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Iteration $x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1})T_{n+1}x_n$ for an Infinite Family of Nonexpansive Maps $\{T_n\}_{n=1}^{\infty}$

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Abstract Under the framework of uniformly smooth Banach spaces, Chang[1] proved in 2006 that the sequence $\{x_n\}$ generated by the iteration $x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1})T_{n+1}x_n$ converges strongly to a common fixed point of a finite family of nonexpansive maps $\{T_n\}$, where $f : C \to C$ is a contraction. However, in this paper, the author considers the iteration in more general case that $\{T_n\}$ is an infinite family of nonexpansive maps, and proves that Chang's result holds still in the setting of reflexive Banach spaces with the weakly sequentially continuous duality mapping.

Keywords infinitely many nonexpansive mappings; contractive mapping; weakly sequential continuity.

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1. Introduction

Let E be a Banach space, C be a nonempty closed convex subset of E, and $\{T_1, T_2, \ldots\}$ be an infinite family of nonexpansive mappings from C into itself. In this paper, we are interested in the following iterative scheme

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1})T_{n+1}x_n, \quad \forall n \geq 0. \tag{1.1}$$

Here, we assume that $x_0 \in E$ is any given initial data, $f : C \to C$ is a given contractive mapping, and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Many authors studied some special cases of iterative scheme (1.1). Indeed, if $f(x) \equiv u \in C$, and T is a nonexpansive mapping on C, a subset of a Hilbert space, then the iteration (1.1) is reduced to the following iteration:

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1})Tx_n, \quad \forall n \geq 0, \tag{1.2}$$

which was firstly introduced and studied by Halpern in 1967[2]. In 1992, Wittmann also considered the iteration (1.2)[4]. He improved and extended the corresponding results of Halpern[2] and Lions[3]. In 1980, Reich[5] extended Halpern's result to all uniformly smooth Banach spaces, and

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in 1994 he extended Wittmann's result to the uniformly smooth space with a weakly sequentially continuous duality mapping (see [6, Theorem and Remark 1]).

If C is a nonempty closed convex subset of a Hilbert space, $T_i : C \to C$ is a nonexpansive mapping for each $i = 1, 2, \ldots, N$, and $f(x) \equiv u$ (a given point in C), then (1.2) is equivalent to the following iteration with a finite family of nonexpansive mapping $\{T_n\}_{n=1}^N$:

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad \forall\, n \geq 0,\ T_n = T_{n (\mathrm{mod}\, N)}, \tag{1.3}$$

which was introduced and studied in Bauschke in 1996[8]. In 2005, Jung[7, Theorem 10] considered it in the framework of uniformly smooth Banach spaces with the weakly sequentially continuous duality mapping, which might imply that the class of uniformly smooth Banach spaces is different from the class of reflexive Banach spaces with the weakly sequentially continuous duality mapping. Indeed, firstly we say, there exist reflexive Banach spaces with the weakly continuous duality mapping, for example, the sequence spaces l^p for all $p \in (1, +\infty)$. Next, we say, not all uniformly smooth Banach spaces admit a weakly continuous duality mapping. Indeed, we can consider L^p in the case of $p \neq 2$. For example, $L^4([0, 2\pi])$ is such a uniformly smooth Banach space that none of its duality mappings is weakly continuous (see, [14, Section 4]). Finally, it is obvious that not all reflexive Banach spaces (with the weakly continuous duality mapping) belong to the class of uniformly smooth Banach spaces.

If $T : C \to C$ is a nonexpansive mapping, and $f : C \to C$ is a contractive mapping, then (1.1) is reduced to the following iteration:

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1}) T x_n, \quad \forall\, n \geq 0, \tag{1.4}$$

which was firstly introduced and studied by Moudafi[9] in the setting of Hilbert spaces. In 2004, Xu[10] extended and improved the result of Moudafi[9] from Hilbert spaces to uniformly smooth Banach spaces.

In 2006, Chang[1] considered the following iteration in uniformly smooth Banach space E:

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad T_n = T_{n (\mathrm{mod}\ N)}, \quad \forall\, n \geq 0, \tag{1.5}$$

where $\{T_1, T_2, \ldots, T_N\}$ is a finite family of nonexpansive mappings on C, a closed convex subset of E.

Now in this paper we extend the iteration (1.1) to a more general case that $\{T_1, T_2, \ldots, T_n, \ldots\}$ is an infinite family of nonexpansive mappings, and study its convergence in the framework of reflexive Banach spaces which admit the weakly sequentially continuous duality mapping.

2. Preliminaries

Throughout this paper, we assume, E is a real Banach space, and E^* is the dual space of E. Suppose that C is a nonempty closed convex subset of E, and that $F(T)$ is the set of fixed points of mapping T. Denote the generalized duality pairing between E and E^* by $\langle \cdot, \cdot \rangle$, and the identity mapping by I. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^*, \langle x, f \rangle = \|x\| \cdot \|f\| = \|f\|^2 = \|x\|^2\}, \quad x \in E. \tag{2.1}$$

If $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \rightharpoonup^* x$) denotes strong (resp., weak and weak*) convergence of the sequence $\{x_n\}$ to x.

Definition 2.1[11] *A Banach space is said to admit a weakly sequentially continuous normalized duality mapping J, if $J : E \to E^*$ is single-valued and weak-weak* sequentially continuous. i.e., if $x_n \rightharpoonup x$ in E, then $J(x_n) \rightharpoonup^* J(x)$ in E^*.*

Definition 2.2 *Let $U = \{x \in E : \|x\| = 1\}$. E is said to be a smooth Banach space, if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for $x, y \in U$.*

Definition 2.3 *(1) A mapping $f : C \to C$ is said to be a Banach contraction on C with a contractive constant $\alpha \in (0,1)$ if $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$.*

(2) Let $T : C \to C$ be a mapping. T is said to be nonexpansive, if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$.

Lemma 2.1[12] *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \le (1 - \lambda_n)a_n + b_n + c_n, \quad \forall\, n \ge n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\} \subset [0,1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $a_n \to 0$ as $n \to \infty$.

Lemma 2.2[13] *Let E be a real Banach space, and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, the following conclusion holds:*

$$\|x\|^2 + 2\langle y, j(x)\rangle \le \|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall\, j(x) \in J(x),\ j(x + y) \in J(x + y).$$

Lemma 2.3[15] *If E admits a weakly sequentially continuous normalized duality mapping, and if $T : E \to E$ is a nonexpansive mapping, then the mapping $I - T$ is demiclosed. That is, for any sequence $\{x_n\}$ in E, if $x_n \rightharpoonup x$ and $(x_n - Tx_n) \to y$, then $(I - T)x = y$.*

3. Main results

Let E be a real Banach space, and C be a nonempty closed convex subset of E. Denote $S = T_1T_2 \cdots T_n \cdots$, where $\{T_1, T_2, \ldots\}$ is an infinite family of nonexpansive mappings from C to itself. Suppose, $f : C \to C$ is any given Banach contractive mapping with a contractive constant $\alpha \in (0,1)$. Define a mapping $S_t^f : C \to C$ by

$$S_t^f(x) = tf(x) + (1 - t)S(x), \quad x \in C. \tag{3.1}$$

Now we can easily know from (3.1) that $S_t^f : C \to C$ is a Banach contraction mapping. Hence, S_t^f has a unique fixed point z_t in C by Banach's Contraction Mapping Principle, which implies that for any given $t \in (0,1)$ there exists a corresponding $z_t \in F(S_t)$ such that z_t is the unique solution in C for the following equation

$$z_t = tf(z_t) + (1 - t)S(z_t). \tag{3.2}$$

Particularly for a single nonexpansive map T, which implies $S = T$ in (3.2). Xu[10, Theorem 4.1] showed in the setting of a uniformly smooth Banach space that $z_t \to z \in F(T)$. Here, z is a solution in $F(T)$ for the following variational inequality:

$$\langle (I-f)z, j(z-u)\rangle \le 0, \quad \forall u \in F(T).$$

Now, for an infinite family of nonexpansive mappings, we give a similar result in the framework of reflexive Banach spaces with the weakly sequentially continuous normalized duality mapping.

Theorem 3.1 *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from E to E^*, and C be a nonempty closed convex subset of E. Assume, $f : C \to C$ is a given Banach contraction with a contractive constant $\alpha \in (0,1)$. Let $\{z_t : t \in (0,1)\}$ be the net defined by (3.2), and $T_i : C \to C$, $i = 1, 2, \ldots$ be nonexpansive mappings satisfying the following conditions:*

(i) $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$;

(ii) $\bigcap_{n=1}^{\infty} F(T_n) = F(T_1T_2\cdots T_n\cdots) = F(T_2T_3\cdots T_n\cdots T_1) = \cdots = F(T_nT_{n+1}\cdots T_2T_1) = \cdots = F(S)$,

where $S = T_1T_2\cdots T_n\cdots$, then, as $t \to 0$, $\{z_t\}$ converges strongly to a common fixed point $q \in \bigcap_{n=1}^{\infty} F(T_n)$ such that q is the unique solution in $\bigcap_{n=1}^{\infty} F(T_n)$ for the following variational inequality

$$\langle (I-f)q, j(q-u)\rangle \le 0, \quad \forall u \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.3}$$

Proof On the one hand, for any $u \in \bigcap_{n=1}^{\infty} F(T_n)$, we can get by (3.2)

$$\|z_t - [tf(z_t) + (1-t)u]\| \le (1-t)\|Sz_t - u\| \le (1-t)\|z_t - u\|. \tag{3.4}$$

On the other hand, we have by Lemma 2.2

$$\begin{aligned}\|z_t - [tf(z_t) + (1-t)u]\|^2 &= \|(1-t)(z_t-u) + t(z_t - f(z_t))\|^2 \\ &\ge (1-t)^2\|z_t - u\|^2 + 2t(1-t)\langle z_t - f(z_t), j(z_t-u)\rangle.\end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$2t(1-t)\langle z_t - f(z_t), j(z_t-u)\rangle \le \|z_t - [tf(z_t)+(1-t)u]\|^2 - (1-t)^2\|z_t-u\|^2 \le 0. \tag{3.6}$$

Then we get by (3.6)

$$\langle z_t - f(z_t), j(z_t-u)\rangle \le 0, \quad \forall u \in \bigcap_{n=1}^{\infty} F(T_n),\ \forall j(z_t-u) \in J(z_t-u). \tag{3.7}$$

Since f is a Banach contraction, we know for any $u \in \bigcap_{n=1}^{\infty} F(T_n)$

$$\langle f(z_t) - f(u), j(z_t-u)\rangle \le \alpha\|z_t - u\|^2. \tag{3.8}$$

Since

$$\begin{aligned}\langle z_t - f(z_t), j(z_t-u)\rangle &= \langle z_t - u + u - f(u) + f(u) - f(z_t), j(z_t-u)\rangle \\ &= \|z_t-u\|^2 + \langle u - f(u), j(z_t-u)\rangle + \langle f(u) - f(z_t), j(z_t-u)\rangle\end{aligned}$$

$$\geq \|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle - \|f(u) - f(z_t)\| \cdot \|z_t - u\|$$
$$\geq (1-\alpha)\|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle, \tag{3.9}$$

we can deduce by (3.7) and (3.9) that

$$(1-\alpha)\|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle \leq 0. \tag{3.10}$$

Now we get by (3.10)

$$(1-\alpha)\|z_t - u\|^2 \leq \langle u - f(u), j(u - z_t)\rangle \leq \|u - f(u)\| \cdot \|u - z_t\|. \tag{3.11}$$

It follows from (3.11) that

$$\|z_t - u\| \leq \frac{\|u - f(u)\|}{1-\alpha}. \tag{3.12}$$

This implies that $\{z_t : t \in (0,1)\}$ is bounded. Thus, both $\{S(z_t) : t \in (0,1)\}$ and $\{f(z_t) : t \in (0,1)\}$ are bounded. Then it follows by (3.2) that

$$\|z_t - S(z_t)\| \leq t\|f(z_t) - S(z_t)\| \to 0, \quad \text{as} \quad t \to 0.$$

Thus,

$$\lim_{t\to 0} \|z_t - S(z_t)\| = 0. \tag{3.13}$$

Next we prove that $\{z_t : t \in (0,1)\}$ is relatively compact. Indeed, since E is reflexive and $\{z_t : t \in (0,1)\}$ is bounded, for any subsequence $\{z_{t_n}\} \subset \{z_t\}$ with $t_n \in (0,1)$, there exists a subsequence of $\{z_{t_n}\}$ (for simplicity we still denote it by $\{z_{t_n}\}$) such that

$$z_{t_n} \rightharpoonup q \quad \text{as} \quad t_n \to 0. \tag{3.14}$$

We can easily see by (3.13)

$$\|z_{t_n} - S(z_{t_n})\| \to 0, \quad \text{as} \quad t_n \to 0,$$

which together with (3.14) and Lemma 2.3 implies $I - S$ has the demiclosed property. Thus,

$$q \in F(S) = \bigcap_{n=1}^{\infty} F(T_n). \tag{3.15}$$

Taking $u = q$ and $t = t_n$ in (3.11), we get

$$\|z_{t_n} - q\|^2 \leq \frac{\langle q - f(q), j(q - z_{t_n})\rangle}{1-\alpha}. \tag{3.16}$$

Since J is weakly sequentially continuous, we know

$$\lim_{t_n\to 0} \|z_{t_n} - q\|^2 \leq \lim_{t_n\to 0} \frac{\langle q - f(q), j(q - z_{t_n})\rangle}{1-\alpha} = 0. \tag{3.17}$$

Finally, we will prove that the entire net $\{z_t, t \in (0,1)\}$ converges strongly to q. Indeed, suppose the contrary that there exists another subsequence $\{z_{t_i}\}$ of $\{z_t\}$ such that $z_{t_i} \to q'$ as $t_i \to 0$. By the same method as given above, we can also prove that $q' \in F(S) = \bigcap_{n=1}^{\infty} F(T_n)$. Now we predict that

$$\langle (I - f)q', j(q' - u)\rangle \leq 0, \quad \forall u \in \bigcap_{n=1}^{\infty} F(T_n).$$

Indeed, for any given $\forall u \in \bigcap_{n=1}^{\infty} F(T_n)$, both $\{z_t - u\}$ and $\{z_t - f(z_t)\}$ are bounded. Then we can deduce by the assumption on the normalized duality mapping J and $\lim_{t_i \to 0} z_{t_i} = q'$ that

$$\begin{aligned} &|\langle (I-f)z_{t_i}, j(z_{t_i} - u)\rangle - \langle (I-f)q', j(q'-u)\rangle| \\ &\quad = |\langle (I-f)z_{t_i} - (I-f)q', j(z_{t_i} - u)\rangle + \langle (I-f)q', j(z_{t_i} - u) - j(q'-u)\rangle| \\ &\quad \le \|(I-f)z_{t_i} - (I-f)q'\| \cdot \|z_{t_i} - u\| + |\langle (I-f)q', j(z_{t_i} - u) - j(q'-u)\rangle| \to 0, \quad \text{as} \quad t_i \to 0. \end{aligned}$$

Hence, we know by (3.7)

$$\langle (I-f)q', j(q'-u)\rangle = \lim_{t_i \to 0} \langle (I-f)z_{t_i}, j(z_{t_i} - u)\rangle \le 0. \tag{3.18}$$

Similarly we can also prove that

$$\langle (I-f)q, j(q-u)\rangle \le 0. \tag{3.19}$$

Now we take $u = q$ in (3.18) and $u = q'$ in (3.19), respectively. Then we can easily see by adding up these two inequalities that

$$\langle (I-f)q - (I-f)q', j(q-q')\rangle \le 0.$$

Thus,

$$\|q-q'\|^2 \le \langle f(q) - f(q'), j(q-q')\rangle \le \alpha \|q-q'\|^2.$$

This implies that $q = q'$, and hence Theorem 3.1 has been proved. $\square$

Theorem 3.2 *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from E to E^*, and C be a nonempty closed convex subset of E. Assume that $f : C \to C$ is a given Banach contraction with a contractive constant $\alpha \in (0,1)$, and $\{z_t : t \in (0,1)\}$ is the net defined by (3.2). Suppose, $\{T_i : C \to C,\ i = 1, 2, \ldots\}$ is an infinite family of nonexpansive mappings satisfying the following conditions:*

(i) $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$;

(ii) $\bigcap_{n=1}^{\infty} F(T_n) = F(T_1 T_2 \cdots T_n \cdots) = F(T_2 T_3 \cdots T_n \cdots T_1) = \cdots = F(T_n T_{n+1} \cdots T_2 T_1) = \cdots = F(S)$,

where $S = T_1 T_2 \cdots T_n \cdots$.

Let $x_0 \in C$ be any given point, and $\{x_n\}$ be generated by the iteration (1.1) and $x_0 \in C$. If, in addition, the following conditions hold:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\|x_n - Sx_n\| \to 0$,

then $\{x_n\}$ generated by $x_0 \in C$ and iteration (1.1) converges strongly to $q = \lim_{t\to 0} z_t$ such that q is the unique solution in $\bigcap_{n=1}^{\infty} F(T_n)$ for the following variational inequality:

$$\langle (I-f)q, j(q-u)\rangle \le 0, \quad \forall u \in \bigcap_{n=1}^{\infty} F(T_n).$$

Proof By Theorem 3.1, we know $\lim\limits_{t\to 0} z_t = q \in \bigcap_{n=1}^{\infty} F(T_n)$. Then we have by (1.1)

$$\|x_{n+1} - q\| \le \alpha_{n+1} \|f(x_n) - q\| + (1-\alpha_{n+1}) \|T_{n+1}(x_n) - q\|$$

$$
\begin{aligned}
&\leq \alpha_{n+1}(\|f(x_n) - f(q)\| + \|f(q) - q\|) + (1 - \alpha_{n+1})\|x_n - q\| \\
&\leq \alpha_{n+1}\|f(q) - q\| + (1 - (1 - \alpha)\alpha_{n+1})\|x_n - q\| \\
&\leq \max\{\|x_n - q\|, \frac{\|f(q) - q\|}{1 - \alpha}\}.
\end{aligned}
$$

Now we can obtain by induction

$$
\|x_n - q\| \leq \max\{\|x_0 - q\|, \frac{\|f(q) - q\|}{1 - \alpha}\}, \quad \text{for all } \ n \geq 0. \tag{3.20}
$$

This proves that $\{x_n\}$ is bounded. Since $\{z_t\}$ is bounded, there exists a constant $M > 0$ such that

$$
\|x_n\| + \|x_n\|^2 + \|q\|^2 + \|q\| + \|z_t\| + \|z_t\|^2 < M, \quad \text{for all } \ n \geq 0 \text{ and } t \in (0, 1).
$$

On the other hand, we can get by (1.1) and Lemma 2.2

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq (1 - \alpha_{n+1})^2\|T_{n+1}x_n - q\|^2 + 2\alpha_{n+1}\langle f(x_n) - q, j(x_{n+1} - q)\rangle \\
&\leq (1 - \alpha_{n+1})^2\|x_n - q\|^2 + 2\alpha_{n+1}\langle f(x_n) - q, j(x_{n+1} - q)\rangle. \qquad (3.21)
\end{aligned}
$$

Since f is a contractive mapping, we obtain

$$
\begin{aligned}
&2\alpha_{n+1}\langle f(x_n) - q, j(x_{n+1} - q)\rangle \\
&\quad = 2\alpha_{n+1}\langle f(x_n) - f(q) + f(q) - q, j(x_{n+1} - q)\rangle \\
&\quad \leq 2\alpha_{n+1}\alpha\|x_n - q\| \cdot \|x_{n+1} - q\| + 2\alpha_{n+1}\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\quad \leq \alpha_{n+1}\alpha(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + 2\alpha_{n+1}\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\quad \leq \alpha_{n+1}\alpha(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + 2\alpha_{n+1}\gamma_{n+1}, \qquad (3.22)
\end{aligned}
$$

where

$$
\gamma_n = \max\{\, 0, \langle f(q) - q, j(x_n - q)\rangle\}, \quad \text{for all } \ n \geq 0. \tag{3.23}
$$

It follows by (3.21) and (3.22) that

$$
\|x_{n+1} - q\|^2 \leq (1 - \alpha_{n+1})^2\|x_n - q\|^2 + \alpha_{n+1}\alpha(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + 2\alpha_{n+1}\gamma_{n+1}.
$$

Thus,

$$
(1 - \alpha_{n+1}\alpha)\|x_{n+1} - q\|^2 \leq ((1 - \alpha_{n+1})^2 + \alpha_{n+1}\alpha)\|x_n - q\|^2 + 2\alpha_{n+1}\gamma_{n+1}.
$$

On the other hand, we know by the condition (a) that there exists a nonnegative integer n_1 such that

$$
2(1 - \alpha)\alpha_{n+1} \in [0, 1) \ \text{ and } \ 1 - \alpha_{n+1}\alpha \geq \frac{1}{2} \ \text{ for all } \ n \geq n_1.
$$

Thereby, we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \frac{((1 - \alpha_{n+1})^2 + \alpha_{n+1}\alpha)}{1 - \alpha_{n+1}\alpha}\|x_n - q\|^2 + \frac{2\alpha_{n+1}\gamma_{n+1}}{1 - \alpha_{n+1}\alpha} \\
&\leq (1 - \frac{2\alpha_{n+1}(1 - \alpha)}{1 - \alpha_{n+1}\alpha})\|x_n - q\|^2 + \frac{\alpha_{n+1}^2}{1 - \alpha_{n+1}\alpha}(\|x_n\| + \|q\|)^2 + \frac{2\alpha_{n+1}\gamma_{n+1}}{1 - \alpha_{n+1}\alpha} \\
&\leq (1 - 2\alpha_{n+1}(1 - \alpha))\|x_n - q\|^2 + \frac{\alpha_{n+1}^2}{1 - \alpha_{n+1}\alpha} \cdot 2(\|x_n\|^2 + \|q\|^2) + \frac{2\alpha_{n+1}\gamma_{n+1}}{1 - \alpha_{n+1}\alpha}
\end{aligned}
$$

$$\leq (1-2(1-\alpha)\alpha_{n+1})\|x_n-q\|^2+4M\alpha_{n+1}^2+4\alpha_{n+1}\gamma_{n+1}$$
$$\leq (1-2(1-\alpha)\alpha_{n+1})\|x_n-q\|^2+4\alpha_{n+1}(M\alpha_{n+1}+\gamma_{n+1}), \quad \forall n>n_1. \tag{3.24}$$

Next we show $\lim_{n\to\infty}\gamma_n=0$.

Indeed, there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle q-f(q), j(q-x_n)\rangle=\lim_{k\to\infty}\langle q-f(q), j(q-x_{n_k})\rangle. \tag{3.25}$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}\subset\{x_{n_k}\}$ such that $x_{n_i}\rightharpoonup x_0$ as $i\to\infty$. Then we obtain by (3.25)

$$\limsup_{n\to\infty}\langle q-f(q), j(q-x_n)\rangle=\lim_{i\to\infty}\langle q-f(q), j(q-x_{n_i})\rangle.$$

It follows by the condition (c) that

$$\|x_{n_i}-S(x_{n_i})\|\to 0, \quad \text{as} \quad i\to\infty.$$

Now we see by the Lemma 2.3 that $x_0\in F(S)=\bigcap_{n=1}^{\infty}F(T_n)$. Therefore, we get by (3.3)

$$\limsup_{n\to\infty}\langle q-f(q), j(q-x_n)\rangle=\lim_{i\to\infty}\langle q-f(q), j(q-x_{n_i})\rangle=\langle q-f(q), j(q-x_0)\rangle\leq 0.$$

This implies that for any given $\varepsilon>0$ there correspondingly exists a positive integer $n_2>n_1$ such that

$$\langle q-f(q), j(q-x_n)\rangle<\varepsilon, \quad \forall n>n_2.$$

Thus, we have $0\leq\gamma_n<\varepsilon$, and hence $\gamma_n\to 0$. Now, we take $\lambda_n=2(1-\alpha)\alpha_{n+1})$, $a_n=\|x_n-q\|^2$, $b_n=4\alpha_{n+1}(M\alpha_{n+1}+\gamma_{n+1})$ and $c_n=0$ for all $n>n_2$. By (3.24) we can conclude by Lemma 2.1 that

$$\lim_{n\to\infty}\|x_n-q\|=0, \quad \text{i.e.,} \quad x_n\to q=\lim_{t\to 0}z_t \quad \text{and} \quad q\in\bigcap_{n=1}^{\infty}F(T_n).$$

This completes the proof of Theorem 3.2. □

Theorem 3.3 *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from E to E^*, C be a nonempty closed convex subset of E, and $f: C\to C$ be a given Banach contraction with a contractive constant $\alpha\in(0,1)$. Suppose that $T: C\to C$ is a nonexpansive mapping with $F(T)\neq\emptyset$, and $x_0\in C$ is any given point. Let $\{x_n\}$ be generated by the iteration (1.4) and $x_0\in C$. If, in addition, the following conditions hold:*

(a) $\lim_{n\to\infty}\alpha_n=0$;

(b) $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(c) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ *or* $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}}=1$,

then $\{x_n\}$ generated by (1.4) converges strongly to $q=\lim_{t\to 0}z_t$ such that q is the unique solution in $F(T)$ for the following variational inequality:

$$\langle (I-f)q, j(q-u)\rangle\leq 0, \quad \forall u\in F(T).$$

Here, the net $\{z_t: t\in(0,1)\}$ is defined by (3.2) in the case that $S=T$.

Proof In Theorem 3.2, we put $T_1 = T_2 = \cdots = T$ and $S = T$. Now, we only need to show that the condition $\|x_n - Tx_n\| \to 0$ holds.

Indeed, similarly to the proof in Theorem 3.2, we can also prove and get (3.20). Thus, $\{x_n\}$ is bounded. So are $\{f(x_n)\}$ and $\{Tx_n\}$. Hence, there exists a constant $M_0 > 0$ such that

$$\|f(x_n)\| + \|Tx_n\| < M_0, \quad \text{for all} \quad n \geq 0.$$

It follows by the iteration (1.4) that

$$\begin{aligned} \|x_n - Tx_n\| &= \|\alpha_n f(x_{n-1}) + (1-\alpha_n)Tx_{n-1} - Tx_n\| \\ &\leq \alpha_n \|f(x_{n-1}) - Tx_n\| + (1-\alpha_n)\|Tx_{n-1} - Tx_n\| \\ &\leq 2\alpha_n M_0 + \|x_{n-1} - x_n\|. \end{aligned} \tag{3.26}$$

On the other hand, we get by (1.4)

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|\alpha_{n+1}(f(x_n) - f(x_{n-1})) + (\alpha_{n+1} - \alpha_n)(f(x_{n-1}) - Tx_{n-1}) + (1-\alpha_{n+1})(Tx_n - Tx_{n-1})\| \\ &\leq \alpha_{n+1}\alpha\|x_n - x_{n-1}\| + |\alpha_{n+1} - \alpha_n| \cdot \|f(x_{n-1}) - Tx_{n-1}\| + (1-\alpha_{n+1})\|x_n - x_{n-1}\| \\ &\leq (1 - (1-\alpha)\alpha_{n+1})\|x_n - x_{n-1}\| + M|\alpha_{n+1} - \alpha_n|. \end{aligned} \tag{3.27}$$

If $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, we can get by (3.27) and Lemma 2.1 that $\|x_{n+1} - x_n\| \to 0$.

If $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, we may take $\lambda_n = (1-\alpha)\alpha_{n+1}$ and

$$b_n = M|\alpha_{n+1} - \alpha_n| = M\alpha_{n+1}|1 - \frac{\alpha_n}{\alpha_{n+1}}| = o(\lambda_n).$$

Then by (3.27) we have $a_n = \|x_n - x_{n-1}\| \to 0$. Hence, it follows by (3.26) that

$$\|x_n - Tx_n\| \leq 2\alpha_n M_0 + \|x_{n-1} - x_n\| \to 0.$$

Thus, the condition (c) in Theorem 3.2 does hold. Now we can finish the proof of Theorem 3.3 by the methods applied in Theorem 3.2. □

Remark (1) Conclusions of Chang[1, Theorem 1] and Xu[10, Theorem 4.2] hold in the setting of uniformly smooth Banach spaces. Now, these conclusions hold still in our Theorems 3.2 and 3.3 under the framework of reflexive Banach spaces with the weakly sequentially continuous duality mapping. Moreover, this paper extends their results from a finite family of nonexpansive maps to an infinite family of ones. In addition, by way of the methods applied in Chang[1, Theorem 1] we can similarly prove that the conditions (a)–(c) of our Theorem 3.2 are in fact necessary and sufficient.

(2) Particularly, in the case that $\{T_n\}$ is a finite family of nonexpansive mappings and $f(x_n) \equiv u$, we have in fact proved by our Theorem 3.2 that the result of Jung[7, Theorem 10] holds still in the setting of reflexive Banach spaces with the weakly sequentially continuous normalized duality mapping, against to the framework of uniformly smooth Banach spaces with the weakly sequentially continuous normalized duality mapping[7, Theorem 10].

(3) This paper is not the unique paper which extends some results from uniformly smooth

Banach spaces to reflexive Banach spaces with the weakly sequentially continuous normalized duality mapping. For example, in 2006, Xu[16, Theorem 3.1] extended similarly the result of Reich[5].

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