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Extended Cesàro Operators between Different Bergman Spaces in the Ball

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Abstract In this paper, we obtain the characterizations on μ for $(p,q)-\varphi$ Carleson measure, and discuss the boundedness (and compactness) of the extended Cesàro operators T_g between different weighted Bergman spaces as some application.

Keywords extended Cesàro operator; Bergman space; normal function.

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1. Introduction

Let $\mathbf{B}$ be the open unit ball of $\mathbf{C^n}$, and let $H(\mathbf{B})$ be the set of all holomorphic functions on $\mathbf{B}$. A positive continuous function φ on $[0,1)$ is called normal if there are three constants $0 \le \delta < 1$ and $-1 < a < b$ such that

$$\frac{\varphi(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to1}\frac{\varphi(r)}{(1-r)^a}=0; \tag{1.1}$$

$$\frac{\varphi(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to1}\frac{\varphi(r)}{(1-r)^b}=\infty. \tag{1.2}$$

We extend it to $\mathbf{B}$ by $\varphi(z)=\varphi(|z|)$. For $0<p<\infty$ the weighted Bergman space $A^p_a(\varphi)$ is the space of all functions $f\in H(\mathbf{B})$ for which

$$\|f\|_{p,\varphi}=\Big(\int_{\mathbf{B}}|f(z)|^p\varphi(z)\mathrm{d}v(z)\Big)^{\frac{1}{p}}<\infty.$$

Moreover, Hu [1] shows that

$$\|f\|_{p,\varphi}\simeq|f(0)|+\Big(\int_{\mathbf{B}}|\Re f(z)|^p(1-|z|^2)^p\varphi(z)\mathrm{d}v(z)\Big)^{\frac{1}{p}} \tag{1.3}$$

for all $f\in H(\mathbf{B})$. Here and afterward, the expression $A(f)\simeq B(f)$ means there exists C such that for all f, $C^{-1}A(f)\le B(f)\le CA(f)$, where C stands for finite positive constant whose value may change from line to line but independent of f.

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For $g \in H(\mathbf{B})$, the extended Cesàro operator T_g on $H(\mathbf{B})$ is defined by

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{\mathrm{d}t}{t}, \quad z \in \mathbf{B},$$

where $\Re g(z) = \sum_{j=1}^n z_j \frac{\partial g(z)}{\partial z_j}$ is the radial derivative of g. The boundedness and compactness of T_g on the Bergman spaces have been characterized by many authors [1–3]. Moreover, the same problems of T_g on many function spaces, such as mixed norm spaces, Hardy spaces, Bloch type spaces, Dirichlet type spaces and Zygmund spaces, have been studied [2, 4–9]. Our work is to obtain the necessary and sufficient condition for g such that T_g is bounded or compact from $A_a^p(\varphi_1)$ to $A_a^q(\varphi_2)$ for all $0 < p$, $q < \infty$.

2. Some preliminary results

For $z \in \mathbf{B}$ and $r > 0$, denote by $E(z, r)$ the Bergman ball on $\mathbf{B}$. It is well known that $|E(z, r)| \simeq (1 - |z|^2)^{n+1}$ and

$$|1 - \langle z, w\rangle| \simeq 1 - |z| \simeq 1 - |w|, \quad \varphi(z) \simeq \varphi(w) \quad \text{for } w \in E(z, r). \tag{2.1}$$

Suppose $0 < p, q < \infty$. A finite positive Borel measure μ on $\mathbf{B}$ is called a $(p, q) - \varphi$ Carleson measure if

$$\sup_{a\in\mathbf{B}} \frac{\mu(E(a, r))}{(1 - |a|)^{\frac{(n+1)q}{p}} \varphi(a)^{\frac{q}{p}}} < \infty.$$

Moreover, if

$$\lim_{|a|\to 1} \frac{\mu(E(a, r))}{(1 - |a|)^{\frac{(n+1)q}{p}} \varphi(a)^{\frac{q}{p}}} = 0,$$

then μ is called a vanishing $(p, q) - \varphi$ Carleson measure.

Lemma 2.1 ([10]) *For any $r > 0$, there exists a sequence $\{a^j\} \subseteq \mathbf{B}$ satisfying:*

(1) $\mathbf{B} = \bigcup_{j=1}^\infty E(a^j, r)$;

(2) There is a positive integer N such that each point in $\mathbf{B}$ belongs to at most N of the sets $E(a^j, 2r)$.

Lemma 2.2 *Let $0 < p \le q < \infty$, and let φ be normal. Suppose μ is a finite positive Borel measure on $\mathbf{B}$, then the following statements are equivalent:*

(1) The identity operator $i : A_a^p(\varphi) \to L^q(\mu)$ is bounded;

(2) μ is a $(p, q) - \varphi$ Carleson measure.

Furthermore,

$$\|i\| \simeq \sup_{a\in\mathbf{B}} \frac{\mu(E(a, r))^{\frac{1}{q}}}{(1 - |a|)^{\frac{n+1}{p}} \varphi(a)^{\frac{1}{p}}}. \tag{2.2}$$

Proof (1)⇒(2). For any $a \in \mathbf{B}$, set

$$f_a(z) = \frac{(1 - |a|^2)^\beta}{\varphi(a)^{\frac{1}{p}} (1 - \langle z, a\rangle)^{\frac{n+1}{p}+\beta}}, \quad z \in \mathbf{B}. \tag{2.3}$$

Here β is large enough. Then, $\|f_a\|_{p,\varphi} \le C$ by [1]. (2.1) yields

$$\frac{\mu(E(a,r))}{(1-|a|)^{\frac{(n+1)q}{p}}\varphi^{\frac{q}{p}}(a)} \le C\int_{\mathbf{B}} |f_a(z)|^q \mathrm{d}\mu(z) \le C\|f_a\|_{p,\varphi}^q \le C. \tag{2.4}$$

(2)⇒(1). For $f \in H(\mathbf{B})$, we have

$$\begin{aligned} |f(z)|^p &\le \frac{C}{|E(z,r)|}\int_{E(z,r)} |f(w)|^p \mathrm{d}v(w) \\ &\simeq \frac{1}{\varphi(z)(1-|z|)^{n+1}}\int_{E(z,r)} |f(w)|^p\varphi(w)\mathrm{d}v(w). \end{aligned}$$

Hence,

$$\sup_{z\in E(a,r)} |f(z)|^p \le \frac{C}{\varphi(a)(1-|a|)^{n+1}}\int_{E(a,2r)} |f(w)|^p\varphi(w)\mathrm{d}v(w). \tag{2.5}$$

This implies

$$\begin{aligned} \int_{\mathbf{B}} |f(w)|^q \mathrm{d}\mu(w) &\le \sum_{j=1}^{\infty}\int_{E(a^j,r)} |f(w)|^q\mathrm{d}\mu(w) \\ &\le \sum_{j=1}^{\infty}\mu(E(a^j,r))\Big(\sup_{w\in E(a^j,r)}|f(w)|^p\Big)^{\frac{q}{p}} \\ &\le \sum_{j=1}^{\infty}\frac{\mu(E(a^j,r))}{\varphi(a^j)^{\frac{q}{p}}(1-|a^j|)^{\frac{(n+1)q}{p}}}\Big(\int_{E(a^j,2r)}|f(w)|^p\varphi(w)\mathrm{d}v(w)\Big)^{\frac{q}{p}} \\ &\le NC\Big(\int_{\mathbf{B}}|f(w)|^p\varphi(w)\mathrm{d}v(w)\Big)^{\frac{q}{p}}. \end{aligned}$$

This, together with (2.4), we have (2.2). □

Lemma 2.3 ([1]) *Let $0<q<p<\infty$, and let φ be normal. Suppose μ is a finite positive Borel measure on $\mathbf{B}$, then a necessary and sufficient condition for a constant $G>0$ to exist such that*

$$\Big(\int_{\mathbf{B}}|f(z)|^q\mathrm{d}\mu(z)\Big)^{\frac{1}{q}} \le G\Big(\int_{\mathbf{B}}|f(z)|^p\varphi(z)\mathrm{d}v(z)\Big)^{\frac{1}{p}}$$

for all $f \in A_a^p(\varphi)$ is that $\int_{\mathbf{B}}\hat{\mu}(z)^s\varphi(z)\mathrm{d}v(z) < \infty$, where $\frac{1}{s}+\frac{q}{p}=1$, $\hat{\mu}(z)=\frac{\mu(E(z,1))}{(1-|z|^2)^{n+1}\varphi(z)}$. Furthermore,

$$\Big(\int_{\mathbf{B}}\hat{\mu}(z)^s\varphi(z)\mathrm{d}v(z)\Big)^{\frac{1}{s}} \le CG^q. \tag{2.6}$$

Lemma 2.4 *Let $0<p\le q<\infty$, and let φ be normal. Suppose μ is a finite positive Borel measure on $\mathbf{B}$, then the following statements are equivalent:*

(1) The identity operator $i: A_a^p(\varphi)\to L^q(\mu)$ is compact;

(2) μ is a vanishing $(p,q)-\varphi$ Carleson measure.

Proof (1)⇒(2). For $a\in\mathbf{B}$, define the test function as (2.3), then $\|f_a\|_{p,\varphi}\le C$, and $\{f_a\}$ converges to 0 uniformly on any compact subset of $\mathbf{B}$ as $|a|\to 1$. It follows that

$$0 \le \frac{\mu(E(a,r))}{(1-|a|)^{\frac{(n+1)q}{p}}\varphi^{\frac{q}{p}}(a)} \le C\int_{\mathbf{B}}|f_a(z)|^q\mathrm{d}\mu(z) \to 0, \quad |a|\to 1.$$

(2)⇒(1). For any $r>0$, by Lemma 2.1, we can choose a sequence $\{a^j\}\subseteq \mathbf{B}$ with $|a^j|\to 1$ as $j\to\infty$ satisfying (i) $\mathbf{B}=\bigcup_{j=1}^{\infty}E(a^j,r)$; (ii) There is a positive integer N such that each point $z\in\mathbf{B}$ belongs to at most N of the sets $E(a^j,2r)$. Then for any $\varepsilon>0$, by (2), there exists a positive integer J_0, if $j>J_0$,

$$\frac{\mu(E(a^j,r))}{(1-|a^j|)^{\frac{(n+1)q}{p}}\varphi^{\frac{q}{p}}(a^j)}<\varepsilon. \tag{2.7}$$

Suppose $\{f_k\}$ is any norm bounded sequence in $A^p_a(\varphi)$ and $f_k\to 0$ uniformly on each compact subsets of $\mathbf{B}$. We claim that $\lim_{k\to\infty}\|f_k\|_{L^q(\mu)}=0$. In fact, by (2.5) we obtain

$$\begin{aligned}\|f_k\|^q_{L^q(\mu)}&\le\Big(\sum_{j=1}^{J_0}+\sum_{j=J_0+1}^{\infty}\Big)\frac{\mu(E(a^j,r))}{\varphi(a^j)^{\frac{q}{p}}(1-|a^j|)^{\frac{(n+1)q}{p}}}\Big(\int_{E(a^j,2r)}|f_k(w)|^p\varphi(w)\mathrm{d}v(w)\Big)^{\frac{q}{p}}\\&=I_1+I_2.\end{aligned}$$

On the one hand, for $1\le j\le J_0$, $E(a^j,2r)$ is a compact subset of $\mathbf{B}$, then $I_1<\varepsilon$ if k is sufficiently large. On the other hand, (2.7) yields

$$I_2\le CN\varepsilon\Big(\int_{\mathbf{B}}|f_k(w|^p\varphi(w)\mathrm{d}v(w)\Big)^{\frac{q}{p}}<C\varepsilon.$$

Therefore, $i:A^p_a(\varphi)\to L^q(\mu)$ is compact. □

3. Main results

Theorem 3.1 *Let $g\in H(\mathbf{B})$, and let φ_1, φ_2 be both normal. Then $T_g:A^p_a(\varphi_1)\to A^q_a(\varphi_2)$ is bounded if and only if*

(i) For $0<p\le q<\infty$, $\sup_{a\in\mathbf{B}}\frac{(1-|a|)^{q-\frac{(n+1)q}{p}}\varphi_2(a)}{\varphi_1^{\frac{q}{p}}(a)}\int_{E(a,r)}|\Re g(z)|^q\mathrm{d}v(z)<\infty$. Moreover,

$$\|T_g\|\simeq\sup_{a\in\mathbf{B}}\frac{(1-|a|)^{1-\frac{n+1}{p}}\varphi_2^{\frac{1}{q}}(a)}{\varphi_1^{\frac{1}{p}}(a)}\Big(\int_{E(a,r)}|\Re g(z)|^q\mathrm{d}v(z)\Big)^{\frac{1}{q}}. \tag{3.1}$$

(ii) For $0<q<p<\infty$,

$$\int_{\mathbf{B}}\frac{|\Re g(z)|^{\frac{pq}{p-q}}(1-|z|^2)^{\frac{pq}{p-q}}\varphi_2^{\frac{p}{p-q}}(z)}{\varphi_1^{\frac{q}{p-q}}(z)}\mathrm{d}v(z)<\infty. \tag{3.2}$$

Moreover,

$$\|T_g\|\simeq\Big(\int_{\mathbf{B}}\frac{|\Re g(z)|^{\frac{pq}{p-q}}(1-|z|^2)^{\frac{pq}{p-q}}\varphi_2^{\frac{p}{p-q}}(z)}{\varphi_1^{\frac{q}{p-q}}(z)}\mathrm{d}v(z)\Big)^{\frac{p-q}{pq}}.$$

Proof First, for $f,g\in H(\mathbf{B})$, by direct calculation we see $\Re(T_gf)(z)=f(z)\Re g(z)$. By (1.3) and $T_gf(0)=0$, the operator $T_g:A^p_a(\varphi_1)\to A^q_a(\varphi_2)$ is bounded if and only if there exists C such that

$$\|T_gf\|^q_{q,\varphi_2}\simeq\int_{\mathbf{B}}|f(z)|^q\Re g(z)|^q(1-|z|^2)^q\varphi_2(z)\mathrm{d}v(z)\le C\|f\|^q_{p,\varphi_1} \tag{3.3}$$

for all $f\in A^p_a(\varphi_1)$. Set $\mathrm{d}\mu_g(z)=|\Re g(z)|^q(1-|z|^2)^q\varphi_2(z)\mathrm{d}v(z)$.

(i) For $0 < p \le q < \infty$, Lemma 2.2 means that (3.3) holds if and only if μ_g is a $(p,q)-\varphi_1$ Carleson measure. Furthermore, (3.1) follows by (2.2).

(ii) For $0 < q < p < \infty$, Lemma 2.3 yields that (3.3) holds if and only if

$$\int_{\mathbf{B}} \hat{\mu_g}(z)^s \varphi_1(z)\mathrm{d}v(z) < \infty.$$

Since $\hat{\mu_g}(z) = \frac{\mu_g(E(z,1))}{(1-|z|^2)^{n+1}\varphi_1(z)} \ge \frac{C|\Re g(z)|^q(1-|z|^2)^q\varphi_2(z)}{\varphi_1(z)}$, together with (2.6), we have

$$\Big(\int_{\mathbf{B}} \frac{|\Re g(z)|^{\frac{pq}{p-q}}(1-|z|^2)^{\frac{pq}{p-q}}\varphi_2^{\frac{p}{p-q}}(z)}{\varphi_1^{\frac{q}{p-q}}(z)}\mathrm{d}v(z)\Big)^{\frac{p-q}{p}} \le C\Big(\int_{\mathbf{B}} \hat{\mu}_g(z)^s\varphi_1(z)\mathrm{d}v(z)\Big)^{\frac{1}{s}} \le C\|T_g\|^q. \quad (3.4)$$

Conversely, (1.3) and Hölder's inequality yield

$$\begin{aligned}\|T_gf\|_{q,\varphi_2}^q &\simeq \int_{\mathbf{B}} |f(z)|^q|\Re g(z)|^q(1-|z|^2)^q\varphi_2(z)\mathrm{d}v(z)\\ &\le \Big\{\int_{\mathbf{B}}\Big[\frac{|\Re g(z)|^q(1-|z|^2)^q\varphi_2(z)}{\varphi_1^{\frac{q}{p}}(z)}\Big]^{\frac{p}{p-q}}\mathrm{d}v(z)\Big\}^{\frac{p-q}{p}} \times \Big\{\int_{\mathbf{B}}[|f(z)|^q\varphi_1^{\frac{q}{p}}(z)]^{\frac{p}{q}}\mathrm{d}v(z)\Big\}^{\frac{q}{p}}\\ &\le \Big\{\int_{\mathbf{B}} \frac{|\Re g(z)|^{\frac{pq}{p-q}}(1-|z|^2)^{\frac{pq}{p-q}}\varphi_2^{\frac{p}{p-q}}(z)}{\varphi_1^{\frac{q}{p-q}}(z)}\mathrm{d}v(z)\Big\}^{\frac{p-q}{p}} \cdot \|f\|_{p,\varphi_1}^q \qquad (3.5)\end{aligned}$$

for any $f \in A_a^p(\varphi_1)$. Furthermore, (3.4) and (3.5) show

$$\|T_g\| \simeq \Big\{\int_{\mathbf{B}} |\Re g(z)|^{\frac{pq}{p-q}}(1-|z|^2)^{\frac{pq}{p-q}}\frac{\varphi_2^{\frac{p}{p-q}}(z)}{\varphi_1^{\frac{q}{p-q}}(z)}\mathrm{d}v(z)\Big\}^{\frac{p-q}{pq}}. \quad \square$$

Theorem 3.2 *Let $g \in H(\mathbf{B})$, and let φ_1, φ_2 be both normal. Then $T_g : A_a^p(\varphi_1) \to A_a^q(\varphi_2)$ is compact if and only if*

(i) For $0 < p \le q < \infty$, $\lim_{|a|\to 1} \frac{(1-|a|)^{q-\frac{(n+1)q}{p}}\varphi_2(a)}{\varphi_1^{\frac{q}{p}}(a)}\int_{E(a,r)} |\Re g(z)|^q\mathrm{d}v(z) = 0$.

(ii) For $0 < q < p < \infty$, (3.2) holds.

Proof (i) Set $\mu_g(z)$ as in Theorem 3.1, then

$$\|T_gf\|_{q,\varphi_2}^q \simeq \int_{\mathbf{B}} |f(z)|^q\Re g(z)|^q(1-|z|^2)^q\varphi_2(z)\mathrm{d}v(z) = \int_{\mathbf{B}} |f(z)|^q\mathrm{d}\mu_g(z).$$

Thus, $T_g : A_a^p(\varphi_1) \to A_a^q(\varphi_2)$ is compact if and only if $i : A_a^p(\varphi) \to L^q(\mu_g)$ is compact, which is equivalent to that μ_g is a vanishing $(p,q)-\varphi_1$ Carleson measure if $0 < p \le q < \infty$ by Lemma 2.4.

(ii) The necessity is clear by Theorem 3.1. We will show the sufficiency. For any $\varepsilon > 0$, by (3.2), there is some $\eta \in (0,1)$ such that

$$\int_{\mathbf{B}\setminus\mathbf{B}_\eta} |\Re g(z)|^{\frac{pq}{p-q}}(1-|z|^2)^{\frac{pq}{p-q}}\frac{\varphi_2^{\frac{p}{p-q}}(z)}{\varphi_1^{\frac{q}{p-q}}(z)}\mathrm{d}v(z) < \varepsilon,$$

where $\mathbf{B}_\eta = \{z \in \mathbf{B} : |z| \le \eta\}$. Given any sequence $\{f_j\} \subseteq A_a^p(\varphi_1)$ satisfying $\|f_j\|_{p,\varphi_1} \le 1$ and $f_j(z) \to 0$ uniformly on compact subsets of $\mathbf{B}$, we will show $\lim_{j\to\infty} \|T_gf_j\|_{q,\varphi_2} = 0$. Similarly to the proof of (3.5), we have

$$\begin{aligned}\|T_g f_j\|_{q,\varphi_2}^q &\simeq \Big(\int_{\mathbf{B}_\eta} + \int_{\mathbf{B}\setminus\mathbf{B}_\eta}\Big)|f_j(z)|^q|\Re g(z)|^q(1-|z|^2)^q\varphi_2(z)\mathrm{d}v(z)\\ &\le C_1\sup_{|z|\le\eta}|f_j(z)|^q + C_2\Big(\int_{\mathbf{B}\setminus\mathbf{B}_\eta}|\Re g(z)|^{\frac{pq}{p-q}}(1-|z|^2)^{\frac{pq}{p-q}}\frac{\varphi_2^{\frac{p}{p-q}}(z)}{\varphi_1^{\frac{q}{p-q}}(z)}\mathrm{d}v(z)\Big)^{\frac{p-q}{p}}\|f_j\|_{p,\varphi_1}^q\\ &< C\varepsilon,\end{aligned}$$

if j is large enough. □

References

[1] HU Zhangjian. *Extended Cesàro operators on Bergman spaces* [J]. J. Math. Anal. Appl., 2004, **296**(2): 435–454.

[2] XIAO Jie. *Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball* [J]. J. London Math. Soc. (2), 2004, **70**(1): 199–214.

[3] LI Songxiao, STEVIĆ S. *Riemann-Stieltjes operators between different weighted Bergman spaces* [J]. Bull. Belg. Math. Soc. Simon Stevin, 2008, **15**(4): 677–686.

[4] HU Zhangjian. *Extended Cesàro operators on mixed norm spaces* [J]. Proc. Amer. Math. Soc., 2003, **131**(7): 2171–2179

[5] STEVIĆ S. *On an integral operator on the unit ball in* $\mathbb{C}^n$ [J]. J. Inequal. Appl., 2005, **1**: 81–88.

[6] ZHANG Xuejun. *Weighted Cesàro operators on Dirichlet type spaces and Bloch type spaces of* C^n [J]. Chinese Ann. Math. Ser. A, 2005, **26**(1): 139–150. (in Chinese)

[7] LI Songxiao. *Volterra composition operators between weighted Bergman spaces and Bloch type spaces* [J]. J. Korean Math. Soc., 2008, **45**(1): 229–248.

[8] AVETISYAN K, STEVIĆ S. *Extended Cesàro operators between different Hardy spaces* [J]. Appl. Math. Comput., 2009, **207**(2): 346–350.

[9] FANG Zhongshan, ZHOU Zehua. *Extended Cesáro operators on Zygmund spaces in the unit ball* [J]. J. Comput. Anal. Appl., 2009, **11**(3): 406–413.

[10] ZHU Kehe. *Spaces of Holomorphic Functions in the Unit Ball* [M]. Springer-Verlag, New York, 2005.