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Generalized Cesàro Operator on Dirichlet Spaces

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Abstract: In this paper, we study the boundedness of the generalized Cesàro operator on the weighted Dirichlet spaces

$$\mathcal{D}_\alpha = \left\{ f \in H(D); ||f||^2_{\mathcal{D}_\alpha} = |f(0)|^2 + \int_D |f'(z)|^2 (1-|z|)^\alpha \mathrm{d}m(z) < +\infty \right\},$$

where $-1 < \alpha < +\infty$ and $H(D)$ is the class of all holomorphic functions on the unit disc D.

Key words: weighted Dirichlet space; generalized Cesàro operator; weighted composition operator; boundedness.
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1. Introduction

Let D be the unit disc in the complex plane $\mathbf{C}$, $B(a,r) = \{z \in \mathbf{C}; |z-a| < r\}$ be the open disk centered at a with radius r, and $\mathrm{d}m(z) = \frac{1}{\pi} r \mathrm{d}r \mathrm{d}\theta$ be the normalized Lebesgue area measure on D. We denote by $H(D)$ the class of all holomorphic functions on D. For $-1 < \alpha < +\infty$ and $0 < p < +\infty$, the weighted Dirichlet spaces $\mathcal{D}_\alpha$ and the weighted Bergman spaces A^p_α are defined respectively by

$$\mathcal{D}_\alpha = \left\{ f \in H(D); ||f||^2_{\mathcal{D}_\alpha} = |f(0)|^2 + \int_D |f'(z)|^2 (1-|z|)^\alpha \mathrm{d}m(z) < +\infty \right\}$$

and

$$A^p_\alpha = \left\{ f \in H(D); ||f||^p_{\alpha,p} = \int_D |f(z)|^p (1-|z|)^\alpha \mathrm{d}m(z) < +\infty \right\}.$$

For each complex γ with $\mathrm{Re}\gamma > -1$ and nonnegative integer k, let A^γ_k be defined as the kth coefficient in the expression

$$\frac{1}{(1-x)^{\gamma+1}} = \sum_{k=0}^{+\infty} A^\gamma_k x^k,$$

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so that $A_k^\gamma = \frac{\Gamma(k+\gamma+1)}{\Gamma(k+1)\Gamma(\gamma+1)}$. For $f(z)=\sum_{n=0}^{+\infty} a_n z^n \in H(D)$, the generalized Cesàro operator is defined by

$$\mathcal{C}^\gamma(f)(z)=\sum_{n=0}^{+\infty}\left(\frac{1}{A_n^{\gamma+1}}\sum_{k=0}^{n}A_{n-k}^{\gamma}a_k\right)z^n. \tag{1.1}$$

A direct calculation shows that

$$\mathcal{C}^\gamma(f)(z)=\frac{\gamma+1}{z^{\gamma+1}}\int_0^z f(\zeta)\frac{(z-\zeta)^\gamma}{(1-\zeta)^{\gamma+1}}\mathrm{d}\zeta. \tag{1.2}$$

This operator was introduced in [1] and was proved to be bounded on the Hardy spaces and the weighted Bergman spaces in [2] and [3]. For $\gamma=0$ we see $\mathcal{C}^0=\mathcal{C}$, the classical Cesàro operator. It is well known that the operator $\mathcal{C}$ is bounded on the Hardy spaces[4–7] and on the Bergman spaces[8], as well as on the Dirichlet spaces $\mathcal{D}_\alpha$ when $0<\alpha<1$[9].

We have $\mathcal{D}_\alpha=H^2$ and $\mathcal{D}_\alpha=A^2_{\alpha-2}$ respectively whenever $\alpha=1$ and $\alpha>1$. Recently in [10], S.Stević has proved that the generalized Cesàro operator is bounded on $\mathcal{D}_\alpha$ for $\alpha>1$. In this case the spaces $\mathcal{D}_\alpha$ are the weighted Bergman spaces $A^2_{\alpha-2}$. The purpose of this paper is to close the gap for the remaining values of the chain of the spaces $\mathcal{D}_\alpha$. Our results will extend the results in [9].

In what follows, C will stand for positive constants whose value may change from line to line but not depend on the functions in $H(D)$. The expression $A\sim B$ means $C^{-1}A\le B\le CA$.

2. A weighted composition operator on $\mathcal{D}_\alpha$

Let $t\in[0,1]$ and

$$\phi_t(z)=\frac{tz}{1-(1-t)z},\quad z\in D.$$

It is clear that for each $t\in[0,1]$, $\phi_t(z)$ maps the unit disk into itself. Denote

$$w_t(z)=\frac{\phi_t(z)}{z}=\frac{t}{1-(1-t)z},\quad z\in D.$$

Obviously, $w_t(z)$ is a holomorphic function of z on D. Following [9], for $f\in H(D)$, we define the weighted composition operator

$$T_t(f)(z)=w_t(z)f(\phi_t(z)),\quad t\in(0,1].$$

We will prove that for each $t\in[0,1]$, T_t is a bounded operator on $\mathcal{D}_\alpha$. To do this, we need some auxiliary results contained in the following lemmas.

Lemma 2.1 *If $f\in\mathcal{D}_\alpha$, $0<\alpha<1$, then* $|f(z)|\le\frac{C}{(1-|z|)^{\frac{\alpha}{2}}}||f||_{\mathcal{D}_\alpha}$.

Proof Since $f(z)-f(0)=\int_0^z f'(\zeta)\mathrm{d}\zeta$, by elementary inequalities we obtain

$$|f(z)|\le|f(0)|+\int_0^{|z|}|f'(\zeta)|\mathrm{d}\zeta$$

for each $z \in D$. On the other hand, by the subharmonicity of $|f'(z)|^2$ we get

$$|f'(z)|^2 \leq \frac{4}{(1-|z|)^2} \int_{B(z,\frac{1-|z|}{2})} |f'(w)|^2 \mathrm{d}m(w).$$

For $w \in B(z, \frac{1-|z|}{2})$, we have $1-|z| \sim 1-|w|$ for each $z \in D$. Then

$$\begin{aligned} |f'(z)|^2 &\leq \frac{C}{(1-|z|)^{2+\alpha}} \int_{B(z,\frac{1-|z|}{2})} |f'(w)|^2 (1-|w|)^\alpha \mathrm{d}m(w) \\ &\leq \frac{C}{(1-|z|)^{2+\alpha}} \int_D |f'(w)|^2 (1-|w|)^\alpha \mathrm{d}m(w) \\ &\leq C \frac{||f||^2_{\mathcal{D}_\alpha}}{(1-|z|)^{2+\alpha}}. \end{aligned}$$

So,

$$\int_0^{|z|} |f'(\zeta)| \mathrm{d}\zeta \leq C||f||_{\mathcal{D}_\alpha} \int_0^{|z|} \frac{1}{(1-|\zeta|)^{\frac{2+\alpha}{2}}} \mathrm{d}\zeta \leq C \frac{||f||_{\mathcal{D}_\alpha}}{(1-|z|)^{\frac{\alpha}{2}}}.$$

Thus, we have

$$|f(z)| \leq |f(0)| + C \frac{||f||_{\mathcal{D}_\alpha}}{(1-|z|)^{\frac{\alpha}{2}}} \leq \frac{C}{(1-|z|)^{\frac{\alpha}{2}}} ||f||_{\mathcal{D}_\alpha}.$$

The lemma is proved.

Lemma 2.2[11] *For any $-1 < \lambda < +\infty$ and any real number $\beta > 0$, set*

$$I_{\lambda,\beta}(z) = \int_D \frac{(1-|w|^2)^\lambda}{|1-z\overline{w}|^{2+\lambda+\beta}} \mathrm{d}m(w), \quad z \in D.$$

Then

$$I_{\lambda,\beta}(z) \sim \frac{1}{(1-|z|^2)^\beta} \quad (|z| \to 1^-).$$

Theorem 2.1 *If $f \in \mathcal{D}_\alpha$, $0 < \alpha < 1$, then $||T_t(f)||_{\mathcal{D}_\alpha} \leq Ct^{\frac{\alpha}{2}} ||f||_{\mathcal{D}_\alpha}$.*

Proof For $f \in \mathcal{D}_\alpha$,

$$\begin{aligned} ||T_t(f)||^2_{\mathcal{D}_\alpha} =& |T_t(f)(0)|^2 + \int_D |(w_t(z) f(\varnothing_t(z)))'|^2 (1-|z|)^\alpha \mathrm{d}m(z) \\ \leq& t^2 |f(0)|^2 + 2 \int_D |w_t'(z)|^2 |f(\varnothing_t(z))|^2 (1-|z|)^\alpha \mathrm{d}m(z) + \\ & 2 \int_D |w_t(z)|^2 |(f(\varnothing_t(z)))'|^2 (1-|z|)^\alpha \mathrm{d}m(z) \\ =& t^2 |f(0)|^2 + 2I_1 + 2I_2. \end{aligned} \quad (2.1)$$

We now estimate the integrals I_1 and I_2. A calculation shows that

$$|w_t'(z)| = \frac{(1-t)t}{|1-(1-t)z|^2} \quad \text{and} \quad \frac{1}{1-|\varnothing_t(z)|} \leq \frac{|1-(1-t)z|}{1-|z|}.$$

By Lemma 2.1, we obtain that

$$|f(\varnothing_t(z))| \leq \frac{C}{(1-|\varnothing_t(z)|)^{\frac{\alpha}{2}}} ||f||_{\mathcal{D}_\alpha}.$$

Thus, by Lemma 2.2 we get

$$\begin{aligned}I_1 =& \int_D |w_t'(z)|^2 |f(\o_t(z))|^2 (1-|z|)^\alpha \mathrm{d}m(z)\\ \leq& C||f||_{\mathcal{D}_\alpha}^2 \int_D |w_t'(z)|^2 \frac{(1-|z|)^\alpha}{(1-|\o_t(z)|)^\alpha} \mathrm{d}m(z)\\ =& C||f||_{\mathcal{D}_\alpha}^2 t^2(1-t)^2 \int_D \frac{1}{|1-(1-t)z|^{4-\alpha}} \mathrm{d}m(z)\\ \leq& C||f||_{\mathcal{D}_\alpha}^2 t^2(1-t)^2 \frac{1}{[1-(1-t)^2]^{2-\alpha}}\\ \leq& Ct^\alpha ||f||_{\mathcal{D}_\alpha}^2.\end{aligned}$$

For the second integral I_2, we have

$$\begin{aligned}I_2 =& \int_D |w_t(z)|^2 |(f(\o_t(z)))'|^2 (1-|z|)^\alpha \mathrm{d}m(z)\\ =& \int_D |w_t(z)|^2 \left(\frac{1-|z|}{1-|\o_t(z)|}\right)^\alpha |(f(\o_t(z)))'|^2 (1-|\o_t(z)|)^\alpha \mathrm{d}m(z).\end{aligned}$$

It is easy to see that

$$|w_t(z)|^2 \left(\frac{1-|z|}{1-|\o_t(z)|}\right)^\alpha \leq \frac{t^2}{|1-(1-t)z|^{2-\alpha}} \leq t^\alpha.$$

Hence,

$$\begin{aligned}I_2 \leq& t^\alpha \int_D |f'(\o_t(z))|^2 |\o_t'(z)|^2 (1-|\o_t(z)|)^\alpha \mathrm{d}m(z)\\ =& t^\alpha \int_{\o_t(D)} |f'(z)|^2 (1-|z|)^\alpha \mathrm{d}m(z)\\ \leq& t^\alpha ||f||_{\mathcal{D}_\alpha}^2\end{aligned}$$

since $\o_t$ is univalent on D for each $t \in [0,1]$.

Therefore, (2.1) gives that

$$\begin{aligned}||T_t(f)||_{\mathcal{D}_\alpha}^2 \leq& t^2|f(0)|^2 + 2Ct^\alpha ||f||_{\mathcal{D}_\alpha}^2 + 2t^\alpha ||f||_{\mathcal{D}_\alpha}^2\\ \leq& 3t^\alpha ||f||_{\mathcal{D}_\alpha}^2 + Ct^\alpha ||f||_{\mathcal{D}_\alpha}^2\\ \leq& Ct^\alpha ||f||_{\mathcal{D}_\alpha}^2.\end{aligned}$$

This ends the proof.

3. Main results

Theorem 3.1 *The generalized Cesàro operator is bounded on D_α for $0 < \alpha < 1$.*

Proof In the integral (1.2) we choose a path of integration between 0 and z as

$$\gamma(t) = \o_t(z) = \frac{tz}{1-(1-t)z}, \quad t \in [0,1],$$

so that $\gamma(0)=0$ and $\gamma(1)=z$. We have

$$\mathcal{C}^{\gamma}(f)(z)=(\gamma+1)\int_0^1\frac{1}{t}T_t(f)(z)(1-t)^{\gamma}\mathrm{d}t.$$

From this we obtain

$$||\mathcal{C}^{\gamma}(f)(z)||^2_{\mathcal{D}_\alpha}=|f(0)|^2+|\gamma+1|^2\int_D\left|\int_0^1\frac{1}{t}T_t(f)'(z)(1-t)^{\gamma}\mathrm{d}t\right|^2(1-|z|)^{\alpha}\mathrm{d}m(z).$$

By the generalized Minkowski inequality and Theorem 2.1, we get that

$$\begin{aligned}
&\int_D\left|\int_0^1\frac{1}{t}T_t(f)'(z)(1-t)^{\gamma}\mathrm{d}t\right|^2(1-|z|)^{\alpha}\mathrm{d}m(z)\\
&\le\left\{\int_0^1\frac{1}{t}\left[\int_D|T_t(f)'(z)|^2(1-|z|)^{\alpha}\mathrm{d}m(z)\right]^{\frac{1}{2}}(1-t)^{\mathrm{Re}\gamma}\mathrm{d}t\right\}^2\\
&\le\left\{\int_0^1\frac{(1-t)^{\mathrm{Re}\gamma}}{t}||T_t||_{\mathcal{D}_\alpha}\mathrm{d}t\right\}^2\\
&\le C||f||^2_{\mathcal{D}_\alpha}\left\{\int_0^1(1-t)^{\mathrm{Re}\gamma}t^{\frac{\alpha}{2}-1}\mathrm{d}t\right\}^2\\
&=CB^2(\mathrm{Re}\gamma+1,\frac{\alpha}{2})||f||^2_{\mathcal{D}_\alpha},
\end{aligned}$$

where $B(\cdot,\cdot)$ is the usual beta function. This implies

$$\begin{aligned}
||\mathcal{C}^{\gamma}(f)||^2_{\mathcal{D}_\alpha}&\le|f(0)|^2+|\gamma+1|^2CB^2(\mathrm{Re}\gamma+1,\frac{\alpha}{2})||f||^2_{\mathcal{D}_\alpha}\\
&\le||f||^2_{\mathcal{D}_\alpha}+|\gamma+1|^2CB^2(\mathrm{Re}\gamma+1,\frac{\alpha}{2})||f||^2_{\mathcal{D}_\alpha}\\
&\le C||f||^2_{\mathcal{D}_\alpha}.
\end{aligned}$$

The proof is completed.

Finally, we give a counterexample to illustrate the generalized Cesàro operator is unbounded on D_α for $-1<\alpha\le 0$. An equivalent norm on $\mathcal{D}_\alpha$, in terms of its Taylor coefficients, is

$$||f||^2_{\mathcal{D}_\alpha}\sim\sum_{n=0}^{+\infty}(n+1)^{1-\alpha}|a_n|^2$$

for $f(z)=\sum_{n=0}^{+\infty}a_nz^n$.

We consider the function $f(z)=1\in\mathcal{D}_\alpha$. From (1.1), we easily get

$$\mathcal{C}^{\gamma}(f)(z)=\sum_{n=0}^{+\infty}\frac{\gamma+1}{n+\gamma+1}z^n.$$

Therefore,

$$\sum_{n=0}^{+\infty}(n+1)^{1-\alpha}|a_n|^2=\sum_{n=0}^{+\infty}\frac{|\gamma+1|^2}{|n+\gamma+1|^2}(n+1)^{1-\alpha}=+\infty.$$

This means that the generalized Cesàro operator is unbounded on D_α for $-1 < \alpha \leq 0$.

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加权 Dirichlet 空间上的一般 Cesàro 算子

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摘要: 对加权 Dirichlet 空间

$$\mathcal{D}_\alpha = \left\{ f \in H(D); ||f||^2_{\mathcal{D}_\alpha} = |f(0)|^2 + \int_D |f'(z)|^2 (1-|z|)^\alpha \mathrm{d}m(z) < +\infty \right\}, \quad -1 < \alpha < +\infty,$$

我们研究了其上一般 Cesàro 算子的有界性. 此处 $H(D)$ 表示复平面单位圆盘 D 上全纯函数的全体.

关键词: 加权 Dirichlet 空间; 一般 Cesàro 算子; 加权复合算子; 有界性.