第3卷 第1期　　数学研究与评论　　Vol.3 No.1
1983年 1月　　JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION　　Jan. 1983

Fixed Points of Order Convex Maps in Ordered Banach Spaces*

Huang Chunchao (黄春朝)

(Shandong University)

In this paper we discuss problems of the existence of fixed points of order convex maps.

Lemma 1 Let (E,P) be an OBS, $S_1=\{x\in P\,|\,\|x\|=1\}$. Suppose $f: S_1\to S_1$ is a completely continuous map. Then f has fixed points in S_1.

Lemma 2 Let (E,P) be an OBS whose positive cone is normal and has nonempty interior. Suppose $f: P\to P$ is a continuous, order increasing convex map, $f(0)=0$; $\exists 0\ll\bar{x}<\bar{y}$ such that $0\leqslant f(\bar{x})\ll\bar{x}$, $f(y)\gg\bar{y}$. Then there exist a convex subset $H\subset P$ and a positive number r such that $H\supset B_r$ ($B_r=\{x\in p\,|\,\|x\|\leqslant r\}$) and $f(H)\subset H$, $f(\partial H)\subset\partial H$.

Lemma 3 For the set H in Lemma 2, $T: x\longmapsto x/\|x\|$ is a homeomorphism from $\partial H\supset\dot{P}$ onto $S_1\cap\mathring{P}$.

Lemma 4 Let $f: \partial H\to\partial H$ be completely continuous, $u\in\mathring{P}$, $\|u\|=1$, then for any $0<\varepsilon<\frac{1}{4}$, there exists $x_\varepsilon\in\partial H\cap\mathring{P}$ such that

$$\frac{f(x_\varepsilon)+2\varepsilon\|f(x_\varepsilon)\|u}{\|f(x_\varepsilon)+2\varepsilon\|f(x_\varepsilon)\|u\|}=\frac{x_\varepsilon}{\|x_\varepsilon\|}. \tag{1}$$

Proof Put $P_\varepsilon(u)=\{x\in P\,|\,x\geqslant\varepsilon\|x\|u\}$, then $P_\varepsilon(u)$ is a cone and $\partial H\cap P_\varepsilon(u)$ is bounded. Obviously $B: x\longmapsto\dfrac{f(T^{-1}x)+2\varepsilon\|f(T^{-1}x)\|u}{\|f(T^{-1}x)+2\varepsilon\|f(T^{-1}x)\|u\|}$ $(x\in S_1\cap P_\varepsilon(u))$ is completely continuous from $S_1\cap P_\varepsilon(u)$ into $S_1\cap P_\varepsilon(u)$.

Therefore, by Lemma 1, there exists $x'_\varepsilon\in S_1\cap P_\varepsilon(u)$ such that

$$\frac{f(T^{-1}x'_\varepsilon)+2\varepsilon\|f(T^{-1}x'_\varepsilon)\|u}{\|f(T^{-1}x'_\varepsilon)+2\varepsilon\|f(T^{-1}x'_\varepsilon)\|u\|}=x'_\varepsilon. \tag{1'}$$

Let $x_\varepsilon=T^{-1}x'_\varepsilon$, then $x'_\varepsilon=Tx_\varepsilon=x_\varepsilon/\|x_\varepsilon\|$. Consequently, x_ε satisfies (1).

Theorem 1 Let $f: P\to P$ be a completely continuous, order increasing convex map, $f(0)=0$. Suppose the following three conditions are satisfied:

(i) $\exists 0\ll\bar{x}<\bar{y}$ such that $f(\bar{x})\ll\bar{x}$, $f(\bar{y})\gg\bar{y}$;

(ii) $\exists u\in\mathring{P}$, $\|u\|=1$ and $a>0$ such that $f^2(x)\geqslant a\|f^2(x)\|u$, $\forall x\in P$;

(iii) There exists a continuous function $\varphi: R_+\to R_+$ such that $f(tx)\leqslant\varphi(t)f(x)$, $\forall x\in P$, $t\in R_+$.

*Received July 16, 1981.

Then f has at least one fixed point in $\mathring{P}$.

Theorem 2 Let $f: P\to P$ be a completely continuous, order increasing convex map, $f(0)=0$. Suppose that the following two conditions are satisfied:

(i)′ $\exists 0\ll\bar{x}<\bar{y}$ such that $f(\bar{x})\ll\bar{x}$, $f(\bar{y})\gg\bar{y}$;

(ii)′ $\exists u\in\mathring{P}, \|u\|=1$ and $a>0$ such that $f(x)\geqslant a\|f(x)\|u$.

Then f has at least one point in $\mathring{P}$.

Proof of theorem 1 and 2 Under the conditions of theorem 1 or theorem 2, it is easy to show, by lemmas 1-4, that for any $0<\varepsilon<\frac{1}{4}$, there exists $x_\varepsilon\in\partial H$ satisfying (1), and that the set $\{x_\varepsilon\}$ is bounded. Hence, since f is completely continuous, the set $\{f(x_\varepsilon)\}$ is compact. Therefore, we can choose a sequence of number $\varepsilon_n\to 0$ as $n\to+\infty$ such that $f(x_{\varepsilon_n})$ and $\|x_{\varepsilon_n}\|$ are both convergent. By (1), x_{ε_n} is also convergent. We assume that $x_{\varepsilon_n}\to x\in P$ as $n\to+\infty$. Now, putting $x_\varepsilon=x_{\varepsilon_n}$ in formula (1), we obtain (letting $n\to+\infty$) the equality $\frac{f(x)}{\|f(x)\|}=\frac{x}{\|x\|}$. Since ∂H is closed and $f(\partial H)\subset\partial H$, it follows that $x\in\partial H$ and $f(x)\in\partial H$. By lemma 3, we obtain $f(x)=x$. By the condition (ii) of theorem 1 or (ii)′ of theorem 2, we have $x\in\mathring{P}$.

Applying Theorems 1--2 to nonlinear integral equations of Hammerstein type, we obtain correspondently the following

Theorem 3 Let Ω be a bounded closed domain in Euclidean space R^n and $p>1$. Suppose that $K(x,y)$ is non-negative and continuous on $\Omega\times\Omega$. and $\int_\Omega K(x,z)K(z,y)\,dz>0$, $\forall(x,y)\in\Omega\times\Omega$. Then equation

$$\int_\Omega K(x,y)\varphi^p(y)\,dy-\varphi(x)=0$$

has at least one positive continuous solution.

Theorem 4 Let Ω be a bounded closed domain in Euclidean space R^n. Suppose $K(x,y)$ is continuous on $\Omega\times\Omega$ and $K(x,y)\geqslant m>0$, and $f: R_+\to R_+$ is a continuous, convex function satisfying

$$\lim_{r\to+0}\frac{f(r)}{r}=0,\qquad \lim_{r\to+\infty}\frac{f(r)}{r}=+\infty.$$

Then equation

$$\int_\Omega K(x,y)f(\varphi(y))\,dy-\varphi(x)=0$$

has at least one positive continuous solution.

I am grateful to Professor Guo Dajun for his guidance.