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Two Summation Rules Using Generalized Stirling Numbers of the Second Kind*

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Abstract. The generalized Stirling numbers of the second kind introduced by B.S. El-Desouky are userd to formulate two summation rules concerning the summation of the form $\sum_{k=0}^{n} F(n,k)k^m$, whereby some new identities are derived.

Keywords Stirling numbers, combinatorial identity, δ^*−transformation.

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1. Introduction

It is known that summations of the form $\sum_{k=0}^{n} F(n,k)k^m$ can be computed using Stirling numbers of the second kind, provided that there can be found a summation formula or a combinatorial identity such as

$$\sum_{k=j}^{n} F(n,k)\binom{k}{j} = \phi(n,j), \quad j \geq 0, \tag{1}$$

where $F(n,k)$ is a bivariate function defined for integers $n, k \geq 0$. For details and various examples, see L.C.Hsu [4].

The object of this note is to formulate two more general summation rules that may be used to compute sums of the forms $\sum_{k=0}^{m} F(n,k)f_m(k)$ and $\sum_{k=0}^{\infty} F(k)f_m(k)$ with $f_m(x)$ denoting an m−th degree polynomial in x. These two summation rules involve generalized Stirling numbers of the second kind recently introduced by B.S. El-Desouky in [3].

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots,)$ be a sequence of complex numbers, and denote

$$(t|\alpha)_n = \prod_{j=0}^{n-1}(t-\alpha_j), \qquad (t|\alpha)_0 = 1.$$

B.S. El-Desouky [3] has defined $S(n,k;\bar{\alpha}) \equiv S(n,k;\alpha_0,\alpha_1,\cdots,\alpha_{n-1})$ as the multiparameter non- central Stirling numbers of the second kind with parameters $\bar{\alpha} \equiv (\alpha_0, \alpha_1 1, \cdots, \alpha_{n-1})$ defined by the following

$$(t|\alpha)_n = \sum_{k=0}^{n} S(n,k;\bar{\alpha})(t)_k, \tag{2}$$

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where $(t)_k = t(t-1)\cdots(t-k+1)$ is the k-th falling factorial of t, $S(0,0;\bar{\alpha}) = 1$ and $S(n,k;\bar{\alpha}) = 0$ for $k > n$.

First Rule *If (1) holds for every $j \geq 0$, then for every $m \geq 0$ we have a summation formula such as*

$$\sum_{k=0}^{n} F(n,k)(k|\alpha)_m = \sum_{j=0}^{m} \phi(n,j) j! S(m,j;\bar{\alpha}). \tag{3}$$

This may be verified at once. In fact, we have

$$\begin{aligned}\sum_{k=0}^{n} F(m,k)(k|\alpha)_m &= \sum_{k=0}^{n}\sum_{j=0}^{m} F(n,k) S(m,j;\bar{\alpha})(k)_j \\ &= \sum_{j=0}^{m} S(m,j;\bar{\alpha}) \sum_{k=0}^{n} F(n,k) j! \binom{k}{j} \\ &= \sum_{j=0}^{m} S(m,j;\bar{\alpha}) j! \phi(n,j).\end{aligned}$$

Clearly, (3) would be particularly useful when n is much larger than m.

2. Applications of the Rule

In [4], there has been listed many combinatorial identities involving the summation of the form $\sum_{k=0}^{n} F(n,k)k^m$. As may be observed, all these examples can be extended to such a case that the factor k^m is replaced by $(k|\alpha)_m$. More precisely, corresponding to the formulas (4)–(9), (13) and (16) in [4], we have the following identities, respectively,

$$\sum_{k=1}^{n} (k|\alpha)_m = \sum_{j=1}^{m} \binom{n+1}{j+1} j! S(m,j;\bar{\alpha}), \tag{4}$$

$$\sum_{k=1}^{n} (k|\alpha)_m \binom{n}{k} p^k q^{n-k} = \sum_{j=1}^{m} \binom{n}{j} p^j j! S(m,j;\bar{\alpha}), \tag{5}$$

where $p + q = 1, p, q > 0$.

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} (k|\alpha)_m = \sum_{j=0}^{m} 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j} j! S(m,j;\bar{\alpha}), \tag{6}$$

$$\sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} (k|\alpha)_m = \sum_{j=0}^{m} 2^{n-2j} \binom{n-j}{j} j! S(m,j;\bar{\alpha}), \tag{7}$$

$$\sum_{k=0}^{n} \binom{n-k}{s} (k|\alpha)_m = \sum_{j=0}^{m} \binom{n+1}{s+j+1} j! S(m,j;\bar{\alpha}), \tag{8}$$

$$\sum_{k=0}^{n} \binom{s+k}{s} (k|\alpha)_m = \sum_{j=0}^{m} \binom{n+1}{j} \binom{n+s+1}{s} \frac{n+1-j}{s+1+j} j! S(m,j;\bar{\alpha}), \tag{9}$$

$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}(k|\alpha)_m=\sum_{j=0}^{m}\binom{a}{j}\binom{a+b-j}{n-j}j!S(m,j;\bar{\alpha}), \tag{10}$$

where a and b are real parameters.

$$\sum_{k=1}^{n}(k|\alpha)_m H_k=\sum_{j=1}^{m}\binom{n+1}{j+1}(H_{n+1}-\frac{1}{j+1})j!S(m,j;\bar{\alpha}), \tag{11}$$

where $H_k := 1+\frac{1}{2}+\cdots+\frac{1}{k}$, $(k\geq 1)$ are harmonic numbers.

Since $(k|0)_m = k^m$ and $S(m,j;\bar{0}) = S(m,j)$, the original Stirling numbers of the second kind, it is clear that the identities listed above will reduce to the original ones in [4] whenever $\alpha=(0,0,\cdots,0)$.

It is also easily seen from (2) that $S(n,n;\bar{\alpha})=1$. Thus, it follows that, for n large,

$$\begin{aligned}\sum_{k=1}^{n}(k|\alpha)_m H_k &\sim \binom{n+1}{m+1}(H_{n+1}-\frac{1}{m+1})m!S(m,m;\bar{\alpha})\\ &\sim \frac{n^{m+1}}{m+1}(H_{n+1}-\frac{1}{m+1})\\ &\sim \frac{n^{m+1}}{m+1}(\log n+\gamma-\frac{1}{m+1}),\end{aligned} \tag{12}$$

where $\gamma=\lim(H_n-\log n)=0.5772\cdots$ is Euler's constant. The above analysis shows that the asymptotic behavior of $\sum_{k=1}^{n}(k|\alpha)_m H_k$ when $n\to\infty$ is independent of the sequence $\alpha=(\alpha_0,\alpha_1,\alpha_2,\cdots,)$.

Also, one may easily observe from (2) that in the case $\alpha=(0,1,2,\cdots,)$, i.e., $\alpha_j=j$, we have

$$S(n,n;\bar{\alpha})=1, S(n,k;\bar{\alpha})=0, k<n.$$

Consequently, (6) and (7) reduce to the pair of Moriarty identities (cf. [2], (2.73), (2.74)) as follows

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{k}{m}=2^{n-2m-1}\binom{n-m}{m}\frac{n}{n-m},$$

$$\sum_{k=0}^{[n/2]}\binom{n+1}{2k+1}\binom{k}{m}=2^{n-2m}\binom{n-m}{m}.$$

where $m<n$. Similarly, (11) gives the following known relation for $\alpha_j=j, j=0,1,2,\cdots,$.

$$\sum_{k=m}^{n}\binom{k}{m}=\binom{n+1}{m+1}(H_{n+1}-\frac{1}{m+1}),$$

Finally, we see that (10) implies the following identity with $a=b=n$.

$$\sum_{k=0}^{n}\binom{n}{k}^2(k|\alpha)_m=\sum_{j=0}^{m}\binom{2n-j}{n}(n)_j S(m,j;\bar{\alpha})$$

This is a general form of an example mentioned in Comtet [1, Ch.5].

3. Rule for infinite sums

We may also formulate a summation rule for infinite series. This is quite similar to the case exhibited in [4].

Second Rule *Let $\{f(k)\}$ be a real-valued sequence. If the δ^*-transformation of $\{f(k)\}$ is given by $\{g(j)\}$, namely, we have that for every $j \geq 0$,*

$$\sum_{k=0}^{\infty} f(k)\binom{k}{j}(-1)^j = g(j) \tag{13}$$

Then for any given $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots)$, we have a summation formula such as

$$\sum_{k=0}^{\infty} f(k)(k|\alpha)_m = \sum_{j=0}^{m}(-1)^j g(j) j! S(m, j; \bar{\alpha}). \tag{14}$$

Notice that the convergence of the series in (13) guarantees that the order of summation can be altered, so that we may verify (14) using the similar procedure as for the verification of (3). The detials are omitted. The final example of [4] is the series with an explicit sum

$$\sum_{k=0}^{\infty} k^m \rho^{-k} F_k = \frac{\rho}{\sqrt{5}} \sum_{j=0}^{m} \left((\frac{a}{\rho - a})^{j+1} - (\frac{b}{\rho - b})^{j+1} \right) j! S(m, j), \tag{15}$$

where $a = (1+\sqrt{5})/2, b = (1-\sqrt{5})/2$, and $F_k = (a^{k+1} - b^{k+1}/\sqrt{5}$ are the well known Fibonacci numbers, and ρ is a positive constant with $\rho > a$. Now, making use of the rule (13)$\Rightarrow$ (14) one may get a new series

$$\sum_{k=0}^{\infty} (k|\alpha)_m \rho^{-k} F_k = \frac{\rho}{\sqrt{5}} \sum_{j=0}^{m} \left((\frac{a}{\rho - a})^{j+1} - (\frac{b}{\rho - b})^{j+1} \right) j! S(m, j; \bar{\alpha}), \tag{16}$$

which will be reduced to (15) when $\alpha = (0, 0, 0, \cdots)$.

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两个涉及第二类Stirling数的求和规则

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摘　要

本文利用第二类Stirling 数给出了两个求和规则，从而重新得到了一些古典组合恒等式的统一证明，并由此给出了一些级数求和的渐近估计.

首届数学哲学与方法论研讨会在天津召开

由著名数学家胡国定、徐利治与台湾数学家李国伟倡导的我国首届数学哲学与方法论研讨会，于1994年11月4日在天津南开数学研究所召开。会议历时四天。来自海峡两岸近三十名专家、学者出席了这次会议。会议着重研讨了在数学研究和数学教育中有重要影响的数学哲学和方法论问题，交流了研究动态和信息，并就今后数学哲学与方法论研究提出了建设性的意见。

出席这次会议的一些著名数学家在会上作了专题报告，引起了与会者的极大兴趣。如吴文俊教授的"漫谈公理化与机械化"、王梓坤教授的"论混沌与随机性"、徐利治教授的"自然数列的二重性与相无限性"、莫绍揆教授的"完备的一阶逻辑与集合论"、胡国定教授的"数学对象的哲学探讨"等等，都是数学研究前沿上的深刻的哲学思考，对数学哲学与方法论研究有着重要的理论意义。台湾中央研究院数学研究所李国伟研究员作了"人脑是不是电脑？——数学哲学与认知哲学交汇的一例"及"证明的流变"等报告，并介绍了海外数学哲学与方法论研究状况。他的报告得到了与会者的高度评价。

这次会议的一个重要议题是数学哲学、数学方法论与数学史、数学教育的关系。与会者从不同角度发表了自己的意见。其中朱梧槚、杨安洲教授等侧重讨论了数学基础中的有关哲学问题；林夏水、郑毓信、袁向东教授等介绍了国内外数学哲学与方法论研究的进展，胡作玄和李文林教授分别探讨了数学方法的层次结构和科学知识增长的数学描述，孙小礼、王前、徐本顺、张鸿庆教授等着重探讨了数学哲学、数学方法论研究与数学教育的关系。大家都感到，数学哲学与方法论研究有着巨大的应用潜力和广阔的发展前景。今后的数学哲学与方法论研究，应加强数学工作者与哲学工作者的思想交流和合作，注重探讨与当代数学研究和数学教育关系密切的理论问题，注重数学哲学与方法论研究成果的应用研究。与会者指出，由徐利治教授倡导的数学方法论研究，不仅在理论上获得了独具特色的成果，而且在应用中产生了明显的社会效益，这项工作是很有意义的，值得深入研究和大力推广。

这次研讨会得到了主办单位南开数学研究所和江苏教育出版社赞助单位的大力支持，取得了圆满的成功。与会者认为，这样的数学哲学与方法论研讨会今后还应继续举办下去，并争取有更多的海内外专家、学者参加，以推动这方面研究的深入开展。（王前、郭永康）