(U)$ is a basis of $g_1g_2(U)/U \cap g_1g_2(U)$ since $g_1(c_2) \notin U$. By our argument,

$$g_1(x_2) = g_1g_2(a_2) \in U + \langle g_1(c_2)\rangle \subseteq U + g_1(U) = g_1(U) + \langle c_2\rangle.$$

Since $g_1(x_2) \notin g_1(U)$, we have $c_2 \in g_1(U) + \langle g_1(x_2)\rangle$. To sum up, we have got $c_2 \in g_1(U) + \langle g_1(x_2)\rangle$ from $c_2 \in U \cap g_2(U)$, $g_1(c_2) \notin U$ and $c_2 \notin g_1(U)$. However, the above condition is equivalent to that $c_2 \in U \cap g_2(U)$, $g_1^{-1}(c_2) \notin U$ and $c_2 \notin g_1^{-1}(U)$, so we have that $c_2 \in g_1^{-1}(U) + \langle g_1^{-1}(x_2)\rangle$. Hence $g_1(c_2) \in U + \langle x_2\rangle$ and $x_1 = g_1(a_1) \in g_1(U_1) + \langle g_1(c_2)\rangle \subseteq U + \langle x_2\rangle$. So $U + \langle x_1\rangle = U + \langle x_2\rangle$.

Now we have indeed proved that $U + \langle x_i\rangle = U + \langle x_j\rangle$ holds whenever $U_i \neq U_j$. Since $U_1 \neq U_2$, we have that $U_i \neq U_1$ or $U_i \neq U_2$ for any U_i. So $U + \langle x_i\rangle = U + \langle x_1\rangle$. From Section 1.3 it follows that $U + \langle x_1\rangle$ is invariant.

The proof is complete.

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向量空间中拟不变元素的结构

冯　红

(大连理工大学应用数学系，116024)

摘 要: 设 V 是有限域上 F 的向量空间，G 是 V 上的线性变换群. 本文讨论了 V 中拟不变元素的结构. 即如果 U 是 V 中的拟不变元，则存在 $g \in G$, 使得 $U \cap g(U)$ 是 G- 不变的，或存在 $x \in V \backslash U$, 使得 $V + \langle x\rangle$ 是 G 不变的.