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Spectral Characterization of the Edge-Deleted Subgraphs of Complete Graph

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Abstract In this paper, we show that some edges-deleted subgraphs of complete graph are determined by their spectrum with respect to the adjacency matrix as well as the Laplacian matrix.

Keywords cospectral graphs; spectra of graph; eigenvalues.

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1. Introduction

Let $G=(V,E)$ be a graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and edge set E. All graphs considered here are simple and undirected. Let $d(v_i)$ denote the vertex degree of v_i. Let $A(G)$ be the (0,1)-adjacency matrix of G. The matrix $L(G)=D(G)-A(G)$ is called the Laplacian matrix of G, where $D(G)$ is the $n\times n$ diagonal matrix with $\{d_1,d_2,\ldots,d_n\}$ as diagonal entries (and all other entries 0). The polynomial $P_{A(G)}(\lambda)=\det(\lambda I-A(G))$ and $P_{L(G)}(\mu)=\det(\mu I-L(G))$ are defined as the characteristic polynomials of the graph G with respect to the adjacency matrix and the Laplacian matrix, respectively, where I is the identity matrix, which can be written as $P_{A(G)}(\lambda)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$ and $P_{L(G}(\mu)=\mu^n+q_1\mu^{n-1}+\cdots+q_n$, respectively. Since both matrices $A(G)$ and $L(G)$ are real and symmetric, their eigenvalues are all real numbers. Assume that $\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_n(G)$ and $\mu_1(G)\geq\mu_2(G)\geq\cdots\geq\mu_n(G)(=0)$ are the adjacency eigenvalues and Laplacian eigenvalues of graph G, respectively. The adjacency spectrum of graph G consists of the adjacency eigenvalues (together with their multiplicities), and the Laplacian spectrum of graph G consists of the Laplacian eigenvalues (together with their multiplicities).

Two graphs are cospectral if they share the same spectrum. A graph G is said to be determined by its spectrum (DS for short) if for any graph H, $P_{A(H)}(\lambda)=P_{A(G)}(\lambda)$(or $P_{L(H)}(\mu)=P_{L(G)}(\mu)$) implies that H is isomorphic to G.

Up to now, only few graphs with very special structures have been proved to be determined by their spectra. So, "which graphs are determined by their spectrum?" [3] seems to be a difficult

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problem in the theory of graph spectrum.

Some known results can be found in [2, 4–8, 10–13].

In this paper, some more special graphs will be discussed. If a graph G is obtained from K_n by deleting one, two, three or four edges, then G must be isomorphic to one of G_{ij} ($i = 1, 2, 3, 4; j = 0, 1, \dots, 10$) as shown in Figure 1.

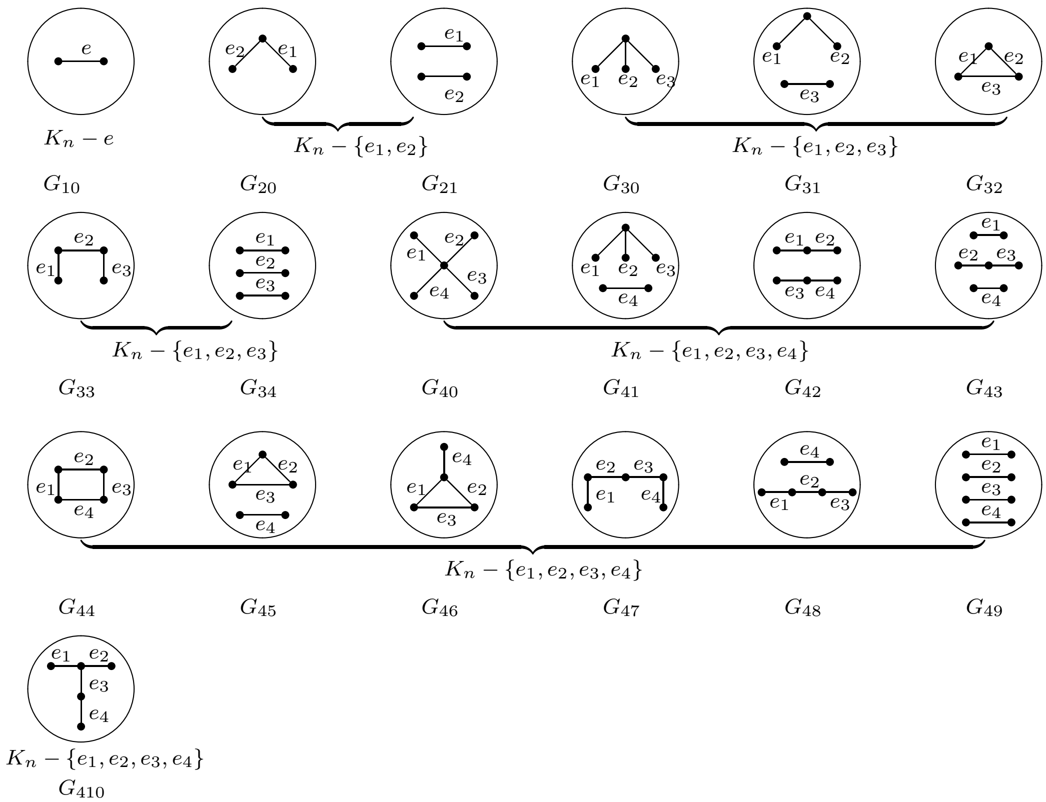

Figure 1 G_{ij} ($i = 1, 2, 3, 4; j = 0, 1, \dots, 10$)

Let $\mathcal{G}$ be a collection consisting of G where G is the graph obtained from the complete graph K_n by deleting one, two, three or four edges, that is, $\mathcal{G} = \{G_{10}, G_{20}, G_{21}, G_{30}, G_{31}, G_{32}, G_{33}, G_{34}, G_{40}, G_{41}, G_{42}, G_{43}, G_{44}, G_{45}, G_{46}, G_{47}, G_{48}, G_{49}, G_{410}\}$. The number of deleted edges is i in K_n. In this paper, we prove that for any graph $G \in \mathcal{G}$, G is determined by its adjacency spectrum and Laplacian spectrum, respectively. That is

Theorem 1.1 *If graph G_i is obtained from K_n ($n \geq i + 2$) by deleting i ($i = 1, 2, 3, 4$) edges, then G_i is determined by its adjacency spectrum.*

Theorem 1.2 *If graph G_i is obtained from K_n ($n \geq i + 2$) by deleting i ($i = 1, 2, 3, 4$) edges, then G_i is determined by its Laplacian spectrum.*

2. Some lemmas

In the section, we will present some lemmas which are required in the proof of the main results.

Lemma 2.1 ([1]) *The coefficients of the characteristic polynomial of a graph G satisfy:*

(1) $a_1 = 0$;

(2) $-a_2$ *is the number of edges of* G;

(3) $-a_3$ *is twice the number of triangles in* G.

Lemma 2.2 ([3, 9]) *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum.*

(i) The number of vertices.

(ii) The number of edges.

(iii) Whether G is regular.

(iv) Whether G is regular with any fixed girth.

For the adjacency matrix the following follows from the spectrum.

(v) The number of closed walk of any length.

(vi) Whether G is bipartite.

For the Laplacian matrix the following follows from the spectrum.

(vii) The number of spanning trees.

(viii) The number of components.

(ix) The sum of the squares of degrees of vertices.

Lemma 2.3 ([9, p. 657]) *Let G be a graph with e edges, x_i vertices of degree i, and y 4-cycles. Then*

$$|w_4(G)| = 2e + 4\sum_i \binom{i}{2} x_i + 8y, \tag{1}$$

where $|w_4(G)|$ is the total number of closed 4-walks in G.

Lemma 2.4 *Let G be a graph with n vertices and $\binom{n}{2} - i$ edges, $i = 1, 2, 3, 4$. If $n \geq 3, 4, 5, 6$ for $i = 1, 2, 3, 4$, respectively, then G has only one connected component.*

Proof Without loss of generality, we take $i = 4$. Assume that G have l ($l > 1$) connected components, that is $G = G_{n_1} \cup G_{n_2} \cup \cdots \cup G_{n_l}$, where $|V(G_{n_i})| = n_i$, $i = 1, 2, \ldots, l$ and $n_1 + n_2 + \cdots + n_l = n$.

$$\begin{aligned}\frac{n(n-1)}{2} - 4 = |E(G)| &= |E(G_{n_1})| + |E(G_{n_2})| + \cdots + |E(G_{n_l})| \\ &\leq \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \cdots + \frac{n_l(n_l-1)}{2},\end{aligned}$$

namely,

$$\sum_{i=1}^{l} n_i^2 + 2\sum_{1\leq i<j\leq l} n_i n_j - 8 = n^2 - 8 \leq \sum_{i=1}^{l} n_i^2,$$

we get

$$\sum_{1\le i<j\le l} n_i n_j \le 4.$$

Since $n \ge 6$, this is a contradiction.

Lemma 2.5 ([1, p. 41]) *If $\overline{G}$ is the complement of G, and G has n vertices, then*

$$\kappa(G) = n^{-2} P_{L(\overline{G})}(n), \tag{2}$$

where $\kappa(G)$ is the number of spanning trees of the graph G.

3. Proofs of Theorems 1.1 and 1.2

It is well known that the complete graph K_n are determined by their adjacency spectrum and Laplacian spectrum. Now we are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Let $G_i \in \mathcal{G}$. Suppose a graph H is cospectral with G_i with respect to the adjacency spectrum. We consider the following cases.

Case 1 i=1. Consider the complete graph K_n by deleting one edge. By Lemma 2.2, H is a graph with n vertices and $\binom{n}{2}-1$ edges. By Lemma 2.4, H has only one connected component, then $H \cong G \cong G_{10}$.

Case 2 i=2. Similarly to Case 1, we have $H \cong G_{20}$ or $H \cong G_{21}$. In view of the fact that $\binom{n}{3} - 2(n-2) + 1$ triangles are contained in G_{20} and $\binom{n}{3} - 2(n-2)$ triangles are contained in G_{21}, by Lemma 2.1(3) or Lemma 2.2(v), G is determined by its adjacency spectrum.

Case 3 i=3. Similarly to Case 1, the H must be isomorphic to one of G_{3j} $(j = 0, 1, 2, 3, 4)$.

There are $\binom{n}{3}-3(n-2)+3$, $\binom{n}{3}-3(n-2)+1$, $\binom{n}{3}-3(n-2)+2$, $\binom{n}{3}-3(n-2)+2$ and $\binom{n}{3}-3(n-2)$ triangles contained in G_{30}, G_{31}, G_{32}, G_{33} and G_{34}, respectively. Obviously, G_{32} and G_{33} have equal triangles. Moreover, there are $2e + 4(3\binom{n-3}{2} + (n-3)\binom{n-1}{2}) + 8(3\binom{n}{4} - 6\binom{n-2}{2} + 3(n-3))$, $2e + 4(2\binom{n-3}{2} + 2\binom{n-2}{2} + (n-4)\binom{n-1}{2}) + 8(3\binom{n}{4} - 6\binom{n-2}{2} + 2(n-3) + 1)$ closed 4-walks in G_{32} and G_{33}, respectively. If G_{32} and G_{33} are cospectral, by Lemma 2.2(v), we have

$$\begin{aligned} &2e + 4\left(3\binom{n-3}{2} + (n-3)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 6\binom{n-2}{2} + 3(n-3)\right) \\ &= 2e + 4\left(2\binom{n-3}{2} + 2\binom{n-2}{2} + (n-4)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 6\binom{n-2}{2} + 2(n-3) + 1\right). \end{aligned}$$

Solving this equation, we get n=3, a contradiction.

Case 4 i=4. Similarly to Case 1, the H must be isomorphic to one of G_{4j} $(j = 0, 1, 2, \ldots, 10)$.

In view of $G_{40} - G_{410}$, there are $\binom{n}{3} - 4(n-2) + 6$, $\binom{n}{3} - 4(n-2) + 3$, $\binom{n}{3} - 4(n-2) + 2$, $\binom{n}{3} - 4(n-2) + 1$, $\binom{n}{3} - 4(n-2) + 4$, $\binom{n}{3} - 4(n-2) + 2$, $\binom{n}{3} - 4(n-2) + 4$, $\binom{n}{3} - 4(n-2) + 3$, $\binom{n}{3}-4(n-2)+2$, $\binom{n}{3}-4(n-2)$ and $\binom{n}{3}-4(n-2)+4$ triangles contained in $G_{40}-G_{410}$, respectively. Obviously, G_{41} and G_{47} have equal triangles, G_{44}, G_{46} and G_{410} have equal triangles, G_{42}, G_{45} and G_{48} have equal triangles. If they are cospectral, we consider the following subcases.

Subcase 1 By Lemma 2.3, we calculate $|w_4(G_{41})|$ and $|w_4(G_{47})|$. We have

$$|w_4(G_{41})| = 2e + 4\left(\binom{n-4}{2} + 5\binom{n-2}{2} + (n-6)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 8\binom{n-2}{2} + 3(n-3) + 6\right) \text{ and}$$

$$|w_4(G_{47})| = 2e + 4\left(3\binom{n-3}{2} + 2\binom{n-2}{2} + (n-5)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 8\binom{n-2}{2} + 3(n-3) + 4\right).$$

By Lemma 2.2(v), we have $|w_4(G_{41})| = |w_4(G_{47})|$, that is

$$\begin{aligned}&2e + 4\left(\binom{n-4}{2} + 5\binom{n-2}{2} + (n-6)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 8\binom{n-2}{2} + 3(n-3) + 6\right)\\ &\quad = 2e + 4\left(3\binom{n-3}{2} + 2\binom{n-2}{2} + (n-5)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 8\binom{n-2}{2} + 3(n-3) + 4\right).\end{aligned}$$

This equation has no solution.

Subcase 2 Similarly to Subcase 1, by Lemma 2.3, we calculate $|w_4(G_{44})|$, $|w_4(G_{46})|$ and $|w_4(G_{410})|$. We have

$$|w_4(G_{44})| = 2e + 4\left(4\binom{n-3}{2} + (n-4)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 8\binom{n-2}{2} + 4(n-3) + 1\right),$$

$$|w_4(G_{46})| = 2e + 4\left(2\binom{n-3}{2} + \binom{n-2}{2} + \binom{n-4}{2} + (n-4)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 8\binom{n-2}{2} + 5(n-3)\right)$$

and

$$|w_4(G_{410})| = 2e + 4\left(\binom{n-4}{2} + \binom{n-3}{2} + 3\binom{n-2}{2} + (n-5)\binom{n-1}{2}\right) + 8\left(3\binom{n}{4} - 8\binom{n-2}{2} + 4(n-3) + 2\right).$$

By Lemma 2.2(v), we have

$$\begin{aligned}&2e + 4(4\binom{n-3}{2} + (n-4)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 4(n-3) + 1)\\ &\quad = 2e + 4(2\binom{n-3}{2} + \binom{n-2}{2} + \binom{n-4}{2} + (n-4)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 5(n-3)),\end{aligned} \tag{3}$$

and

$$\begin{aligned}&2e + 4(2\binom{n-3}{2} + \binom{n-2}{2} + \binom{n-4}{2} + (n-4)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 5(n-3))\\ &\quad = 2e + 4(\binom{n-4}{2} + \binom{n-3}{2} + 3\binom{n-2}{2} + (n-5)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 4(n-3) + 2),\end{aligned} \tag{4}$$

and

$$\begin{aligned}&2e + 4(\binom{n-4}{2} + \binom{n-3}{2} + 3\binom{n-2}{2} + (n-5)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 4(n-3) + 2)\\ &\quad = 2e + 4(4\binom{n-3}{2} + (n-4)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 4(n-3) + 1).\end{aligned} \tag{5}$$

Solving the equation (3), we get n=3, a contradiction with $n \geq 6$. Solving the equation (4), we get n=4, a contradiction with $n \geq 6$. The equation (5) has no solution.

Subcase 3 Similarly to Subcase 1, by Lemma 2.3, we calculate $|w_4(G_{42})|$, $|w_4(G_{45})|$ and $|w_4(G_{48})|$. We have

$$|w_4(G_{42})| = 2e + 4(2\binom{n-3}{2} + 4\binom{n-2}{2} + (n-6)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 2(n-3) + 8),$$

$$|w_4(G_{45})| = 2e + 4(3\binom{n-3}{2} + 2\binom{n-2}{2} + (n-5)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 3(n-3) + 6) \text{ and}$$

$$|w_4(G_{48})| = 2e + 4(2\binom{n-3}{2} + 4\binom{n-2}{2} + (n-6)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 2(n-3) + 7).$$

By Lemma 2.2(v), we have

$$\begin{aligned}&2e + 4(2\binom{n-3}{2} + 4\binom{n-2}{2} + (n-6)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 2(n-3) + 8)\\ &\quad = 2e + 4(3\binom{n-3}{2} + 2\binom{n-2}{2} + (n-5)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 3(n-3) + 6),\end{aligned} \tag{6}$$

$$\begin{aligned}&2e + 4(3\binom{n-3}{2} + 2\binom{n-2}{2} + (n-5)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 3(n-3) + 6)\\ &\quad = 2e + 4(2\binom{n-3}{2} + 4\binom{n-2}{2} + (n-6)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 2(n-3) + 7),\end{aligned} \tag{7}$$

$$\begin{aligned}&2e + 4(2\binom{n-3}{2} + 4\binom{n-2}{2} + (n-6)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 2(n-3) + 7)\\ &= 2e + 4(2\binom{n-3}{2} + 4\binom{n-2}{2} + (n-6)\binom{n-1}{2}) + 8(3\binom{n}{4} - 8\binom{n-2}{2} + 2(n-3) + 8).\end{aligned} \tag{8}$$

Solving the equation (6), we get n=4, a contradiction with $n \geq 6$. Solving the equation (7), we get n=3, a contradiction with $n \geq 6$. The equation (8) has no solution.

In what follows, we prove Theorem 1.2. To this end, we need the following Lemmas.

Lemma 3.1 *Let $d^2(G) = \sum_{i=1}^n d_i^2(G)$. Then*

$d^2(G_{30}) = (n-4)(n-1)^2 + 3(n-2)^2 + (n-4)^2 = n^3 - 2n^2 - 11n + 24;$
$d^2(G_{31}) = (n-5)(n-1)^2 + 4(n-2)^2 + (n-3)^2 = n^3 - 2n^2 - 11n + 20;$
$d^2(G_{32}) = (n-3)(n-1)^2 + 3(n-3)^2 = n^3 - 2n^2 - 11n + 24;$
$d^2(G_{33}) = (n-4)(n-1)^2 + 2(n-2)^2 + 2(n-3)^2 = n^3 - 2n^2 - 11n + 22;$
$d^2(G_{34}) = (n-6)(n-1)^2 + 6(n-2)^2 = n^3 - 2n^2 - 11n + 18.$
$d^2(G_{40}) = (n-5)(n-1)^2 + 4(n-2)^2 + (n-5)^2 = n^3 - 2n^2 - 15n + 36.$
$d^2(G_{41}) = (n-6)(n-1)^2 + 5(n-2)^2 + (n-4)^2 = n^3 - 2n^2 - 15n + 30.$
$d^2(G_{42}) = (n-6)(n-1)^2 + 4(n-2)^2 + 2(n-3)^2 = n^3 - 2n^2 - 15n + 28.$
$d^2(G_{43}) = (n-7)(n-1)^2 + 6(n-2)^2 + (n-3)^2 = n^3 - 2n^2 - 15n + 26.$
$d^2(G_{44}) = (n-4)(n-1)^2 + 4(n-3)^2 = n^3 - 2n^2 - 15n + 32.$
$d^2(G_{45}) = (n-5)(n-1)^2 + 2(n-2)^2 + 3(n-3)^2 = n^3 - 2n^2 - 15n + 30.$
$d^2(G_{46}) = (n-4)(n-1)^2 + (n-2)^2 + 2(n-3)^2 + (n-4)^2 = n^3 - 2n^2 - 15n + 34.$
$d^2(G_{47}) = (n-5)(n-1)^2 + 2(n-2)^2 + 3(n-3)^2 = n^3 - 2n^2 - 15n + 30.$
$d^2(G_{48}) = (n-6)(n-1)^2 + 4(n-2)^2 + 2(n-3)^2 = n^3 - 2n^2 - 15n + 28.$
$d^2(G_{49}) = (n-8)(n-1)^2 + 8(n-2)^2 = n^3 - 2n^2 - 15n + 24.$
$d^2(G_{410}) = (n-5)(n-1)^2 + 3(n-2)^2 + (n-3)^2 + (n-4)^2 = n^3 - 2n^2 - 15n + 32.$

Proof By simple calculation, we can obtain the results. □

Lemma 3.2 Let G is a graph. If $\kappa(G)$ is the number of spanning trees of the graph G, then

$\kappa(G_{30}) = n^{n-5}(n-1)^2(n-4);$
$\kappa(G_{32}) = n^{n-5}((n-2)^3 - 3n + 8);$
$\kappa(G_{41}) = n^{n-8}(n^6 - 8n^5 + 21n^4 - 22n^3 + 8n^2);$
$\kappa(G_{42}) = n^{n-8}(n^6 - 8n^5 + 22n^4 - 24n^3 + 9n^2);$
$\kappa(G_{44}) = n^{n-6}(n^4 - 8n^3 + 20n^2 - 16n);$
$\kappa(G_{45}) = n^{n-7}(n^5 - 8n^4 + 21n^3 - 18n^2);$
$\kappa(G_{47}) = n^{n-7}(n^5 - 8n^4 + 21n^3 - 20n^2 + 5n);$
$\kappa(G_{48}) = n^{n-8}(n^6 - 8n^5 + 22n^4 - 23n^3 + 5n^2 + 2n);$
$\kappa(G_{410}) = n^{n-7}(n^5 - 8n^4 + 20n^3 - 18n^2 + 5n).$

Proof Without loss of generality, we calculate only $\kappa(G_{30})$. Since

$$\overline{G}_{30} = K_{1,3} \cup (n-4)K_1,$$

it follows

$$P_{L(\overline{G}_{30})}(\mu) = \mu^{n-3}(\mu-1)^2(\mu-4).$$

By Lemma 2.5, we have

$$\kappa(G_{30}) = n^{-2}P_{L(\overline{G}_{30})}(n) = n^{n-5}(n-1)^2(n-4).$$

Similarly to the calculation of $\kappa(G_{30})$, we can get other $\kappa(G_{ij})$ in the Lemma. □

Proof of Theorem 1.2 Let $G_i \in \mathcal{G}$. Suppose a graph H is cospectral with G_i with respect to the Laplacian spectrum. We consider the following cases.

Case 1 $i = 1$. Considering the complete graph K_n by deleting one edge leads to the conclusion obviously.

Case 2 $i = 2$. Consider the complete graph K_n by deleting two edges. By Lemma 2.2, H is a graph with n vertices and $\binom{n}{2}$-2 edges. By Lemma 2.2(viii), H has only one connected component, then $H \cong G_{20}$ or $H \cong G_{21}$. We prove G_{20} and G_{21} are not Laplacian cospectral. Suppose that G_{20} and G_{21} are Laplacian cospectral. By Lemma 2.2(ix), graphs G_{20} and G_{21} have the same sum of the squares of degrees of vertices. We have the following equation

$$2(n-2)^2 + (n-3)^2 + (n-1)^2 = 4(n-2)^2,$$

which has no solution, a contradiction.

Case 3 $i = 3$. Similarly to Case 2, consider the complete graph K_n by deleting three edges. The H must be isomorphic to one of G_{3j} $(j = 0, 1, 2, 3, 4)$. By Lemma 3.1, we know that only graphs G_{30} and G_{32} have the same sum of the squares of degrees of vertices. If G_{30} and G_{32} are cospectral with respect to the Laplacian spectrum, then by Lemma 2.2(vii) G_{30} and G_{32} have the same number of apanning trees, but by Lemma 3.2 we know that $\kappa(G_{30}) \neq \kappa(G_{32})$ for any n. So G_{30} and G_{32} are not cospectral with respect to the Laplacian spectrum.

Case 4 $i = 4$. Similarly to Case 2, consider the complete graph K_n by deleting four edges. The H must be isomorphic to one of G_{4j} $(j = 0, 1, 2, \ldots, 10)$. By Lemma 3.1, we have 3 subcases as follows.

Subcase 1 The graphs G_{41}, G_{45} and G_{47} have the same sum of the squares of degrees of vertices. But by Lemma 3.2, we have $\kappa(G_{41}) \neq \kappa(G_{45}) \neq \kappa(G_{47})$ for $n \geq 3$. So G_{41}, G_{45} and G_{47} are not cospectral with respect to the Laplacian spectrum.

Subcase 2 Only the graphs G_{42} and G_{48} have the same sum of the squares of degrees of vertices. But by Lemma 3.2, we have $\kappa(G_{42}) \neq \kappa(G_{48})$ for any n. So G_{42} and G_{48} are not cospectral with respect to the Laplacian spectrum.

Subcase 3 Only the graphs G_{44} and G_{410} have the same sum of the squares of degrees of vertices. But by Lemma 3.2, we have $\kappa(G_{44}) \neq \kappa(G_{410})$ for any n. So G_{44} and G_{410} are not cospectral with respect to the Laplacian spectrum.

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