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Surface Sum of Heegaard Splittings

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Abstract Suppose $M_i = V_i \cup W_i$ $(i = 1, 2)$ are Heegaard splittings. A homeomorphism $f : F_1 \to F_2$ produces an attached manifold $M = M_1 \cup_{F_1=F_2} M_2$, where $F_i \subset \partial_- W_i$. In this paper we define a surface sum of Heegaard splittings induced from the Heegaard splittings of M_1 and M_2, and give a sufficient condition when the surface sum of Heegaard splitting is stabilized. We also give examples showing that the surface sum of Heegaard splittings can be unstabilized. This indicates that the surface sum of Heegaard splittings and the amalgamation of Heegaard splittings can give different Heegaard structures.

Keywords Heegaard splitting; stabilized; amalgamation.

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1. Introduction

A compression body C is a 3-manifold obtained by attaching 2-handles to $F \times I$, along a collection of pairwise disjoint simple closed curves on $F \times \{0\}$. Then cap off any resulting 2-sphere boundary components with 3-balls, where F is a connected closed surface.

A Heegaard splitting of a 3-manifold M is a pair (V, W) where V, W are compression bodies such that $M = V \cup W$ and $V \cap W = \partial_+ V = \partial_+ W = S$. In this case, we call S the Heegaard surface.

In [3], Qiu and Ma defined connected sum and boundary connected sum of Heegaard splittings. In this paper, we define a surface sum of Heegaard splitting along an arbitrary orientable surface instead of a 2-sphere or a disk.

A Heegaard splitting $M = V \cup_S W$ is said to be stabilized if there are two properly embedded disks $D_1 \subset V$ and $D_2 \subset W$ such that ∂D_1 intersects ∂D_2 at only one point; otherwise, it is said to be unstabilized.

A Heegaard splitting $M = V \cup W$ is said to be primitively relative to F, where $F \subset \partial_- W$, if there is a meridian disk D of V and a spanning annulus A of W such that ∂D intersects $\partial_1 A$ at one point and $\partial_2 A \subset \mathrm{int} F$.

Note If F is a 2-sphere or disk, then $\partial_2 A$ bounds a disk in F. Hence $\partial_1 A$ bounds a disk in W, which means that $M = V \cup W$ is stabilized.

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In Section 2, we defined the surface sum of Heegaard splittings induced by two sub-Heegaard splittings. In Section 3, there are examples showing that the surface sum of unstabilized Heegaard splittings is stabilized, and also examples showing that surface sum of unstablized Heegaard splittings is unstabilized. In the present paper, we shall prove the following theorem:

Theorem 1 *Suppose $M_i = V_i \cup W_i$ $(i = 1, 2)$ are unstabilized Heegaard splittings, and $F_i \subset \partial_- W_i$. The surface sum of Heegaard splittings of $M = M_1 \cup_{F_1=F_2} M_2 = V_i \cup W_i$ is stabilized if one of $M_i = V_i \cup W_i$ is primitively relative to F_i.*

We conjecture that the inverse is also true.

Conjecture 1 Suppose $M_i = V_i \cup W_i$ $(i = 1, 2)$ are unstabilized Heegaard splittings, and $F_i \subset \partial_- W_i$. The surface sum of $M = M_1 \cup_{F=F_1=F_2} M_2 = V_i \cup W_i$ is stabilized if and only if one of $M_i = V_i \cup W_i$ is primitively relative to F_i.

Note (1) If F is a 2-sphere, Conjecture 2 becomes Gordon's conjecture. It is recently proved by Qiu and Scharlemann[1] and also by Bachman[2].

(2) If F is a disk, Conjecture 1 becomes Conjecture 3. Qiu and Ma[3] announced that it is also true.

(3) If F is an annulus, Conjecture 1 somehow is relative to the so-called tunnel number Conjecture.

Conjecture 2 The connected sum of two Heegaard splittings is stabilized if and only if one of the two factors is stabilized.

Conjecture 3 A Heegaard splitting obtained by boundary connected sums and self-boundary connected sums is stabilized if and only if one of the factors is stabilized.

Although we review the results and conceptions in [1] and [3], the results in this paper are self-contained.

2. Surface sum of Heegaard splittings

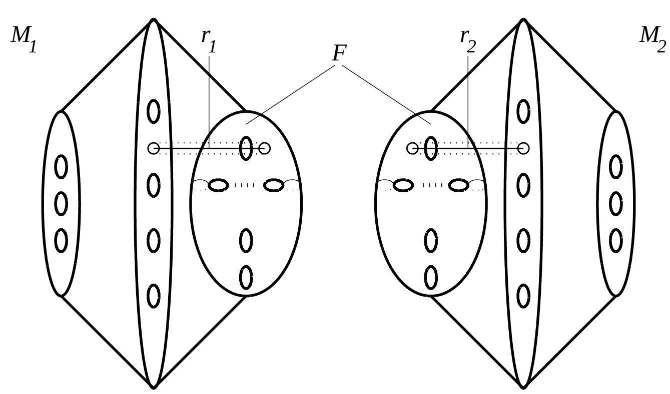

Figure 1 Surface sum of Heegaard splittings

Lemma 1 *Suppose $M_i = V_i \cup W_i$ $(i = 1, 2)$ are Heegaard splittings. Suppose $F_i \in \partial_- W_i$. A homeomorphism $f : F_1 \to F_2$ induces an attached manifold $M = M_1 \cup_{F_1=F_2} M_2$. Let r_i be an unknotted arc in W_i and $\partial_1 r \subset \partial_+ W_1$, $\partial_2 r_1 = \partial_1 r_2 \subset intF$, $\partial_2 r_2 \subset \partial_+ W_2$. Let $r = r_1 \cup r_2$. Denote by $N(r)$ the regular neighborhood of r in M. Let $V = V_1 \cup N(r) \cup V_2$, $W = M - V = W_1 \cup W_2 - N(r)$ (see Figure 1). Then $M = V \cup W$ is a Heegaard splitting of M.*

Proof It is easy to see that $V = V_1 \cup N(r) \cup V_2$ is always a compression body. The prove of that $W = M - V = W_1 \cup W_2 - N(r)$ is also a compression body are divided into two cases:

Case 1 F is a bounded surface.

We will prove that $W = M - V = W_1 \cup W_2 - N(r)$ is a compression body by induction on the Euler characteristic number of F.

Firstly, suppose $\chi(F) = 1$, which means F is a disk. Then $W = M - V = W_1 \cup W_2 - N(r)$ is obviously a compression body. Hence $M = V \cup W$ is a Heegaard splitting of M. It is also the definition of ∂-connected sum of $M_1 = V_1 \cup W_1$ and $M_2 = V_2 \cup W_2$ in [3].

Then, suppose when $\chi(F) = k$ $(k = 1, 0, -1, -2, \ldots)$, $W = W_1 \cup_F W_2 - N(r)$ is always a compression body. Now suppose $\chi(F) = k - 1$.

M can be reconstructed by attaching M_1, M_2 and $F \times I$. That is to say that $M = M_1 \cup (F \times I) \cup M_2$. Then, $r' = \partial_2 r_1 \times I$ is an unknotted arc of $F \times I$ and $\partial_1 r' = \partial_2 r_1$, $\partial_2 r' = \partial_1 r_2$. We also use the symbol $r = r_1 \cup r' \cup r_2$. Let $W = W_1 \cup (F \times I) \cup W_2 - N(r)$.

Let α be a nonseparating arc on F which cut F into F' such that $\chi(F) = k$. Furthermore, we can choose that $N(\alpha \times I)$ is disjoint from r'. Let $W' = W_1 \cup (F' \times I) \cup W_2 - N(r)$. Then $W = W' \cup N(\alpha \times I)$. Under this hypothesis, we know $W' = W_1 \cup (F' \times I) \cup W_2 - N(r)$ is a compression body. Since $W = W' \cup N(\alpha \times I)$, where $N(\alpha \times I)$ is a 2-handle attaching to $\partial_- W'$, then W is a compression body.

Case 2 F is a closed surface.

Reconstruct M, r and W as in the proof in Case 1. Let D be a disk contained in F such that $D \times I$ is disjoint from r. Let $W' = W_1 \cup [(F - D) \times I] \cup W_2$. By the proof in Case 1, W' is a compression body. Since $W = W' \cup (D \times I)$, in which $D \times I$ is a 3-handle attached on $\partial_- W'$, W is a compression body. □

We call the Heegaard splitting $M = V \cup W$ defined as in Lemma 1 the surface sum of the two Heegaard splittings $M_i = V_i \cup W_i$ $(i = 1, 2)$ along $F = F_1 = F_2$.

Proof of Theorem 1 Suppose $M_1 = V_1 \cup W_1$ is primitively related to F, that is, there is a spanning annulus A_1 in W_1 and a disk D in V_1 such that $\partial D \cap \partial_1 A$ is a single point and $\partial_2 A \subset F_1$. Suppose $A_2 \subset W_2$ is a spanning annulus such that $\partial_2 A_1 = \partial_2 A_2$. Let r be an essential arc in $A = A_1 \cup_{\partial_2 A_1 = \partial_2 A_2} A_2$. By the proof of Lemma 1, $V = V_1 \cup N(r) \cup V_2$ is a compression body and $W = W_1 \cup W_2 - \eta(r)$ is a compression body and $M = V \cup W$ is the surface sum of Heegaard splitting. Then D is an essential disk in V and $D' = A - \eta(r)$ is also an essential disk in W such that $\partial D \cap \partial D'$ is a single point. Hence $M = V \cup W$ is stabilized. □

3. Examples

Example 1 We know that there are infinitely many tunnel number one knots k_1 and k_2 which have the property that $t(k_1\#k_2)=2$. Let $E(k_i)=V_i\cup W_i$ be the genus two Heegaard splitting of the knot complement of k_i in S^3 for $i=1,2$. Suppose W_i is the compression body containing the torus boundary components of $E(k_i)$. Hence $E(k_1)=M_1=V_1\cup W_1$ and $E(k_2)=V_2\cup W_2$ are unstabilized. By Theorem 1.4 in [4] and Proposition 2.1 in [4], $E(k_i)=V_i\cup W_i$ are relative to ∂_-W_i. So the surface sum of the Heegaard splitting of $E(k_i)$ is stabilized for $i=1,2$. □

There are also examples showing that there are surface sum of unstabilized Heegaard splittings which are also unstabilized.

Example 2 Suppose W_1 is a compression body obtained by attaching one 1-handle to a torus $\times I$, and M_1 is the manifold obtained by attaching a 2-handle on W_1 along the curve c_1 as shown in Figure 2. Thus M has a Heegaard splitting $M_1=W_1\cup V_1$, where V_1 is compression body obtained by attaching a 1-handle on a torus $\times I$. Let $M_2=W_2\cup V_2$ be another copy of M_1. It is easy to see that both $M_i=W_i\cup V_i$ are unstabilized Heegaard splittings. Let M be the manifold obtained by attaching M_1 and M_2 along ∂_-W_1 and ∂_-W_2 by the identity map. Proposition 1 shows that the surface sum of Heegaard splittings of $M_1=W_1\cup V_1$ and $M_2=W_2\cup V_2$ along $\partial_-W_1=\partial_-W_2$ is unstabilized.

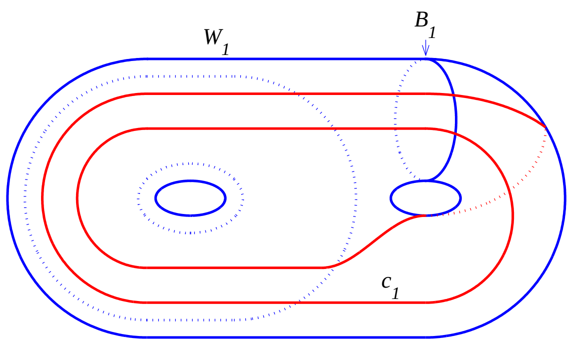

Figure 2 The Heegaard splitting of M_1

Proposition 1 *Suppose the Heegaard splitting $M=W\cup V$ is the surface sum of $M_1=W_1\cup V_1$ and $M_2=W_2\cup V_2$ in Example 2. Then $M=W\cup V$ is unstabilized.*

Proof Let D_1 be the disk in V_1 bounded by c_1. Let E_1 and F_1 be two spanning annuli in W_1 such that $E_1\cap F_1=r_1$ and ∂D_1 intersects ∂E_1 and ∂F_1 with two points (see Figure 3). Then r_1 is an unknotted arc in W_1. Since M_2 is a copy of M_1, we can find E_2, F_2 and r_2 in W_2 and D_2 in V_2. Let $r=r_1\cup r_2$. Then $V=V_1\cup N(r)\cup V_2$ is a compression body with basic disks D_1, D_2 and D_3, where D_3 is a meridian disk in $N(r)$. Furthermore $W=W_1\cup W_2-\eta(r)$ is a handlebody and B_1, B_2, $E_1\cup E_2-\eta(r)$ and $F_1\cup F_2-\eta(r)$ are four basic disks. We can see that each boundary of the basic disks in V intersects each boundary of basic disks in W with even points, thus the Heegaard splitting $M=W\cup V$ is unstabilized.

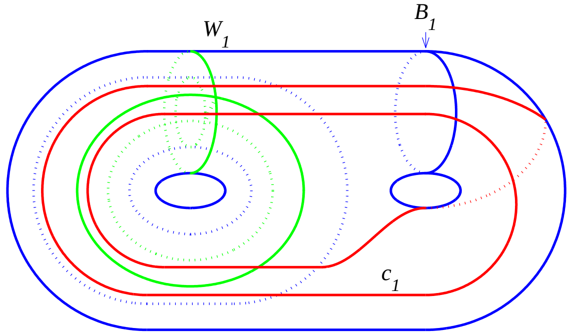

Figure 3 Basis disks

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