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Some Properties on the Category C-c′*

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0. In [1], we have shown the cusp catastrophy is the general mathematical model of the treatment based on syndrome-differentiation, but the following phenomenons often happen: An old man may die without any local pathological change; A yong man can live well with some serious local pathological changes. In mechanics, there are many similar facts. Then, confronts us a new problem: What is a local catastrophe? What is the total catastrophe? How are they connected?

We call this problem the systemetical catastróphe problem (S.C.P.), and the theory (S.C.T.).

We shall show some works about S.C.T., though they are preliminary and very rough. Clearly, it is not possible to write all of the works by one paper. In this note, we shall give some facts only about the category **C**-c′ which is used for describing a changing process of a system. The other works will be published in succession.

In the natural world, the "determining relation" among things usually are "n to 1. It would seem that we should pay attention to n-ary operations, even though the n-ary operation is not important for the present mathematics. Some of the categories relative to the n-ary operation are (**P**)(CQN)-categories having terminal objects (see [2],[3],[4],[5]). On the other hand, in this note we shall show a "regular case" of the categroy **C**-c′, which is a (**P**)-category. In addition, there are many categories satisfying Axiom (**P**), Axiom (U), or Axiom (CQN) (see [6]), and there are several subcategories of **C**-c′, which are (U) or (CQN). Then we put forward the problem NHA (see [6] or [7]). We hope NHA could give us a way to study the categories, that is, we hope NHA could become a part of S.C.T.

In this note, the categories are always small, the topologies of the countable sets are always discrete. We agree on the following appointments:

Each bijective map from a countable set to a set ($\subset R^n$) consisting of its isolated points, such as the set $\{(1,0,\cdots,0),\ (1/2,0,\cdots,0),\cdots,\ (\frac{1}{n},\ 0,\cdots,0),\cdots\}$, is a "homeomorphism".

The number zero 0 is regarded as a higher order-infinitesimal of any infinitesimal appearing in the same process. So any map φ: A→B can be regarded as

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a c^r-map, where $A \subset R^n$ consists of its isolated points and B is a subset of R^m. Hence a countable set can be regarded as a c^r-manifold, and any map defined on it to a manifold can be regarded as a c^r-map.

We have the fact that any set consisting of its isolated points in a c^r-manifold is a submanifold of the manifold (see[8, p.13, let $k=0$]).

1 . Definition. A category $\mathscr{A}$ is called a φ-c-c^r categoty, if

(i) every nonempty Hom-set is a disjoint union of differentiable manifolds, every term of the union is called a component of the Hom-set;

(ii) exists a function $\varphi \in \mathrm{SBH}\mathscr{A} = \mathrm{BH}_I(\mathscr{A}, \mathscr{A})$ (see[9]or[10]) such that the map $\bar{\varphi}$: $(B, C) \times (A, B) \to (A, C)$: $\langle \alpha, \beta \rangle \mapsto \alpha \cdot \varphi\beta$ is c^r-differentiable for every product differential structure, respectively. This means that $\forall \langle \alpha, \beta \rangle \in (B, C) \times (A, B)$ ($\exists$ M a component of (B, C), $\exists$ N a component of (A, B), $\exists$ W a component of (A, C): $\langle \alpha, \beta \rangle \in M \times N \wedge \alpha \cdot \varphi\beta \in W \wedge \bar{\varphi}$ is c^r at $\langle \alpha, \beta \rangle$).

Clearly, a Lie-group is an I-c-c^r category.

2 . Definition. Let $\mathscr{A}$ and $\mathscr{B}$ be two c-c^r categories. A functor F: $\mathscr{A} \to \mathscr{B}$ is called to be c^k, if $\forall m \in \mathscr{A}(A, A')$ ($\exists$ M a component of $\mathscr{A}(A, A')$, $\exists$ N a component of $\mathscr{B}(FA, FA')$: $m \in M \wedge F(m) \in N \wedge F$ is c^k at m).

From now on, the meaning of a c^k-map is the same as the above. We shall write that F: $(A, A') \to (B, B')$ is c^r to represent the above condition, which will not provoke any confusion.

3. Definition. The category consisting of all of the c-c^k categories and all of the c^r functors is denoted as **C**-c^r.

It is clear that the definition is reasonable.

Exists a terminal object 1 in **C**-c^r. which consists of (1) object: a; (2) $(a, a) = \{1_a\}$; and (3) $1_a \cdot 1_a = 1_a$. The empty category ϕ is regarded as a c-c^r category also, so in **C**-c^r exists a initial object.

4 .Proposition. If $\mathscr{A}$ is a φ-c-c^r category, then for any Hom-set (A , B) the function φ: $(A, B) \to (A, B)$ is a c^r-map.

Proof Let u: $(A, B) \to (B, B) \times (A, B)$: $\alpha \mapsto \langle 1_B, \alpha \rangle$, then at any point of (A, B) u is c^r, and hence we say u: $(A, B) \to (B, B) \times (A, B)$ is c^r, so is the function: $(A, B) \xrightarrow{u} (B, B) \times (A, B) \xrightarrow{\bar{\varphi}} (A, B)$. Therefore, $\bar{\varphi}u = \varphi$ is a c^r-map, q.e.d.

5 .Proposition If $\mathscr{A}$ is a φ-c-c^r category and φ is nilpotent, then $\varphi^{-1} \in c^r$.

Proof $\exists n \in N$: $\varphi^{n+1} = e$ (the identity map) $\vdash \varphi^n = \varphi^{-1} \in c^r$.

6 .Proposition If $\mathscr{A}$ is φ-c-c^r and φ^{-1} is c^r, then φ^0: $(B, C) \times (A, B) \to (A, C)$: $\langle \beta, \alpha \rangle \mapsto \beta\alpha$ is c^r.

Proof $\bar{\varphi}$: $(B, C) \times (A, B) \to (A, C)$ is $c^r \vdash \langle \beta, \alpha \rangle \xmapsto{\bar{\varphi}} \beta\varphi(\alpha) \xmapsto{\varphi} \beta\alpha$ is c^r.

7 Corollary If $\mathscr{A}$ is a φ-c-c^r category with $\varphi^{-1} \in c^r$, and if the c^r-map ψ - SBH $\mathscr{A}$, then $\mathscr{A}$ is ψ-c-c^r.

Specially, an $I_{\mathscr{A}}$-c-c$'$ category must be ψ-c-c$'$ for any c$'$-map $\psi \in \mathrm{SBH}\mathscr{A}$.

8. Example. If R is a ring equiped with a differential structure, then the map: $R \times R \to R: \langle b, a\rangle \mapsto b(-a)$ is c$'$ $\Leftrightarrow$ the map: $R \times R \to R: \langle b, a\rangle \mapsto ba$ and the map: $R \to R: a \mapsto -a$ are c$'$.

9. Definition A functor $T: \mathscr{A} \to \mathscr{B}$ is called quasifull on objects, if for any object $B \in \mathrm{ob}\mathscr{B}$ there is an object $A \in \mathrm{ob}\mathscr{A}$ such that $TA \cong B$.

10. If $F: \mathscr{A} \to \mathscr{B}$ is full and faithful as well as quasifull on objects, then [11, Th.1 in p.91] tells us that exists a functor $G: \mathscr{B} \to \mathscr{A}$ such that $\langle G, F\rangle: \mathscr{B} \to \mathscr{A}$ is an adjoint equivalence. Hence [10] shows that for any $\psi \in \mathrm{BH}_F(\mathscr{A}, \mathscr{B})$ the G-inverse of ψ exists, $\overline{\psi}^{1}_{G} \in \mathrm{BH}_G(\mathscr{B}, \mathscr{A})$. We have the following

Lemma If $F: \mathscr{A} \to \mathscr{B}$ is quasifull on objects as well as full and faithful, then for any $\psi \in \mathrm{BH}_F(\mathscr{A}, \mathscr{B})$ the self bijective half-functor $\varphi \in \mathrm{SBH}\mathscr{A}$ induces a function $\lambda^{\psi}_{\varphi} \in \mathrm{SBH}\mathscr{B}$ as follows:

$$\forall\ g \in (B_1, B_2),\quad \lambda^{\psi}_{\varphi}(g) = h_{B_2}(\psi(\varphi(\overline{\psi}^{1}_{G}(g))))h_{B_1}^{-1},$$

where $h = (h_B)_{B \in \mathrm{ob}\mathscr{B}}, FG \xrightarrow{\cdot} I$ is the natural isomorphism.

Proof Let $g \in (B_1, B_2)$ and $g' \in (B_2, B_3)$. We have

$$\begin{aligned}\lambda^{\psi}_{\varphi}(g'\ g) &= h_{B_3}(\psi(\varphi(\overline{\psi}^{1}_{G}(g'\ g))))h_{B_1}^{-1} = h_{B_3}(\psi(\varphi(Gg' \cdot \overline{\psi}^{1}_{G}(g))))h_{B_1}^{-1}\\ &= h_{B_3}(\psi(Gg' \cdot \varphi(\overline{\psi}^{1}_{G}(g))))h_{B_1}^{-1} = h_{B_3}(FGg' \cdot \psi(\varphi(\overline{\psi}^{1}_{G}(g))))h_{B_1}^{1}\\ &= (h_{B_3}(FGg')h_{B_2}^{-1}) \cdot (h_{B_2}(\psi(\varphi(\overline{\psi}^{1}_{G}(g))))h_{B_1}^{-1}) = g' \cdot \lambda^{\psi}_{\varphi}(g), \quad q.e.d.\end{aligned}$$

11 Lemma If $F: \mathscr{A} \to \mathscr{B}$ is quasifull on objects as well as full and faithful, and if $\psi \in \mathrm{BH}_F(\mathscr{A}, \mathscr{B})$ and $\varphi \in \mathrm{SBH}\mathscr{A}$, then $\forall\ u \in (GB_1, GB_2)(\lambda^{\psi}_{\varphi}(h_{B_2}\psi(u)h_{B_1}^{-1}) = h_{B_2}\psi(\varphi(u))h_{B_1}^{-1})$.

$$\begin{tikzcd}
 & (FGB_1, FGB_2) \arrow[d, "h_{B_2} - h_{B_1}^{-1}"] \\
(GB_1, GB_2) \arrow[ur, "\psi"] \arrow[d, "\varphi"'] & (B_1, B_2) \arrow[l, "\overline{\psi}^{-1}_{G}"'] \arrow[d, "\lambda^{\psi}_{\varphi}"] \\
(GB_1, GB_2) \arrow[dr, "\psi"'] & (B_1, B_2) \\
 & (FGB_1, FGB_2) \arrow[u, "h_{B_2} - h_{B_1}^{-1}"']
\end{tikzcd}$$

Fig.

Proof Given a morphism $g \in (B_1, B_2)$, we have

$$h_{B_2}(\psi(\overline{\psi}^{1}_{G}(g)))h_{B_1}^{-1} \overset{[10]}{=} h_{B_2}(\psi(\psi^{-1}(FG(g))))h_{B_1}^{-1} = h_{B_2}(FG(g))h_{B_1}^{-1} = g.$$

Hence $\lambda^{\psi}_{\varphi} \cdot (h_{B_2} - h_{B_1}^{-1}) \cdot \psi \cdot \overline{\psi}^{1}_{G} = \lambda^{\psi}$ $= (h_{B_2} - h^{-1}) \cdot \psi \cdot \varphi \cdot \overline{\psi}^{1}_{G}$. Since $\overline{\psi}^{1}_{G}$ is surjective, $(h_{B_2} - h_{B_1}^{-1}) \cdot \psi \cdot \varphi = \lambda^{\psi}_{\varphi} \cdot (h_{B_2} - h_{B_1}^{-1}) \cdot \psi$. So that $\forall\ u \in (GB_1, GB_2)(\lambda^{\psi}_{\varphi}(h_{B_2}\psi(u)h_{B_1}^{-1}) =$

$$h_{B_2}\psi(\varphi(u))h_{B_1}^{-1}), \quad q.e.d.$$

12 Proposition If $\mathscr{A}$ is a φ-c-c$'$ category, and if $F: \mathscr{A} \to \mathscr{B}$ is quasifull on objects as well as full and faithful, then the category $\mathscr{B}$ becomes a λ^{r}_{φ}-c-c$'$ category according to a topology generated by F.

Proof I The hom-set (B_1, B_2) has a defferential structure. In face, from [9] we know that for any $\psi \in \mathrm{BH}_F(\mathscr{A}, \mathscr{B})\{h_{B_2}\psi(U)h_{B_1}^{-1}\}_{U \in \tau(GB_1, GB_2)} =$ $\tau(B_1, B_2)$ is a topology generated by ψ.

Suppose $\{(\psi_i, U_i)\}_{i \in I}$ is a differential structure of (GB_1, GB_2). We define

$\bar{\bar{\varphi}}_i: h_{B_2}\psi(U_i)h_{B_1}^{-1} \to \mathrm{R}^n$: $h_{B_2}\psi(u_i)h_{B_1}^{-1} \mapsto \varphi_i(u_i)$. Since $\bar{\bar{\varphi}}_i$ factors as $h_{B_2}\psi(U_i)h_{B_1}^{1} \xrightarrow{h_{B_2}^{-1}-h_{B_1}}$ $(U_i \xrightarrow{\psi^{-1}} U_i \xrightarrow{\varphi_i} \varphi_i(U_i)$, $\bar{\bar{\varphi}}_i$ is a homeomorphism (see [9, Lemma 3.2 and 3.3]). Therefore, $(\bar{\bar{\varphi}}_i, h_{B_2}\psi(U_i)h_{B_1}^{-1}$ is a chart. In addition, ψ is surjective, so $\bigcup_{i\in I}(h_{B_2}\psi(U_i)h_{B_2}^{-1}) = (B_1, B_2)$. It is easy to prove $\{(\bar{\bar{\varphi}}_i, h_{B_2}\psi(U_i)h_{B_1}^{-1})\}_{i\in I}$ is a differential structure of (B_1, B_2).

2. $\mathscr{B}$ is a λ_φ^F-c-c^r category. Let $\psi = F \in BH_F(\mathscr{A}, \mathscr{B})$. Lemma 10 shows $\lambda_\varphi^F \in SBH\mathscr{B}$. Exists $\overline{F'_G} \in BH_G(\mathscr{B}, \mathscr{A})$. Let $B_1, B_2, B_3 \in \mathrm{ob}\mathscr{B}$. The crux of the proof is to show that $\forall v \in (GB_2, GB_3)$, $\forall u \in (GB_1, GB_2)(h_{B_3}F(v)h_{B_2}^{-1}) \cdot \lambda_\varphi^F(h_{B_2}F(u)h_{B_1}^{-1}) = h_{B_3} \cdot F(v\varphi(u))h_{B_1}^{-1})$. Lemma 11 shows it is true, q.e.d.

13 Lemma If $\varphi_i \in SBH\mathscr{A}_i$, $i \in I$, then $\underset{i\in I}{\times}\varphi_i \in SBH\underset{i\in I}{\times}\mathscr{A}_i$, where

$$\times\varphi_i: \times(A_i, A_i') \to \times(A_i, A_i'): \langle f_i\rangle_{i\in I} \mapsto \langle\varphi_i f_i\rangle_{i\in I}.$$

Proof $(\times\varphi_i)(\langle f_i\rangle \cdot \langle g_i\rangle) = (\times\varphi_i)(\langle f_i g_i\rangle) = \langle\varphi_i(f_i g_i)\rangle = \langle(\varphi_i f_i)g_i\rangle = \langle\varphi_i f_i\rangle \cdot \langle g_i\rangle = (\times\varphi_i)(\langle f_i\rangle)\cdot\langle g_i\rangle$, q.e.d.

14 Lemma If $\mathscr{A}_i$ is φ_i-c-c^r, $i \in I$ a finite set, then $\underset{i\in I}{\times}\mathscr{A}_i$ is $\underset{i\in I}{\times}\varphi_i$-c-c^r.

Proof Let $i = 1, 2$; and let $A_i \in \mathrm{ob}\mathscr{A}_i$. The hom-sets of $\underset{i\in I}{\times}\mathscr{A}_i$ have the product differential stuctures (see [8, p.13]). We have

$$\overline{\times\varphi_i}: (\langle A_1', A_2'\rangle, \langle A_1'', A_2''\rangle) \times (\langle A_1, A_2\rangle, \langle A_1', A_2'\rangle) \to (\langle A_1, A_2\rangle, \langle A_1'', A_2''\rangle):$$
$$\langle\langle f', g'\rangle, \langle f, g\rangle\rangle \mapsto \langle f', g'\rangle \cdot (\times\varphi_i)(\langle f, g\rangle) = \langle f' \cdot \varphi_1 f, g' \cdot \varphi_2 g\rangle.$$

Therefore, $\times\varphi_i \in$ c^r, q.e.d.

15 Proposition The category **C**-c^r has finite products.

Proof Suppose $\mathscr{A}_i$ is φ_i-c-c^r, $i \in I$ a finite set. Let $\lambda_i: \times\mathscr{A}_i \to \mathscr{A}_i: \lambda_i \cdot (\langle A_i\rangle) = A_i$; $\lambda_i(\langle f_i\rangle) = f_i$. Clearly, λ_i is a c^r-functor.

Suppose $\forall i \in I$ ($\exists$ a c^r-functor F_i: $\mathscr{D} \to \mathscr{A}_i$), we define a functor F: $\mathscr{D} \to \times\mathscr{A}_i$ as follows:

$$F(D) = \langle F_i(D)\rangle;\quad F(h) = \langle F_i(h).$$

It is easy to know that F is a c^r-functor and $\lambda_i F = F_i$, q.e.d.

16 There are many regular cases in **C**-c^r, such as the category consisting of

(1) objects, φ-c-c^r category with φ nilpotent, and

(2) $(\mathscr{A}, \mathscr{B}) = \{F \mid F: \mathscr{A} \to \mathscr{B}$ is full and faithful as well as quasifull on objects $\wedge$ the differential structure of $\mathscr{B}(B_1, B_2)$ is one generated by $F\}$.

We call it the category **RC**-c^r. It satisfies the axiom (**P**) The proof is not difficult, its sketch is the following:

If there is a commutative diagram

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "H"] \arrow[d, "S"'] & \mathscr{C} \arrow[d, "G"] \\
\mathscr{A} \arrow[r, "F"'] & \mathscr{B}
\end{tikzcd}$$

then we have a nonempty category   , which consists of

(1) objects, every elements of the class $\{\langle A, C\rangle \mid A\in \mathrm{ob}\mathscr{A}, C\in \mathrm{ob}\mathscr{C}, FA=GC\}$, and

(2) $(\langle A, C\rangle, \langle A', C'\rangle) = \coprod_{v\in(FA,\,FA')} F^{-1}_{|(A,\,A')}(v)\times G^{-1}_{|(C,\,C')}(v)$

Given a commutative diagram

$$\begin{tikzcd} \mathscr{K} \arrow[r, "M"] \arrow[d, "L"'] & \mathscr{C} \arrow[d, "G"] \\ \mathscr{A} \arrow[r, "F"'] & \mathscr{B} \end{tikzcd}$$

we define a functor U: $\mathscr{K}\to\mathscr{M}$ as follows:

$\forall\, O\in \mathrm{ob}\,\mathscr{K}$, $U(O)=\langle L(O), M(O)\rangle$ and $\forall\, h$: $O\to O'$, $U(h)=\langle L(h), M(h)\rangle$.

U is full and faithful as well as quasifull on objects.

The differential structure of $\mathscr{M}$ is defined to be the structure generated by U (see Proposition 12), and $\mathscr{M}$ is I-c-c′ (see Proposition 5, 6 and 12).

Next, we define two functors:

P: $\mathscr{M}\to\mathscr{C}$: $\langle A, C\rangle\mapsto C$; $\langle f, g\rangle\mapsto g$ and Q: $\mathscr{M}\to\mathscr{A}$: $\langle A, C\rangle\mapsto A$; $\langle f, g\rangle\mapsto f$.

Clearly P and Q are full and faithful as well as quasifull on objcts.

Finally, the differential structure of $\mathscr{C}$ coincides with the structure generated by P. The proof is completed.

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