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Bipartite Graphs $K_{n,n+r}-A$ ($|A|\leq 3$) Determined by Their Cycle Length Distributions

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Abstract The cycle length distribution of a graph G of order n is a sequence $(c_1(G),\ldots,c_n(G))$, where $c_i(G)$ is the number of cycles of length i in G. In general, the graphs with cycle length distribution $(c_1(G),\ldots,c_n(G))$ are not unique. A graph G is determined by its cycle length distribution if the graph with cycle length distribution $(c_1(G),\ldots,c_n(G))$ is unique. Let $K_{n,n+r}$ be a complete bipartite graph and $A\subseteq E(K_{n,n+r})$. In this paper, we obtain: Let $s>1$ be an integer. (1) If $r=2s, n>s(s-1)+2|A|$, then $K_{n,n+r}-A$ ($A\subseteq E(K_{n,n+r}), |A|\leq 3$) is determined by its cycle length distribution; (2) If $r=2s+1, n>s^2+2|A|$, $K_{n,n+r}-A$ ($A\subseteq E(K_{n,n+r}), |A|\leq 3$) is determined by its cycle length distribution.

Keywords cycle length distribution; bipartite graphs.

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1. Introduction

Let G be a graph of order n. The cycle length distribution, denoted by CLD, of G is a sequence $(c_1(G),c_2(G),\ldots,c_n(G))$, where $c_i(G)$ is the number of cycles of length i in G. For a simple graph G, define $c_1(G)=c_2(G)=0$. In general, the graphs G with CLD $(c_1(G),c_2(G),\ldots,c_n(G))$ are not unique. A graph G is determined by its CLD if the CLD $(c_1(G),c_2(G),\ldots,c_n(G))$ of G determines uniquely the graph G. Then it is natural to ask what graphs are determined by their CLDs.

A graph $G=(V,E)$ is called a bipartite graph if its vertex set $V(G)$ can be partitioned into two parts V_1,V_2 such that every edge has one end in V_1 and one in V_2. A bipartite graph G in which every two vertices from different partition classes are adjacent is called complete. Let $K_{n,m}$ denote a complete bipartite graph with $|V_1|=n$ and $|V_2|=m$. Without loss of generality, assume that $n\leq m$ in this paper.

In [2, 3], Wang and Shi obtained

$$G=K_{n,r}-A\ (A\subseteq E(K_{n,r}),\ |A|\leq 1,\ n\leq r\leq \min(n+6,2n-3)),$$

$$G=K_{n,r}-A\ (A\subseteq E(K_{n,r}),\ |A|=2,\ n\leq r\leq \min(n+6,2n-5)),$$

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$$G = K_{n,r} - A \ (A \subseteq E(K_{n,r}),\ |A| \le 3,\ n \le r \le \min(n+6, 2n-7))$$

are determined by their CLDs. In [4], the authors improved the results of Wang and Shi and obtained: If $n \ge 9 + 2|A|$, then the bipartite graphs $G = K_{n,n+7} - A \ (A \subseteq E(K_{n,n+7}), |A| \le 3)$ are determined by their CLDs.

In this paper, we improve the above results and obtain the following result.

Theorem 1 *Let $s > 1$ be an integer.*

(1) If $r = 2s$, $n > s(s-1) + 2|A|$, then $K_{n,n+r} - A \ (A \subseteq E(K_{n,n+r}), |A| \le 3)$ is determined by its CLD.

(2) If $r = 2s+1$, $n > s^2 + 2|A|$, then $K_{n,n+r} - A \ (A \subseteq E(K_{n,n+r}), |A| \le 3)$ is determined by its CLD.

2. The proof of Theorem 1

In the following, we always use A to denote a subset of the edge set of $K_{n,m}$, i.e., $A \subseteq E(K_{n,m})$. Let $X_j = \{G | G = K_{n,m} - A,\ |A| = j\}$, $m_j = \min_{G \in X_j} c_4(G)$, $M_j = \max_{G \in X_j} c_4(G)$.

Lemma 1[3] *If $n \ge j \ge 2$, then*

$$m_j = \binom{n}{2}\binom{m}{2} - j\binom{n-1}{1}\binom{m-1}{1} + \binom{j}{2},$$
$$M_j = \binom{n}{2}\binom{m}{2} - j\binom{n-1}{1}\binom{m-1}{1} + \binom{j}{2}(m-1).$$

Lemma 2[5] *If $j \ge 2$, $n \ge j(j+1)/2 + 2$, then $M_{j+1} < m_j$.*

Lemma 3[2] *Let $G \in X_j$. If $m \ge n \ge j+2$, then, in the CLD of G, $c_{2n}(G) \ne 0$.*

We distinguish three cases to prove Theorem 1 according to the order of $|A|$.

Lemma 4 *Let $s > 1$ be an integer. If n and r are integers with*

$$n > \begin{cases} s(s-1), & r = 2s, \\ s^2, & r = 2s+1, \end{cases}$$

then $G = K_{n,n+r}$ is determined by its CLD. Moreover, the CLD of G satisfies

$$c_i(G) = \begin{cases} \frac{1}{2}\binom{n}{p}\binom{n+r}{p} p[(p-1)!]^2, & i = 2p,\ p = 2, \dots, n; \\ 0, & \text{otherwise}. \end{cases}$$

Proof Firstly, we determine the CLD of $G = K_{n,n+r}$. Since $G = K_{n,n+r}$ is a simple bipartite graph, $c_1(G) = c_2(G) = 0$, $c_{2p+1}(G) = 0$, for $p = 1, \dots, n-1$ and $c_i(G) = 0$ for $2n < i \le 2n+r$. For any $i = 2p$ $(p = 2, \dots, n)$, $K_{n,n+r}$ has $\binom{n}{p}\binom{n+r}{p}$ subgraphs $K_{p,p}$ of order i, while each $K_{p,p}$ has $\frac{1}{2}p[(p-1)!]^2$ cycles of length i. Therefore

$$c_i(G) = \frac{1}{2}\binom{n}{p}\binom{n+r}{p} p[(p-1)!]^2.$$

In the following, we will prove that $G = K_{n,n+r}$ is determined by its CLD by contradiction.

Clearly, the graphs satisfying the CLD given by the lemma must be bipartite graphs of order $2n + r$. Assume that there exists a graph $G' \neq K_{n,n+r}$ with the same CLD as G. Then $G' = K_{n,n+r} - A$, where $|A| \geq 1$, or $G' = K_{n+k,n+r-k} - A$, where $|A| \geq 0$ and $0 < k \leq \lfloor \frac{r}{2} \rfloor$.

Case 1 $G' = K_{n,n+r} - A, |A| \geq 1$. By Lemma 1, it is clear that $c_4(G') < \binom{n}{2}\binom{n+r}{2} = c_4(G)$, a contradiction.

Case 2 $G' = K_{n+k,n+r-k} - A$, $|A| \geq 0, 0 < k \leq \lfloor \frac{r}{2} \rfloor$. Let $|A| = j$. If $n + k \geq j + 2$, then $0 \leq j \leq k+n-2$. By Lemma 3, $c_{2n+2k}(G') \neq 0$, which contradicts $c_i(G) = 0$ for all $j > 2n$. Hence $G' \in \{K_{n+k,n+r-k} - A \mid |A| = j \geq n+k-1\}$. Clearly, $c_4(G') \leq \max_{|A|=j=n+k-1} c_4(K_{n+k,n+r-k} - A)$. By Lemma 1,

$$\begin{aligned} c_4(G') \leq & \binom{n+k}{2}\binom{n+r-k}{2} - (n+k-1)\binom{n+k-1}{1}\binom{n+r-k-1}{1} + \\ & \binom{n+k-1}{2}(n+r-k-1) \\ = & \binom{n+k}{2}\binom{n+r-k-1}{2}. \end{aligned}$$

If $c_4(G') < c_4(G)$, then we have a desired contradiction. Let

$$H(k) = \binom{n+k}{2}\binom{n+r-k-1}{2} - \binom{n}{2}\binom{n+r}{2}.$$

Then, to show that $c_4(G') < c_4(G)$, it suffices to show $H(k) < 0$. In the following, we will show that $H(k) < 0$.

$$\begin{aligned} H(k) - H(k-1) &= \binom{n+k}{2}\binom{n+r-k-1}{2} - \binom{n+k-1}{2}\binom{n+r-k}{2} \\ &= (n+k-1)(n+r-k-1)(\frac{r}{2} - k). \end{aligned}$$

Hence $H(k)$ increases on $[1, \lfloor \frac{r}{2} \rfloor]$.

If $r = 2s$, then

$$\begin{aligned} H(s) &= \binom{n+s}{2}\binom{n+s-1}{2} - \binom{n}{2}\binom{n+2s}{2} \\ &= -\frac{1}{4}[2(n+s)^3 - 2(s^2+2)(n+s)^2 + 2(s^2+1)(n+s) + s^4 - s^2]. \end{aligned}$$

Let

$$f(x) = 2x^3 - 2(s^2+2)x^2 + 2(s^2+1)x + s^4 - s^2.$$

Now we prove that $f(x) > 0$ for $x > s^2$. Since

$$\begin{aligned} f(s^2+1) &= 2(s^2+1)^3 - 2(s^2+2)(s^2+1)^2 + 2(s^2+1)(s^2+1) + s^4 - s^2 \\ &= s^4 - s^2 > 0, \text{ for } \forall s > 1, \end{aligned}$$

to verify that $f(x) > 0$, it suffices to show that $f(x)$ increases on $x > s^2$. Solving the equation

$$f'(x) = 6x^2 - 4(s^2+2)x + 2(s^2+1) = 0$$

gives the solutions

$$x_{1,2} = \frac{s^2 + 2 \pm \sqrt{s^4 + s^2 + 1}}{3}.$$

Clearly, $x_1, x_2 \in (0, s^2)$. Hence $f'(x) > 0$ if $x > s^2$, that is, $f(x)$ is an increasing function on $x > s^2$. Therefore, $f(n+s) > 0$ for $n > s(s-1)$, that is, $H(s) < 0$. Since $H(k)$ is an increasing function on $1 \le k \le s$, $H(k) < 0$.

If $r = 2s + 1$, then

$$\begin{aligned} H(s) &= \binom{n+s}{2}\binom{n+s}{2} - \binom{n}{2}\binom{n+2s+1}{2} \\ &= -\frac{1}{4}[2(n+s)^3 - 2(s^2+s+1)(n+s)^2 + s^2(s+1)^2]. \end{aligned}$$

Let

$$f(x) = 2x^3 - 2(s^2+s+1)x^2 + s^2(s+1)^2.$$

We prove that $f(x) > 0$ for $x > s^2 + s$. Since

$$\begin{aligned} f(s^2+s+1) &= 2(s^2+s+1)^3 - 2(s^2+s+1)(s^2+s+1)^2 + s^2(s+1)^2 \\ &= s^2(s+1)^2 > 0, \text{ for } \forall s > 1, \end{aligned}$$

to verify that $f(x) > 0$, it suffices to show that $f(x)$ increases on $x > s^2 + s$. Solving the equation

$$f'(x) = 6x^2 - 4(s^2+s+1)x = 0$$

gives the solutions

$$x_1 = 0, \ x_2 = \frac{2(s^2+s+1)}{3} < s^2 + s.$$

Hence $f'(x) > 0$ for $x > s^2 + s$, that is, $f(x)$ increases on $x > s^2 + s$. Therefore, $f(n+s) > 0$ for $n > s^2$, that is, $H(s) < 0$. Since $H(k)$ is an increasing function on $1 \le k \le s$, we have $H(k) < 0$. $\square$

Lemma 5 *Let $s > 1$ be an integer. If n and r are integers with*

$$n > \begin{cases} s(s-1)+2, & r = 2s \\ s^2 + 2, & r = 2s+1 \end{cases},$$

then $G = K_{n,n+r} - A$ ($|A| = 1$) is determined by its CLD. Moreover, the CLD of G satisfies

$$c_i(G) = \begin{cases} \frac{1}{2}\binom{n}{p}\binom{n+r}{p}p[(p-1)!]^2 - \binom{n-1}{p-1}\binom{n+r-1}{p-1}[(p-1)!]^2, & i = 2p, p = 2, \cdots, n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof Firstly, we determine the CLD of $G = K_{n,n+r} - A$. Let $A = \{e\}$ and denote $G = K_{n,r} - A = K_{n,r} - e$. Since G is a simple bipartite graph, $c_1 = c_2 = 0$, $c_{2p+1} = 0$ for $p = 1, \ldots, n-1$ and $c_i = 0$ for $\forall i > 2n$. For $i = 2p, p = 2, \ldots, n$, By Lemma 4,

$$c_i(K_{n,n+r}) = \frac{1}{2}\binom{n}{p}\binom{n+r}{p}p[(p-1)!]^2.$$

Since $K_{n,n+r}$ has

$$\binom{n-1}{p-1}\binom{n+r-1}{p-1}$$

subgraphs $K_{p,p}$ of order i which contain the edge e as an edge, while each $K_{p,p}$ has $[(p-1)!]^2$ cycles of length i which contain the edge e, $K_{n,n+r}$ has

$$\binom{n-1}{p-1}\binom{n+r-1}{p-1}[(p-1)!]^2$$

cycles of length i which contain the edge e. Hence $K_{n,n+r}-e$ has

$$c_i(G)=\frac{1}{2}\binom{n}{p}\binom{n+r}{p}p[(p-1)!]^2-\binom{n-1}{p-1}\binom{n+r-1}{p-1}[(p-1)!]^2$$

cycles of length i.

In the following, we prove that G is determined by its CLD by contradiction. Suppose that $G'\neq K_{n,r}-e$ is a graph with CLD $(c_1(G),\dots,c_{2n+r}(G))$, then G' must be a bipartite graph of order $2n+r$. By Lemma 4, $G'\neq K_{n,n+r}$. Hence $G'=K_{n,n+r}-A, |A|\ge 2$ or $G'=K_{n+k,n+r-k}-A$, $|A|\ge 0$, and $k\le\lfloor\frac{r}{2}\rfloor$.

Case 1 $G'=K_{n,n+r}-A$, $|A|\ge 2$. By Lemma 1, $c_4(G')\le\binom{n}{2}\binom{n+r}{2}-2\binom{n-1}{1}\binom{n+r-1}{1}+(n+r-1)$. But $c_4(G)=\binom{n}{2}\binom{n+r}{2}-\binom{n-1}{1}\binom{n+r-1}{1}>c_4(G')$, a contradiction.

Case 2 $G'=K_{n+k,n+r-k}-A,\ |A|=j\ge 0,\ k\le\lfloor\frac{r}{2}\rfloor$. With a similar discussion to the Case 2 of Lemma 4, we have

$$c_4(G')\le\max_{|A|=j=n+k-1}C_4(K_{n+k,n+r-k}-A)$$

and

$$c_4(G')\le\binom{n+k}{2}\binom{n+r-k-1}{2}.$$

Let

$$H(k)=\binom{n+k}{2}\binom{n+r-k-1}{2}-\binom{n}{2}\binom{n+r}{2}+\binom{n-1}{1}\binom{n+r-1}{1}.$$

If $H(k)<0$, then $c_4(G')<c_4(G)$, we have a desired contradiction.

Clearly, the function $H(k)$ defined here differs only a constant from the function $H(k)$ defined in the proof of Lemma 4, hence we have $H(k)$ increases on $k\in[1,\lfloor\frac{r}{2}\rfloor]$.

If $r=2s$, then

$$\begin{aligned}H(s)&=\binom{n+s}{2}\binom{n+s-1}{2}-\binom{n}{2}\binom{n+2s}{2}+\binom{n-1}{1}\binom{n+2s-1}{1}\\&=-\frac{1}{4}[2(n+s)^3-2(s^2+4)(n+s)^2+2(s^2+5)(n+s)+s^4+3s^2-4].\end{aligned}$$

Hence if $H(s)<0$, we have $H(k)<0$. Let $f(x)=2x^3-2(s^2+4)x^2+2(s^2+5)x+s^4+3s^2-4$. Solving the equation $f'(x)=6x^2-4(s^2+4)x+2(s^2+5)=0$ gives the solutions $x_{1,2}=\frac{s^2+4\pm\sqrt{s^4+5s^2+1}}{3}<s^2+2$. Hence $f'(x)>0$ if $x>s^2+2$, and $f(x)$ increases on $x>s^2+2$. Since $f(s^2+3)=s^4+7s^2+8>0$, $f(x)>0$ if $x>s^2+2$. Therefore $f(n+s)>0$ if $n>s(s-1)+2$. The result follows from $H(s)=-\frac{1}{4}f(n+s)$.

If $r = 2s+1$, then

$$H(s) = \binom{n+s}{2}\binom{n+s}{2} - \binom{n}{2}\binom{n+2s+1}{2} + \binom{n-1}{1}\binom{n+2s}{1}$$
$$= -\frac{1}{4}[2(n+s)^3 - 2(s^2+s+3)(n+s)^2 + 4(n+s) + (s^2+s)(s^2+s+4)].$$

Hence if $H(s) < 0$, we have $H(k) < 0$. Let $f(x) = 2x^3 - 2(s^2+s+3)x^2 + 4x + (s^2+s)(s^2+s+4)$. Solving the equation $f'(x) = 6x^2 - 4(s^2+s+3)x + 4 = 0$ gives the solutions $x_{1,2} = \frac{s^2+s+5\pm\sqrt{(s^2+s+3)^2-3}}{3} < s^2+s+2$. Hence $f'(x) > 0$ if $x > s^2+s+2$, that is, $f(x)$ increases on $x > s^2+s+2$. Since $f(s^2+s+3) = 4(s^2+s+3) + (s^2+s)(s^2+s+4) > 0$, $f(x) > 0$ if $x > s^2+s+2$. Therefore $f(n+s) > 0$ if $n > s^2+2$. The result follows from $H(s) = -\frac{1}{4}f(n+s)$. $\square$

Lemma 6 *Let $s > 1$ be an integer. n and r are integers with*

$$n > \begin{cases} s(s-1)+4, & r = 2s \\ s^2+4, & r = 2s+1 \end{cases}.$$

Then $G = K_{n,n+r} - A$ ($|A| = 2$) is determined by its CLD.

Proof Since $|A| = 2$, the subgraphs induced by A in $K_{n,n+r}$ have three configurations (as shown in Figure 1), denoted by H_1, H_2, H_3, respectively.

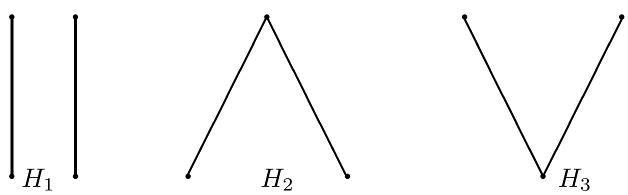

Figure 1 Three configurations induced by A

Let $G_i = K_{n,n+r} - E(H_i)$, $i = 1,2,3$. We prove that each G_i is determined by its CLD. Firstly, we prove that G_1, G_2, G_3 have different CLDs. It is easy to compute that

$$c_4(G_1) = \binom{n}{2}\binom{n+r}{2} - 2\binom{n-1}{1}\binom{n+r-1}{1} + 1,$$
$$c_4(G_2) = \binom{n}{2}\binom{n+r}{2} - 2\binom{n-1}{1}\binom{n+r-1}{1} + n - 1,$$
$$c_4(G_3) = \binom{n}{2}\binom{n+r}{2} - 2\binom{n-1}{1}\binom{n+r-1}{1} + n + r - 1.$$

Hence $c_4(G_1) < c_4(G_2) < c_4(G_3)$, G_1, G_2, G_3 have different CLDs. Next we prove that $G = G_i$ is determined by its CLD. Suppose to the contrary that $G' \neq G_i$, $i = 1,2,3$ is a graph with the same CLD as G. Then G' is a bipartite graph of order $2n+r$. By Lemmas 4 and 5, $G' \neq K_{n,n+r} - A$ ($|A| = 0,1$). Hence $G' = K_{n,n+r} - A, |A| \geq 3$ or $G' = K_{n+k,n+r-k} - A$, $|A| \geq 0$, and $k \leq \lfloor \frac{r}{2} \rfloor$.

Case 1 $G' = K_{n,n+r} - A$, $|A| \geq 3$. By Lemma 2, $c_4(G') \leq M_3 < m_2 \leq c_4(G)$, a contradiction.

Case 2 $G'=K_{n+k,n+r-k}-A$, $|A|\ge 0$, and $k\le\lfloor\frac{r}{2}\rfloor$. With a similar discussion to the Case 2 in the proof of Lemma 4, we have

$$c_4(G')\le\max_{|A|=j=n+k-1}C_4(K_{n+k,n+r-k}-A)$$

and

$$c_4(G')\le\binom{n+k}{2}\binom{n+r-k-1}{2}.$$

Since $c_4(G_1)<c_4(G_2)<c_4(G_3)$, to prove that $G=G_i$ is determined by its CLD, it suffices to prove that $c_4(G')<c_4(G_1)$. Let

$$H(k)=\binom{n+k}{2}\binom{n+r-k-1}{2}-\binom{n}{2}\binom{n+r}{2}+2\binom{n-1}{1}\binom{n+r-1}{1}-1.$$

Then it suffices to prove that $H(k)<0$. Similarly to the proof of Lemma 4, we have $H(k)$ increases on $k\in[1,\lfloor\frac{r}{2}\rfloor]$. Hence it suffices to prove that $H(\lfloor\frac{r}{2}\rfloor)<0$.

If $r=2s$, then

$$\begin{aligned}H(s)&=\binom{n+s}{2}\binom{n+s-1}{2}-\binom{n}{2}\binom{n+2s}{2}+2\binom{n-1}{1}\binom{n+2s-1}{1}-1\\&=-\frac{1}{4}[2(n+s)^3-2(s^2+6)(n+s)^2+2(s^2+9)(n+s)+s^4+7s^2-4].\end{aligned}$$

Let

$$f(x)=2x^3-2(s^2+6)x^2+2(s^2+9)x+s^4+7s^2-4.$$

Solving the equation $f'(x)=6x^2-4(s^2+6)x+2(s^2+9)=0$, we have $x_{1,2}=\frac{(s^2+6)\pm\sqrt{s^4+9s^2+9}}{3}<s^2+4$. Hence $f'(x)>0$ if $x>s^2+4$, that is, $f(x)$ increases on $x>s^2+4$. Since $f(s^2+5)=s^4+15s^2+36>0$, $f(n+s)>0$ if $n>s(s-1)+4$. The result follows from $H(s)=-\frac{1}{4}f(n+s)$.

If $r=2s+1$, then

$$\begin{aligned}H(s)&=\binom{n+s}{2}\binom{n+s}{2}-\binom{n}{2}\binom{n+2s+1}{2}+2\binom{n-1}{1}\binom{n+2s}{1}-1\\&=-\frac{1}{4}[2(n+s)^3-2(s^2+s+5)(n+s)^2+8(n+s)+(s^2+s)(s^2+s+8)+4].\end{aligned}$$

Let

$$f(x)=2x^3-2(s^2+s+5)x^2+8x+(s^2+s)(s^2+s+8)+4.$$

Solving the equation $f'(x)=6x^2-4(s^2+s+5)x+8=0$ gives the solutions $x_{1,2}=((s^2+s+5)\pm\sqrt{(s^2+s+5)^2-12})/3<s^2+s+4$. Hence $f'(x)>0$ if $x>s^2+s+4$, that is, $f(x)$ increases on $x>s^2+s+4$. Since $f(s^2+s+5)=8(s^2+s+5)+(s^2+s)(s^2+s+8)+4>0$, $f(n+s)>0$ if $n>s^2+4$. The result follows from $H(s)=-\frac{1}{4}f(n+s)$. □

Lemma 7 *Let $s>1$ be an integer. n and r are integers with*

$$n>\begin{cases}s(s-1)+6, & r=2s;\\ s^2+6, & r=2s+1.\end{cases}$$

Then $G=K_{n,n+r}-A$ ($|A|=3$) is determined by its CLD.

Proof Since $|A|=3$, the subgraphs induced by A in $K_{n,n+r}$ have six configurations (as shown in Figure 2), denoted by H_1,H_2,H_3,H_4,H_5,H_6, respectively.

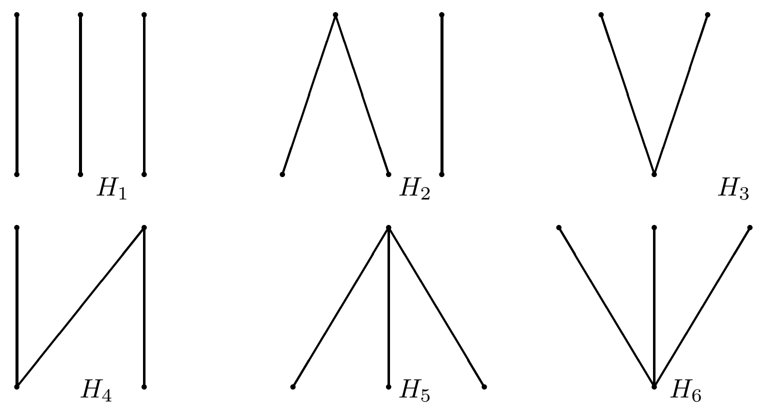

Figure 2 Six configurations induced by A

Let $G_i=K_{n,n+r}-E(H_i)$, $i=1,2,3,4,5,6$. We prove that each G_i is determined by its CLD. It is easy to compute that

$$\begin{aligned}
c_4(G_1)&=\binom{n}{2}\binom{n+r}{2}-3\binom{n-1}{1}\binom{n+r-1}{1}+3,\\
c_4(G_2)&=\binom{n}{2}\binom{n+r}{2}-3\binom{n-1}{1}\binom{n+r-1}{1}+(n-1)+2,\\
c_4(G_3)&=\binom{n}{2}\binom{n+r}{2}-3\binom{n-1}{1}\binom{n+r-1}{1}+(n+r-1)+2,\\
c_4(G_4)&=\binom{n}{2}\binom{n+r}{2}-3\binom{n-1}{1}\binom{n+r-1}{1}+(2n+r-1)-1,\\
c_4(G_5)&=\binom{n}{2}\binom{n+r}{2}-3\binom{n-1}{1}\binom{n+r-1}{1}+3(n-1),\\
c_4(G_6)&=\binom{n}{2}\binom{n+r}{2}-3\binom{n-1}{1}\binom{n+r-1}{1}+3(n+r-1).
\end{aligned}$$

Clearly, $c_4(G_1)<c_4(G_2)<c_4(G_3)<c_4(G_4)<c_4(G_5)<c_4(G_6)$. Hence G_1,G_2,G_3,G_4,G_5,G_6 have different CLDs. Suppose that $G'\neq G_i$, $i=1,2,3,4,5,6$ is a graph with the same CLD as G. Then G' must be a bipartite graph of order $2n+r$. By Lemmas 4, 5 and 6, $G'\neq K_{n,n+r}-A$ $(|A|=0,1,2)$. Hence $G'=K_{n,n+r}-A, |A|\geq 4$ or $G'=K_{n+k,n+r-k}-A$, $|A|\geq 0$, and $k\leq\lfloor\frac{r}{2}\rfloor$.

Case 1 $G'=K_{n,n+r}-A$, $|A|\geq 4$. By Lemma 2, $c_4(G')\leq M_4<m_3\leq c_4(G)$, a contradiction.

Case 2 $G'=K_{n+k,n+r-k}-A$, $|A|\geq 0$, and $k\leq\lfloor\frac{r}{2}\rfloor$. Similarly to the Case 2 in the proof of Lemma 4, we have

$$c_4(G')\leq\max_{|A|=j=n+k-1}C_4(K_{n+k,n+r-k}-A)$$

and

$$c_4(G')\le \binom{n+k}{2}\binom{n+r-k-1}{2}.$$

Since $c_4(G_1)<c_4(G_2)<\cdots<c_4(G_6)$, to prove that $G=G_i$ is determined by its CLD, it suffices to show that $c_4(G')<c_4(G_1)$. Let

$$H(k)=\binom{n+k}{2}\binom{n+r-k-1}{2}-\binom{n}{2}\binom{n+r}{2}+3\binom{n-1}{1}\binom{n+r-1}{1}-3.$$

Hence it suffices to show $H(k)<0$. Similarly to Lemma 4, we have $H(k)$ increases on $k\in[1,\lfloor\frac{r}{2}\rfloor]$. Hence it suffices to show $H(\lfloor\frac{r}{2}\rfloor)<0$.

If $r=2s$, then

$$\begin{aligned}H(s)&=\binom{n+s}{2}\binom{n+s-1}{2}-\binom{n}{2}\binom{n+2s}{2}+3\binom{n-1}{1}\binom{n+2s-1}{1}-3\\&=-\frac{1}{4}[2(n+s)^3-2(s^2+8)(n+s)^2+2(s^2+13)(n+s)+s^4+11s^2].\end{aligned}$$

Let

$$f(x)=2x^3-2(s^2+8)x^2+2(s^2+13)x+s^4+11s^2.$$

Solving the equation $f'(x)=6x^2-4(s^2+8)x+2(s^2+13)=0$, we have $x_{1,2}=\frac{(s^2+8)\pm\sqrt{s^4+13s^2+25}}{3}<s^2+6$. Hence $f'(x)>0$ if $x>s^2+6$, that is, $f(x)$ increases on $x>s^2+6$. Since $f(s^2+7)=s^4+23s^2+84>0$, $f(n+s)>0$ if $n>s(s-1)+6$. The result follows from $H(s)=-\frac{1}{4}f(n+s)$.

If $r=2s+1$, then

$$\begin{aligned}H(s)&=\binom{n+s}{2}\binom{n+s}{2}-\binom{n}{2}\binom{n+2s+1}{2}+3\binom{n-1}{1}\binom{n+2s}{1}-3\\&=-\frac{1}{4}[2(n+s)^3-2(s^2+s+7)(n+s)^2+12(n+s)+(s^2+s)(s^2+s+12)+12].\end{aligned}$$

Let

$$f(x)=2x^3-2(s^2+s+7)x^2+12x+(s^2+s)(s^2+s+12)+12.$$

Solving the equation $f'(x)=6x^2-4(s^2+s+7)x+12=0$, we have $x_{1,2}=\frac{(s^2+s+7)\pm\sqrt{(s^2+s+7)^2-18}}{3}<s^2+s+6$. Hence $f'(x)>0$ if $x>s^2+s+6$, that is, $f(x)$ increases on $x>s^2+s+6$. Since $f(s^2+s+7)=12(s^2+s+7)+(s^2+s)(s^2+s+12)+12>0$, $f(n+s)>0$ if $n>s^2+6$. The result follows from $H(s)=-\frac{1}{4}f(n+s)$. □

Theorem 1 follows directly from Lemmas 4, 5, 6 and 7.

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