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Maximum Genus and Girth of a Graph *

HUANG Yuan-qiu[1], *LIU Yan-pei*[2]

(1. Dept. of Math., Hunan Normal University, Changsha 410081;
2. Dept. of Math., Northern Jiaotong University, Beijing 100044)

Abstract: It is known (for example, see [4]) that the maximum genus of a graph is mainly determined by the Betti deficiency of the graph. In this paper, we establish a best upper bound on the Betti deficiency of a graph bounded by its independence number and girth, and thus immediately obtain a new result on the maximum genus.

Key words: maximum genus; Betti deficiency; independece number; girth.

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1. Introduction

We consider only finite, simple, connected graphs. For terminology and notation not defined here, we refer to [1].

Let $G=(E,V)$ be a graph. The maximum genus, denoted by $r_M(G)$, of a graph G is the maximum integer number k with the property that there exists a cellular embedding of G on the orientable surface (a compact and connected 2-manifold without boundary) of genus k.

Since any cellular embedding of G must have at least one face, the Euler polyhedral equation implies an upper bound on the maximum genus

$$r_M(G) \le [\frac{|E(G)|-|V(G)|+1}{2}].$$

The mumber $|E(G)|-|V(G)|+1$ is known as the Betti number (or cycle rank) of the graph G, and is denoted by $\beta(G)$. A graph G is said to be upper embeddable if $r_M(G)=[\frac{\beta(G)}{2}]$ exactly.

For details concerning the maximum genus of graphs, the read may refer to [3] or Chapter 12 of [2].

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Biography: HUANG Yuan-qiu (1966-), male, born in Anxiang county, Hunan province. Ph.D.

The maximum genus (also, upper embeddability) of graphs has received considerable attention after Nordhaus, Stewar and White[3] defined it. One of the most interesting question is the relationship between maximum genus and other graph invariants, or to study the lower bound on the maximum genus of variety of classes of graphs. It sees that to show that a graph is upper embeddable is equivalent to deriving a lower bound $r_M(G) = [\frac{\beta(G)}{2}]$ for the maximum genus. Many papers, such as [4–8], have shown various classes of graphs that are upper embeddable. M.Skoviera[10] has given the lower bound of the maximum genus of a graph with diameter two. R.Nedeal and M.Skoviera[11] have proved that a graph without loops is upper embeddable if it can be cellularly embedded on some surface with the size of each face not exceeding five. In particular, combining with some known results, recently Chen, et al., in [12] have respectively given the best lower bound on the maximum genus of a k-vertex connected (also, k-edge-connected) simple graph with minimum degree at least three, for $k = 1, 2, 3$, and $k \geq 4$.

The number of vertices in a maximum independent set of a graph G is called the independece number of G, denoted by $\alpha(G)$. The girth, usually denoted by $g(G)$, of G is the length of shortest cycle in G. If G has no cycles, define $g(G) = \infty$.

Just stated as before, the maximum genus $r_M(G)$ of a graph is mainly determined by its Betti deficiency $\xi(G)$ (we will give its definition in the next section). In this paper, our main results are the following two theorems.

Theorem 1 *Let G be a graph. Then* $\xi(G) \leq \frac{2\alpha(G)}{g(G)-1}$.

Theorem 2 *Let G be a graph. Then* $r_M(G) \geq \frac{\beta(G)-m}{2}$, *where* $m = \frac{2\alpha(G)}{g(G)-1}$.

In the last section, we show the bounds in the two theorems above are best possible.

2. Two previous results on $r_M(G)$

In the study of the maximum genus, we shall mention two basic and useful results. In order to restate them, we first explain some notations.

Let G be a graph and let T be a spanning tree of G. We let $\xi(G,T)$ denote the mumber of components of $G \backslash E(T)$ with odd size, i.e. odd number of edges (for any subset $X \subseteq E(G)$, $G \backslash X$ denotes the graph obtained from G by removing all edges in X), and call $\xi(G) = \min_T \xi(G,T)$ the Betti deficiency of G, where T is taken over all spanning trees of G. Again, for any subset $A \subseteq E(G)$, denote by $c(G \backslash A)$ the number of components of $G \backslash A$, and by $b(G \backslash A)$ the number of components of $G \backslash A$ with odd Betti number.

Xuong[4] has first formulated the maximum genus, and also presented a necessary and sufficient condition on the upper embeddability of a graph as follows:

Theorem A *Let G be a graph. Then*

(1) $r_M(G) = \frac{\beta(G)-\xi(G)}{2}$;

(2) G is upper embeddable if and only if $\xi(G) \leq 1$.

Clearly, Theorem A(1) makes clear that the maximum genus $r_M(G)$ is mainly determined by the invariant $\xi(G)$, on which, afterwards, [9] gave another complete combinatorial characterization, i.e., the following theorem.

Theorem B *Let G be a graph. Then* $\xi(G) = \max_{A \subseteq E(G)}\{c(G \backslash A) + b(G \backslash A) - |A| - 1\}$.

3. Lemmas and the proofs of the theorems

Lemma 1 *Let G be a graph with a cut-edge e, and let G_1 and G_2 be two components of $G\backslash e$. Then $\xi(G)=\xi(G_1)+\xi(G_2)$.*

Proof By the definition of $\xi(G)$, there exists a spanning tree T of G so that $\xi(G)=\xi(G,T)$. Assume that T_1 and T_2 are two components of $T\backslash e$, where T_1 and T_2 may be viewed as two spanning trees of G_1 and G_2, respectively. We then observe that a component in $G\backslash E(T)$ is with odd size if and only if it is so in either $G_1\backslash E(T_1)$ or $G_2\backslash E(T_2)$. This thus implies the following:

$$\xi(G_1)+\xi(G_2)\le \xi(G_1,T_1)+\xi(G_2,T_2)=\xi(G,T)=\xi(G).$$

Conversely, let T_1 and T_2 be respectively two spanning trees of G_1 and G_2. Then $T=T_1\cup T_2\cup e$ is a spanning tree of G. By the same reason as above, we have

$$\xi(G)\le \xi(G,T)=\xi(G_1,T_1)+\xi(G_2,T_2)=\xi(G_1)+\xi(G_2).$$

Thereby, the proof is complete. □

Lemma 2 *Let G be a graph with $\xi(G)\ge 2$. Then there exists an edge $e\in E(G)$ such that*

(1) either $\xi(G)\le\xi(G\backslash e)-1$, where $G\backslash e$ is connected; or

(2) $\xi(G)=\xi(G')+\xi(F)$ with the property that $\xi(F)\le 1$, where G' and F are the two components of $G\backslash e$.

Proof By Theorem B, there exists a subset $A\subseteq E(G)$ such that

$$\xi(G)=c(G\backslash A)+b(G\backslash A)-|A|-1. \qquad (*)$$

Then we first note that $c(G\backslash A)\ge 2$. Otherwise, if it is not the case, since obviously $b(G\backslash A)\le c(G\backslash A)$, it easily follows from $(*)$ that $\xi(G)\le 1$, contradicting the assumptation $\xi(G)\ge 2$. And, we also get that $A\ne\emptyset$, since G is connected and $c(G\backslash A)\ge 2$.

We now deal with the following two cases:

Case 1. There exists an edge $e\in A$ such that $G\backslash e$ is connected. In this case, set $A'=A\backslash\{e\}\subseteq E(G\backslash e)$. We then see that all the components of $(G\backslash e)\backslash A'$ are identical with those of $G\backslash A$. Hence, combining Theorem B and $(*)$, we have

$$\begin{aligned}\xi(G\backslash e)&\ge c((G\backslash e)\backslash A')+b((G\backslash e)\backslash A')-|A'|-1\\&=c(G\backslash A)+b(G\backslash A)-|A|-1+1\\&=\xi(G)+1,\end{aligned}$$

which shows the conclusion (1) holds.

Case 2. For any edge $e\in A, G\backslash e$ is disconnected. In this case, we first prove the following two claims.

Claim 1. For any edge $e\in A$, the two end vertices of e are in two distinct components of $G\backslash A$.

Proof By contradiction. Assume that there exists $e \in A$ whose two end vertices, say x_1 and x_2, are in the same component F of $G\backslash A$. Since F is connected, there exists a path in F from x_1 to x_2, and thus e lies on a cycle of F (also, of G), contradicting the case that $G\backslash e$ is disconnected.

Claim 2. There exists a component F of $G\backslash A$ such that there is a unique edge $e \in A$, connecting F with another component of $G\backslash A$.

Proof Assume to contrary that Claim 2 is invaild. Since G is connected, this implies that for any component F of $G\backslash A$, there are at least two edges $e_1, e_2 \in A$, connecting F with other components of $G\backslash A$. Then, we can get a cycle of G, composed of some edges of A, and possibly some edges of components of $G\backslash A$. It thus follows that there exists a edge $e \in A$ whose removal preserves the connectivity. A similar contradiction to Claim 1!

Now, choose e and F such ones as in Claim 2. In fact, Claim 2 says that e is a cut-edge of G and that F is one component of $G\backslash e$. Let G' be the other component of $G\backslash e$. Since e is a cut-edge of G, by Lemma 1,

$$\xi(G) = \xi(G') + \xi(F).$$

Put $A' = E(G') \cap A$. By Claim 1, $E(F) \cap A = \emptyset$, and so $|A'| = |A\backslash\{e\}| = |A| - 1$. We also note that all the components of $G\backslash A$, except for F, are those of $G'\backslash A'$, and thus $c(G'\backslash A') = c(G\backslash A) - 1$. On the other hand, no matter whether $\beta(F)$ is odd or not, we have $b(G'\backslash A') \geq b(G\backslash A) - 1$. Therefore, by Theorem B,

$$\begin{aligned} \xi(G') &\geq c(G'\backslash A') + b(G'\backslash A') - |A'| - 1 \\ &\geq c(G\backslash A) + b(G\backslash A) - |A| - 1 - 1 \\ &= \xi(G) - 1. \end{aligned}$$

Considering that $\xi(G) = \xi(G') + \xi(F)$, we immediately get that $\xi(F') \leq 1$, which implies the conclusion 2 holds. □

Lemma 3 *Let G be a graph and let $G\backslash e$ be connected for some edge e. Then*
(1) $g(G) \leq g(G\backslash e)$; (2) $\alpha(G\backslash e) \leq \alpha(G) + 1$.

Proof By the definition of girth, (1) is direct. The truth of (2) is also clear from the fact that adding an edge joining two vertices in a graph leads to the independence number decreasing at most one.

Lemma 4 *Let G be a graph but not a tree, and let x be any vertex of G. Then there exists an independence set $J \subseteq V(G)$ such that $x \notin J$ and $\frac{2|J|}{g(G)-1} \geq 1$.*

Proof Since G is not a tree, there exists a cycle C with the length $g(G)$ in G. Let $C = y_1 y_2 \cdots y_k y_1$, where $k = g(G)$. By the definition of girth, any two vertices y_i and y_j, but not successive on C, are nonadjacent. If $x \in V(C)$, we may choose an independent set $J = \{y_1, y_3, \cdots, y_{2i+1}, \cdots, y_{k-1}\}$ when k is even, or $\{y_1, y_3, \cdots, y_{2i+1}, \cdots, y_k\}$ when k

is odd. Clearly, J is what we need. If $x \in V(G)$, we may relabel the vertices on C, and it easily sees that the conclusion holds as well. □

Now, we return to the proofs of our Theorems.

Proof of Theorem 1 The method is by induction on the number of edges of G. We first see that if G is a tree (also, implying $g(G) = \infty$), by the definition of $\xi(G)$, then $\xi(G) = 0$, and the conclusion is trivival. If $\xi(G) = 1$, clearly G is not a tree, and thus it is easily verified from Lemma 4 that the conclusion is also correct.

In the following, we may assume that $\xi(G) \geq 2$ (necessarily G is not a tree), and furthermore assume that the conclusion of the theorem holds for a graph with less number of edges than that of G. We shall consider two cases according to Lemma 2.

Case 1. There exists an edge $e \in E(G)$ such that $\xi(G) \leq \xi(G\backslash e) - 1$, where $G\backslash e$ is connected. Then we have the following inequalities:

$$\begin{aligned} \xi(G\backslash e) - 1 &\leq \frac{2\alpha(G\backslash e)}{g(G\backslash e) - 1} - 1 \quad \text{(by the inductive hypothesis)} \\ &\leq \frac{2\alpha(G\backslash e) + 2}{g(G) - 1} - 1 \quad \text{(by Lemma 3)} \\ &\leq \frac{2\alpha(G\backslash 2) + g(G) - 3}{g(G) - 1} \\ &\leq \frac{2\alpha(G)}{g(G) - 1} \quad \text{(because } G \text{ is simple and } g(G) \geq 3). \end{aligned}$$

Therefore, $\xi(G) \leq \frac{2\alpha(G)}{g(G)-1}$.

Case 2. There exists an edge $e \in E(G)$ such that $\xi(G) = \xi(G') + \xi(F)$ with the property $\xi(F) \leq 1$, where G' and F are the two components of $G\backslash e$. In this case, we note that there are the following facts:

$$g(G) \leq g(G'), g(G) \leq g(F), \text{ and } \alpha(G') \leq \alpha(G).$$

On the other hand, by the inductive hypothesis, we have

$$\xi(G') \leq \frac{2\alpha(G')}{g(G') - 1}.$$

If $\xi(F) = 0$, it immediately follows that

$$\xi(G) = \xi(G') \leq \frac{2\alpha(G')}{g(G')} \leq \frac{2\alpha(G)}{g(G) - 1}.$$

If $\xi(F) = 1$, obviously F is not a tree. Let x be one and vertex of e in F. According to Lemma 4 there exits an independent set J of F such that $x \notin J$ and $1 \leq \frac{2|J|}{g(F)-1}$. Furthermore, since e is a cut-edge joining G' and F, and $x \notin J$, then $\alpha(G') + |J| \leq \alpha(G)$. Thus, we also have the following:

$$\xi(G) = \xi(G') + 1 \leq \frac{2\alpha(G')}{g(G') - 1} + \frac{2|J|}{g(F) - 1} \leq \frac{2\alpha(G)}{g(G) - 1}.$$

Thereby, by the inductive hypothesis the proof is finished. □

Proof of Theorem 2 Combined Theorem A(1) and Theorem 1, it is straightforward.

Since the girth $g(G) \geq 2$ for any simply graph G, the following result is immediate.

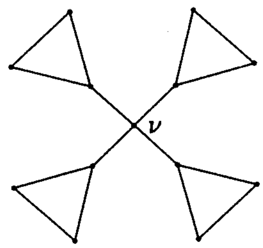
Corollary *Let G be a graph. Then $\xi(G) \leq \alpha(G)$ and $r_M(G) \geq \frac{\beta(G)-\alpha(G)}{2}$.*

4. The bounds in the Theorems

In the following, we shall give two facts to show that the bound in Theorem 1 is best possible, and hence so is in Theorem 2 because of Theorem A(1).

Fact 1. The bound in Theorem 1 is achieved by any a k-cycle C with the odd length k. This is because $\xi(C) = 1$ and $\alpha(C) = \frac{1}{2}(k-1) = \frac{1}{2}(g(C)-1)$.

We now give another fact. Let G be a star, i.e., a complete bipartitie graph $K_{1,m}$, where the vertex adjacent to all the other m vertices is called the central vertex, denoted by v. Let $v_1, v_2, \cdots, v_m$ be the m vertices adjacent to v. We produce a new graph by replacing each v_i, for $1 \leq i \leq m$, with a k-cycle C_i with the odd length $k(\geq 3)$. Denote the resulting graph by N_k^m. The following Figure depicts a graph N_k^m for $m = 4$ and $k = 3$.



Fact 2. For any small positive number ε, there exists infinitly many graphs N_k^m such that

$$\xi(N_k^m) + \varepsilon > \frac{\alpha(N_k^m)}{g(N_k^m) - 1}.$$

We illustrate it by the following. First, we observe that for any a spanning tree T of $N_k^m, \xi(N_k^m, T) = m$, and thus $\xi(N_k^m) = m$. On the other hand, clearly $g(N_k^m) = k$, and furthermore it easily shows that $\alpha(N_k^m) = \frac{k-1}{2}m + 1$. Therefore, we have

$$\frac{2\alpha(N_k^m)}{g(N_k^m) - 1} = \frac{m(k-1)+2}{k-1} = m + \frac{2}{k-1}.$$

Since $\lim_{k\to\infty} \frac{2}{k-1} = 0$, the truth is clear.

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图的最大亏损及围长

黄元秋[1], 刘彦佩[2]

(1. 湖南师范大学数学系，长沙 410081; 2. 北方交通大学数学系，北京 100044)

摘 要: 图的最大亏损主要由其参数 Betti 亏数确定 (例如，见 [3]). 本文给出了由图的独立数及围长所确定的 Betti 亏数的一个最好上界，从而即可得到关于图的最大亏格的一个新结果.