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# A New GLKKM Theorem in L-Convex Spaces with the Application to Fixed Points

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**Abstract** In this paper a new GLKKM theorem in L-convex spaces is established. As applications, a new variational inequality, section theorem, maximal element theorem and fixed point theorem are obtained in L-convex spaces.

**Keywords** L-convex space; variational inequality; section; maximal element; fixed point.

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## 1. Introduction

In 1998, Ben-El-Mechaiekh, et al.[1] introduced and studied the abstract convexity concept and the L-convexity structure on topological spaces. Recently, Ding[2] studied the class KKM$(X, Y)$ of mappings and Himmelberg type fixed point theorems. Ding[3] introduced the GLKKM mapping, and obtained some GLKKM theorems, Ky Fan matching theorems, fixed point theorems and a minimax inequality in L-convex spaces. Ding[4] proved a continuous selection theorem, coincidence theorems, fixed point theorems, a minimax inequality and existence theorems of solutions for generalized equilibrium problems in L-convex spaces. Ding[5] yielded some KKM theorems, coincidence theorems and some fixed point theorems in L-convex spaces. Lu and Tang[6] established an intersection theorem, fixed point theorem, maximal element theorem, coincidence theorem, minimax inequalities and saddle point theorem in L-convex spaces. In this paper, a new GLKKM theorem in L-convex spaces is established. As applications, a new variational inequality, section theorem, maximal element theorem and fixed point theorem are obtained in L-convex spaces. These results unify, improve and generalize some recent results in the reference therein.

## 2. Preliminaries

Let $X$ be a nonempty set. We denote by $\mathcal{F}(X)$ and $2^X$ the family of all nonempty finite subsets of $X$ and the family of all subsets of $X$, respectively. Let $X$, $Y$ be two nonempty sets

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and $F : X \to 2^Y$ a mapping. Then the mapping $F^* : Y \to 2^X$ is defined by $F^*(y) := X \setminus F^{-1}(y)$ for each $y \in Y$. Let $(X, \Gamma)$ be an L-convex space[1−6]. A set $D \subset X$ is said to be L-convex if for each $A \in \mathcal{F}(D)$, $\Gamma(A) \subset D$. We denote by $\mathcal{L}(X)$ the family of all nonempty L-convex subsets of $X$.

Following [1]–[6], let $X$ be a nonempty set and $(Y, \Gamma)$ be an L-convex space. A mapping $G : X \to 2^Y$ is said to be a GLKKM mapping if for each $\{x_1, \ldots, x_n\} \in \mathcal{F}(X)$, there exists $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$ such that for any nonempty subset $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$, we have $\Gamma(\{y_{i_1}, \ldots, y_{i_k}\}) \subset \bigcup_{j=1}^{k} G(x_{i_j})$. When $(Y, \Gamma)$ is a hyperconvex space, H-space, G-convex space or G-H-convex space, the above definition was given by Kirk, et al.[7], Chang and Ma[8], Ding[9], Tan[10] and Verma[11], respectively.

**Definition 2.1**[12] *Let $X$ be a nonempty set, $(Y, \Gamma)$ an L-convex space and $\gamma \in R$ a real number. A function $f : X \times Y \to \overline{R} := R \cup \{\pm\infty\}$ is said to be generalized $\gamma$-L-diagonally quasiconcave (resp., generalized $\gamma$-L-diagonally quasiconvex) in $x$ if for each $\{x_1, \ldots, x_n\} \in \mathcal{F}(X)$, there exists $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$ such that for each $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$ and for each $y \in \Gamma(\{y_{i_1}, \ldots, y_{i_k}\})$, $\min_{1 \le j \le k} f(x_{i_j}, y) \le \gamma$ (resp., $\max_{1 \le j \le k} f(x_{i_j}, y) \ge \gamma$).*

**Lemma 2.1**[12] *Let $X$ be a nonempty set, $(Y, \Gamma)$ an L-convex space and $\gamma \in R$ a real number. Then a function $f : X \times Y \to \overline{R}$ is generalized $\gamma$-L-diagonally quasiconcave (resp., generalized $\gamma$-L-diagonally quasiconvex) in $x$ if and only if the mapping $F : X \to 2^Y$ defined by $F(x) := \{y \in Y : f(x, y) \le \gamma\}$ (resp., $F(x) := \{y \in Y : f(x, y) \ge \gamma\}$) for each $x \in X$ is a GLKKM mapping.*

**Remark 2.1** As shown in Remark 1.2 of Wen[12], Definition 2.1 unifies and generalizes Definition 3.1 of Ding[3], the definition of the hyper $\gamma$-generalized quasiconcave (resp., hyper $\gamma$-generalized quasiconvex) of Kirk, et al.[7] and the other corresponding definitions.

Now, we introduce the following definitions.

**Definition 2.2** *Let $\{G_i\}_{i \in I}$ be a family of subsets of a topological space $X$ and $K$ be a nonempty compact subset of $X$. $\{G_i\}_{i \in I}$ is said to be weakly transfer compactly open (resp., weakly transfer compactly closed ) relative to $K$ if $\{G_i \cap K\}_{i \in I}$ is transfer open (resp., transfer closed).*

**Definition 2.3** *Let $X$ be a nonempty set, $Y$ a topological space and $K$ a nonempty compact subset of $Y$. A mapping $G : X \to 2^Y$ is said to be weakly transfer compactly open valued (resp., weakly transfer compactly closed valued) relative to $K$ if the family $\{G(x)\}_{x \in X}$ is weakly transfer compactly open (resp., weakly transfer compactly closed) relative to $K$.*

**Remark 2.2** Clearly, each transfer compactly open valued (resp., transfer compactly closed valued) mapping (see Ding[5, p 420] and Definition 1 of Lu and Tang[6]) is weakly transfer compactly open valued (resp., weakly transfer compactly closed valued), but the inverse is not true in general.

**Definition 2.4** *Let $X$ be a nonempty set, $Y$ a topological space, $K$ a nonempty compact subset* *of $Y$ and $\gamma \in R$ a real number. A function $f : X \times Y \to \overline{R}$ is said to be weakly $\gamma$-transfer compactly lower semicontinuous (in short, w.$\gamma$-t.c.l.s.c) (resp., weakly $\gamma$-transfer compactly upper semicontinuous (in short, w.$\gamma$-t.c.u.s.c)) relative to $K$ in $y$ if for all $x \in X$ and $y \in K$, $f(x, y) > \gamma$ (resp., $f(x, y) < \gamma$) implies that there exist a relatively open neighborhood $N(y)$ of $y$ in $K$ and $x' \in X$ such that $f(x', z) > \gamma$ (resp., $f(x', z) < \gamma$) for all $z \in N(y)$.*

**Remark 2.3** Obviously, a $\gamma$-transfer compactly lower semicontinuous (resp., $\gamma$-transfer compactly upper semicontinuous) function (see [12, Definition 1.3]) is w.$\gamma$-t.c.l.s.c (resp., w.$\gamma$-t.c.u.s.c), but the inverse is not true in general. Hence, Definition 2.4 unifies and generalizes Definition 2.6 of Kirk, et al.[7], Definition 1.3 of Wen[12], Definition 4.1 of Ding[13], Definition 1.5(4) of Zhang[14], Definition (4) of Wu[15, p 285], Definition 8 of Tian[16] and Definition 2.6 of Ding[17].

We have the following lemma.

**Lemma 2.2** *Let $X$ be a nonempty set, $Y$ a topological space, $K$ a nonempty compact subset of $Y$ and $\gamma \in R$ a real number. A function $f : X \times Y \to \overline{R}$ is w.$\gamma$-t.c.l.s.c (resp., w.$\gamma$-t.c.u.s.c) relative to $K$ in $y$ if and only if the mapping $F : X \to 2^Y$ defined by $F(x) := \{y \in Y : f(x, y) \leq \gamma\}$ (resp., $F(x) := \{y \in Y : f(x, y) \geq \gamma\}$ for each $x \in X$ is weakly transfer compactly closed valued relative to $K$.*

**Remark 2.4** Lemma 2.2 improves and generalizes Lemma 3 of Lu and Tang[6], Lemma 2.7 of Kirk, et al.[7], Lemma 1.4 of Wen[12] and the corresponding result of Ding[17, p 25].

**Definition 2.5** *Let $X$ be a topological space, $Y$ a nonempty set, $K$ a nonempty compact subset of $X$. A mapping $F : X \to 2^Y$ is said to be have the weakly compactly local intersection property relative to $K$ if for each $x \in K$ with $F(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of $x$ such that $\bigcap_{z \in N(x) \cap K} F(z) \neq \emptyset$.*

**Remark 2.5** A mapping with compactly local intersection property (see [5, p420]) has the weakly compactly local intersection property, but the inverse is not true in general.

The following result is the improving version of Lemma 1.3 of Wen[12] and Lemma 2.1 of Ding[18].

**Lemma 2.3** *$X$ be a topological space, $Y$ a nonempty set, $K$ a nonempty compact subset of $X$ and $G : X \to 2^Y$ be a mapping such that $G(x) \neq \emptyset$ for each $x \in K$. Then the following conditions are equivalent:*

*(a) $G$ has the weakly compactly local intersection property relative to $K$;*

*(b) For each $y \in Y$, there exists an open subset $O_y$ of $X$ (which may be empty) such that $O_y \cap K \subset G^{-1}(y)$ and $K = \bigcup_{y \in Y} (O_y \cap K)$;*

*(c) There exists a mapping $F : X \to 2^Y$ such that for each $y \in Y$, $F^{-1}(y)$ is open or empty in $X$, $F^{-1}(y) \cap K \subset G^{-1}(y)$, and $K = \bigcup_{y \in Y} (F^{-1}(y) \cap K)$;*

(d) *For each* $x \in K$*, there exists* $y \in Y$ *such that* $x \in \mathrm{cint}_X G^{-1}(y) \cap K$ *and*

$$K = \bigcup_{y \in Y} (\mathrm{cint}_X G^{-1}(y) \cap K) = \bigcup_{y \in Y} (G^{-1}(y) \cap K);$$

(e) $G^{-1}$ *is weakly transfer compactly open valued relative to* $K$ *on* $X$.

## 3. Main results

**Theorem 3.1** *Let* $X$ *be a nonempty set,* $(Y, \Gamma)$ *an L-convex space,* $K$ *a nonempty compact subset of* $Y$, $G : X \to 2^Y$ *a GLKKM mapping with weakly transfer compactly closed values relative to* $K$*. Suppose there exists* $M \in \mathcal{F}(X)$ *such that* $\bigcap_{x \in M} \mathrm{cl}G(x) \subset K$*. Then* $\bigcap_{x \in X} G(x) \neq \emptyset$.

**Proof** Note that the mapping $\mathrm{cl}G$ is closed valued, and hence $\mathrm{cl}G$ is compactly closed valued. Since $G$ is a GLKKM mapping, $\mathrm{cl}G$ is also a GLKKM mapping. By virtue of Theorem 2.1 of Ding[3], $\{\mathrm{cl}G(x)\}_{x \in X}$ has the finite intersection property. Note that $K$ is compact and $\bigcap_{z \in M} \mathrm{cl}G(z) \subset K$. Then $\{\mathrm{cl}G(x) \cap \bigcap_{z \in M} \mathrm{cl}G(z)\}_{x \in X}$ is a family of closed subsets with the finite intersection property in $K$. By the compactness of $K$, we have $\bigcap_{x \in X} \mathrm{cl}G(x) = \bigcap_{x \in X} (\mathrm{cl}G(x) \cap \bigcap_{z \in M} \mathrm{cl}G(z)) \neq \emptyset$.

Now, define a mapping $F : X \to 2^Y$ by $F(x) := G(x) \cap \bigcap_{z \in M} G(z)$ for each $x \in X$. Since $G$ is weakly transfer compactly closed valued relative to $K$ and for each $x \in X$, $F(x) := G(x) \cap \bigcap_{z \in M} G(z) \subset \bigcap_{z \in M} \mathrm{cl}G(z) \subset K$, $F$ is transfer closed valued. In virtue of Lemma 2.4 of Kirk, et al.[7], we have $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} \mathrm{cl}F(x)$. Hence,

$$\begin{aligned} \bigcap_{x \in X} G(x) &= \bigcap_{x \in X} (G(x) \cap \bigcap_{z \in M} G(z)) \\ &= \bigcap_{x \in X} F(x) = \bigcap_{x \in X} \mathrm{cl}F(x) \\ &= \bigcap_{x \in X} (\mathrm{cl}G(x) \cap \bigcap_{z \in M} \mathrm{cl}G(z)) \\ &\neq \emptyset. \end{aligned}$$

□

**Remark 3.1** If $Y$ is compact, by letting $K = Y$, then the condition that there exists $M \in \mathcal{F}(X)$ such that $\bigcap_{z \in M} \mathrm{cl}G(z) \subset K$ is satisfied trivially. Therefore, Theorem 3.1 unifies, improves and generalizes Theorem 2.2(1) of Ding[3], Corollary 2.6 of Kirk, et al.[7], Theorem 1 of Chang and Ma[8], Theorem 2.3(1) and 2.4 of Tan[10], Theorem 2.1 of Verma[11], Theorem 3.2(i) of Ding[17], Theorem 4 of Khamsi[19], the KKM Theorem of Park[20], Theorem 1.1 of Chowdhury[21], Theorem 1.1 of Chowdhury, et al.[22], Theorem 2.2 of Verma[23] and Theorem 1 of Park[24].

**Theorem 3.2** *Let* $X$ *be a nonempty set,* $(Y, \Gamma)$ *an L-convex space,* $\gamma \in R$ *a real number,* $K$ *a nonempty compact subset of* $Y$ *and* $f, g : X \times Y \to \overline{R}$ *two functions such that*

(1) *There exists* $M \in \mathcal{F}(X)$ *such that* $\bigcap_{x \in M} \mathrm{cl}_Y \{y \in Y : g(x, y) \leq \gamma\} \subset K$*;*

(2) $g(x, y)$ *is generalized* $\gamma$*-L-diagonally quasiconcave in* $x$*;*

(3) *For each* $(x, y) \in X \times Y$, $g(x, y) \leq \gamma$ *implies* $f(x, y) \leq \gamma$*;*

(4) $g(x, y)$ *is w.*$\gamma$*-t.c.l.s.c. relative to* $K$ *in* $y$.

*Then there exists $y^* \in Y$ such that $f(x, y^*) \leq \gamma$ for all $x \in X$.*

**Proof** Define two mappings $F, G : X \to 2^Y$ by $F(x) := \{y \in Y : f(x, y) \leq \gamma\}$ and $G(x) := \{y \in Y : g(x, y) \leq \gamma\}$ for each $x \in X$, respectively. Then by (1), there exists $M \in \mathcal{F}(X)$ such that $\bigcap_{x \in M} \mathrm{cl}G(x) \subset K$. By (2) and Lemma 2.1, $G$ is a GLKKM mapping. By (3), for each $x \in X$, $\{y \in Y : g(x, y) \leq \gamma\} \subset \{y \in Y : f(x, y) \leq \gamma\}$, hence, $G(x) \subset F(x)$. By (4) and Lemma 2.2, $G$ is weakly transfer compactly closed valued relative to $K$. In virtue of Theorem 3.1, $\emptyset \neq \bigcap_{x \in X} G(x) \subset \bigcap_{x \in X} F(x)$. Take $y^* \in \bigcap_{x \in X} F(x)$. Then $f(x, y^*) \leq \gamma$ for all $x \in X$. $\square$

**Remark 3.2** If $Y$ is compact, by letting $K = Y$, then the condition (1) is satisfied trivially. If $f(x, y)$ is lower semicontinuous in $y$, the condition (4) is certainly satisfied. Therefore, Theorem 3.2 unifies, improves and generalizes Theorem 3.5 of Ding[3], Theorem 2.8 of Kirk, et al.[7], Theorems 3.1, 3.2, 3.3 of Tan[10], Theorems 3.1, 3.2, 3.3 of Verma[11], Theorem 4 of Tian[16], Theorem 4.2 of Ding[17], Theorems 2.1, 2.2 of Chowdhury, et al.[22], Theorem 2.3 of Verma[25], Theorems 2.1, 2.2 of Verma[26], Theorem 2.1 and Corollaries 2.2, 2.3, 2.4 of Tan, et al.[27] and Theorem 2.11.15 of Yuan[28].

**Theorem 3.3** *Let $(X, \Gamma)$ be an L-convex space, $K$ a nonempty compact subset of $X$ and $A, B \subset X \times X$ two subsets such that*

*(1) The mapping $G : X \to 2^X$ defined by $G(x) := \{y \in X : (x, y) \in B\}$ for each $x \in X$ is weakly transfer compactly closed valued relative to $K$;*

*(2) For each $x \in X$, $(x, x) \in B$;*

*(3) For each $y \in X$, $\{x \in X : (x, y) \notin B\} = \emptyset$ or $\in \mathcal{L}(X)$;*

*(4) There exists $M \in \mathcal{F}(X)$ such that $\bigcap_{x \in M} \mathrm{cl}_X\{y \in X : (x, y) \in B\} \subset K$;*

*(5) $B \subset A$.*

*Then there exists $y^* \in X$ such that $X \times \{y^*\} \subset A$.*

**Proof** Define two functions $f, g : X \times X \to R$ by

$$f(x, y) := \begin{cases} 1, & \text{if } (x, y) \notin A, \\ 0, & \text{if } (x, y) \in A, \end{cases}$$

and

$$g(x, y) := \begin{cases} 1, & \text{if } (x, y) \notin B, \\ 0, & \text{if } (x, y) \in B, \end{cases}$$

for each $(x, y) \in X \times X$, respectively. Note that for each $x \in X$, $G(x) := \{y \in X : (x, y) \in B\} = \{y \in X : g(x, y) \leq 0\}$. Then by (1) and Lemma 2.2, $g(x, y)$ is w.0-t.c.l.s.c. relative to $K$ in $y$. By (2), we have

(a) for each $x \in X$, $g(x, x) \leq 0$.

By (3), we have

(b) for each $y \in X$, $\{x \in X : g(x, y) > 0\} = \emptyset$ or $\in \mathcal{L}(X)$.

By (4), there exists $M \in \mathcal{F}(X)$ such that $\bigcap_{x \in M} \mathrm{cl}_X\{y \in X : g(x, y) \leq 0\} \subset K$. By (5), for each $(x, y) \in X \times X$, $g(x, y) \leq 0$ implies $f(x, y) \leq 0$. We claim that $g(x, y)$ is generalized 0-L-diagonally quasiconcave in $x$. Otherwise, there exists $\{x_1, \ldots, x_n\} \in \mathcal{F}(X)$, for each $\{y_1, \ldots, y_n\} \in \mathcal{F}(X)$, there exist $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$ and $y \in \Gamma(\{y_{i_1}, \ldots, y_{i_k}\})$ such that $min_{1\leq j\leq k} g(x_{i_j}, y) > 0$. Especially, for $\{x_1, \ldots, x_n\} \in \mathcal{F}(X)$, there exist $\{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\}$ and $y_0 \in \Gamma(\{x_{i_1}, \ldots, x_{i_k}\})$ such that $\min_{1\leq j\leq k} g(x_{i_j}, y_0) > 0$, and then, by (b), $\{x_{i_1}, \ldots, x_{i_k}\} \subset \{x \in X : g(x, y_0) > 0\} \in \mathcal{L}(X)$. Thus $y_0 \in \Gamma(\{x_{i_1}, \ldots, x_{i_k}\}) \subset \{x \in X : g(x, y_0) > 0\}$. Hence $g(y_0, y_0) > 0$, which contradicts (a). Finally, in virtue of Theorem 3.2, there exists $y^* \in X$ such that $f(x, y^*) \leq 0$ for all $x \in X$, which implies that $X \times \{y^*\} \subset A$.

**Remark 3.3** If $X$ is compact, by letting $K = X$, then the condition (4) is satisfied trivially. If $X$ is a hyperconvex metric space, $(X, \Gamma)$ is an L-convex space, certainly. If $G$ is transfer open valued or transfer compactly open valued, the condition (1) is satisfied trivially. If $\{x \in X : (x, y) \notin B\}$ is hyperconvex, of course, $\{x \in X : (x, y) \notin B\}$ is L-convex. Therefore, Theorem 3.3 improves and generalizes Theorem 3.2 of Kirk, et al.[7] and Theorem 3 of Chen and Shen[29] in several aspects.

**Theorem 3.4** *Let $(X, \Gamma)$ be an L-convex space, $K$ a nonempty compact subset of $X$ and $P, Q : X \to 2^X$ two mappings such that*

*(1) $P^{-1}$ is weakly transfer compactly open valued relative to $K$;*

*(2) For each $x \in X$, $x \notin P(x)$;*

*(3) For each $x \in X$, $P(x) = \emptyset$ or $\in \mathcal{L}(X)$;*

*(4) There exists $M \in \mathcal{F}(X)$ such that $\bigcap_{x\in M} \mathrm{cl}P^*(x) \subset K$;*

*(5) For each $x \in X$, $Q(x) \subset P(x)$.*

*Then there exists $x^* \in X$ such that $Q(x^*) = \emptyset$.*

**Proof** Let $A := \{(x, y) \in X \times X : x \notin Q(y)\}$ and $B := \{(x, y) \in X \times X : x \notin P(y)\}$. Then by (1), $P^* : X \to 2^X$ defined by $P^*(x) = X \setminus P^{-1}(x) = \{y \in X : (x, y) \in B\}$ for each $x \in X$ is weakly transfer compactly closed valued relative to $K$. By (2), for each $x \in X$, $(x, x) \in B$. By (3), for each $y \in X$, $\{x \in X : (x, y) \notin B\} = P(y) = \emptyset$ or $\in \mathcal{L}(X)$. By (4), there exists $M \in \mathcal{F}(X)$ such that $\bigcap_{x\in M} \mathrm{cl}_X\{y \in X : (x, y) \in B\} \subset K$. By (5), $B \subset A$. Hence, in virtue of Theorem 3.3, there exists $y^* \in X$ such that $X \times \{y^*\} \subset A$, which implies $Q(y^*) = \emptyset$. $\square$

**Remark 3.4** As shown in Remark 3.3, Theorem 3.4 improves and generalizes Theorem 3.4 of Kirk, et al.[7] and Theorem 2 of Wu[15] in several aspects.

Obviously, Theorem 3.4 implies the following Fan-Browder type fixed point theorem.

**Theorem 3.5** *Let $(X, \Gamma)$ be an L-convex space, $K$ a nonempty compact subset of $X$ and $P : X \to 2^X \setminus \{\emptyset\}$ a mapping such that*

*(1) $P$ satisfies one of the conditions $(a) \sim (e)$ in Lemma 2.3;*

*(2) For each $x \in X$, $P(x) \in \mathcal{L}(X)$;*

*(3) There exists $M \in \mathcal{F}(X)$ such that $\bigcap_{x\in M} \mathrm{cl}P^*(x) \subset K$.*

*Then there exists $x^* \in X$ such that $x^* \in P(x^*)$.*

**Remark 3.5** Theorem 3.5 unifies, improves and generalizes Theorem 3.1 of Kirk, et al.[7], Lemma 2.2 of Zhang[14], Lemma 1 of Wu[15], Theorem 3 of Park[20], Theorem 2.3-A of Chowdhury, et al.[22], Theorem 2.4 of Verma[23], Theorems 2, 3, 4, 8 of Park[24], Corollary 2 and Corollary 3 of Chen and Shen[29], Theorem 2 of Horvath[30, p 350], Theorem 3.6 of Yuan[31], Corollary 2.3 of Tarafdar[32], Theorem 2.1, Corollaries 2.1, 2.2, 2.3 of Tarafdar[33] and Theorem 4.1 of Watson[34].

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