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Note about Fixed Points of Scott Continuous Self-Mappings

Xiao Yong XI[1,2,*], **Yong Ming LI**[3]

1. *College of Mathematics and Information Science, Shaanxi Normal University, Shaanxi* 710062*, P. R. China;*
2. *College of Mathematics Science, Xuzhou Normal University, Jiangsu* 221009*, P. R. China;*
3. *College of Computer Science, Shaanxi Normal University, Shaanxi* 710062*, P. R. China*

Abstract It is discussed in this paper that under what conditions, for a continuous domain L, there is a Scott continuous self-mapping $f: L \to L$ such that the set of fixed points $\mathrm{fix}(f)$ is not continuous in the ordering induced by L. For any algebraic domain L with a countable base and a smallest element, the problem presented by Huth is partially solved. Also, an example is given and shows that there is a bounded complete domain L such that for any Scott continuous stable self-mapping f, $\mathrm{fix}(f)$ is not the retract of L.

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1. Introduction and preliminaries

A poset L is called DCPO if every directed subset of L has the supremum. For all $x, y \in L, x \ll y$ if and only if for every directed subset E of L satisfying $y \le \bigvee E$, there is $e \in E$ such that $x \le e$. For any $x \in L$, we write $\Downarrow x = \{y \in L | y \ll x\}$. A poset is called to be continuous if for any $x \in L, \Downarrow x$ is directed and $x = \vee \Downarrow x$. A continuous DCPO is also called a continuous domain. For any $x \in L$, x is called a compact element if $x \ll x$. The set of all compact elements in L is denoted by $K(L)$. A DCPO L is called an algebraic domain if $\Downarrow x \cap K(L)$ is directed and $\sup(\Downarrow x \cap K(L)) = x$ for every $x \in L$. A DCPO L is called L-domain if for any $x \in L$, $\downarrow x$ is a complete lattice. A DCPO L is called a bounded complete domain if each pair of elements of L has a least upper bound. Suppose that D, L are DCPO. If the mapping $f: D \to L$ preserves the supremum of directed subsets, then f is called Scott continuous. If there are Scott continuous mappings $r: D \to L$ and $s: L \to D$ such that $r \circ s = \mathrm{id}_L$, then L is called the retract of D. If the Scott continuous self-mapping $f: L \to L$ satisfies $f \circ f = f$, then $\mathrm{fix}(f) = \{x \in L | f(x) = x\}$ is a continuous domain with respect to the induce order of L and $\mathrm{fix}(f)$ is the retract of L. In

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* Corresponding author
E-mail address: littlebrook@sina.com (X. Y. XI)

general, $\mathrm{fix}(f)$ is not continuous. In the web page [1], Huth gave a continuous domain L and a Scott continuous self-mapping $f : L \to L$ satisfying that $\mathrm{fix}(f)$ is not continuous. He also put forward the following problem:

Problem For which classes of continuous domains L are there Scott continuous self-maps $f : L \to L$ such that $\mathrm{fix}(f)$ are not continuous on the induced order?

If L is an algebraic domain with a countable base and the smallest element, this problem is answered in this paper.

On the other hand, Kou [2] considered those continuous domains L and Scott continuous self-mappings $f : L \to L$ such that $\mathrm{fix}(f)$ is continuous, and discussed whether $\mathrm{fix}(f)$ is the retract of L if $\mathrm{fix}(f)$ is continuous. Kou proved the following result.

Theorem 1.1 ([2]) *Suppose that L is a continuous L-domain and $f : L \to L$ is a stable mapping. Then* $\mathrm{fix}(f)$ *is a continuous domain; if L is simultaneously a complete lattice, then* $\mathrm{fix}(L)$ *is the retract of L.*

For the case of L-domain, f is a stable mapping if and only if f preserves directed supremum and compatible infimum. Kou [2] constructed a continuous L-domain and a stable mapping $f : L \to L$ such that $\mathrm{fix}(f)$ is not the retract of L. In general, an L-domain is not necessarily a bounded complete domain. For example, $L = \{\perp, a, b, c, d\}$ ordered by: $\perp$ is the smallest element; a, b are incomparable; c, d are incomparable; $a < c, d$ and $b < c, d$. L is an L-domain, but L is not bounded complete since a, b have no supremum though a, b are bounded. For a bounded completely continuous domain L and stable mapping $f : L \to L$, is $\mathrm{fix}(f)$ the retract of L? In this paper, an example is given and shows the answer is no.

2. Main results

Huth [1] gave a continuous domain D and a Scott continuous self-mapping $f : D \to D$ such that $\mathrm{fix}(f)$ is not continuous. But his example is somewhat complex and refers to the other results. We give a simple example as follows.

Example 2.1 Let $D = \{a_0, a_1, a_2, \ldots, a_n, \ldots, a_\infty, b_0, b_1, b_2, \ldots, b_n, \ldots, b_\infty\}$ ordered by:

$$\forall\ i \in \mathbf{N},\ a_i < a_{i+1} < a_\infty,\ b_i < b_{i+1} < b_\infty,\ a_i < b_i,\ a_\infty < b_\infty.$$

Then there is a Scott continuous self-mapping $f : D \to D$ such that $\mathrm{fix}(f)$ is not continuous on the induced order of D.

Define $f : D \to D$ as follows:

$$f(a_0) = a_0, f(a_\infty) = a_\infty, f(b_\infty) = b_\infty, f(a_i) = a_{i-1}, i \geq 1,\ f(b_i) = b_i, \forall\ i \in \mathbf{N}.$$

Then $\mathrm{fix}(f) = \{a_0, a_\infty, b_0, b_1, \ldots, b_n, \ldots, b_\infty\}$. Obviously, $\mathrm{fix}(f)$ is not continuous on the induced order.

At the end of the paper [1], Huth proposed the following problem: For which classes of continuous domains D are there Scott continuous self-maps $f : D \to D$ such that $\mathrm{fix}(f)$ is not continuous on the induced order? The following theorem gives a partial answer.

Theorem 2.2 *Suppose that D is an algebraic domain with a smallest element and a countable base. Then there is a Scott continuous self-map $f : D \to D$ such that $\mathrm{fix}(f)$ is not continuous on the induced order if and only if there is $x, y \in D$ such that $x < y$, but $x \ll_D y$ does not hold.*

Proof We firstly show necessity. Suppose there is a Scott continuous self-map $f : D \to D$ such that $\mathrm{fix}(f)$ is not continuous on the induced order. Since $\mathrm{fix}(f)$ is not continuous, there is $x \in \mathrm{fix}(f)$ such that $\mathrm{fix}(f)$ is not continuous at x, therefore, $x \ll_{\mathrm{fix}(f)} x$ does not hold. Then there is a directed subset $\{x_i\}_{i\in I} \subseteq \mathrm{fix}(f)$ such that $\bigvee_{i\in I} x_i \geq x$ and $\forall\ i \in I, x_i \not\geq x$. Let $y = \bigvee_{i\in I} x_i$. If $y > x$, then $x \ll_{\mathrm{fix}(f)} y$ dose not hold. Otherwise, there is $i_0 \in I$ such that $x \leq x_{i_0}$, contradiction follows. So $x \ll_D y$ does not hold. If $y = x$, then there is x_{i_0} such that $x_{i_0} < x$, but $x_{i_0} \ll_D x$ does not hold. Otherwise, suppose for any x_i, $x_i \ll_D x$. Since $\{x_i\}_{i\in I}$ is directed, $\mathrm{fix}(f)$ is continuous at x. This contradicts the continuity of $\mathrm{fix}(f)$ at x. Necessity is proved.

Then we show sufficiency. Suppose that there are $x, y \in D$ such that $x < y$, but $x \ll y$ does not hold. Since D has a countable base, there is a countable ascending chain $\{y_i\}_{i\in I}$ such that $\forall\ i \in I, y_i \in K(D), \bigvee_{i\in I} y_i = y$, but $\forall\ i \in I, x \not\leq y_i$. Similarly, there is a countable ascending chain $\{x_j\}_{j\in J}$ such that $\forall\ j \in J, x_j \in K(D), \bigvee_{j\in J} x_j = x$. Let $x_0 = a_0 = \perp$, $y_0 = b_0$. We can suppose that $b_0 \not\leq x$. Take x_{j_1} such that $x_{j_1} > \max\{x_j \mid x_j \leq y_0, j \in J\}$. Let $x_{j_1} = a_1$. Since $a_1 \ll x < y$, there is $i_1 \in I$ such that $a_1 < y_{i_1}$. Take x_{j_2} such that $x_{j_2} > \max\{x_j \mid x_j \leq y_{i_1}, j \in J\}$. Let $x_{j_2} = a_2, y_{i_1} = b_1$. Inductively, for any $n \in \mathbf{N}$, a_n and b_n are defined. Let $x = a_\infty, y = b_\infty$. We give a copy of the domain in Example 2.1. Denote this copy as D_1. We project D onto D_1 and define a map $f_1 : D \to D$ as follows :

$$\forall\ x \in D, f_1(x) = \bigvee_{D_1}\{e \in C \mid e \leq x\},$$

in which $C = \{a_0, a_1, \ldots, a_n, \ldots, b_0, b_1, \ldots, b_n, \ldots\}$.

Let $h = f \circ f_1$, in which f is the map in Example 2.1. Then $\mathrm{fix}(h)$ is not continuous on the induced order. The proof is completed. $\Box$

We give a bounded complete continuous domain L and a stable map $f : L \to L$ such that $\mathrm{fix}(f)$ is not the retract of L.

Example 2.3 Let $L = \{a_0, a_1, \ldots, a_n, \ldots, a_\infty, b_1, b_2, \ldots, b_n, \ldots, b_\infty\}$ ordered by: a_0 is the smallest element; $\forall\ i \in \mathbf{N}, a_i < b_i, a_\infty < b_\infty, a_0 < a_1 < \cdots < a_n < \cdots < a_\infty$, where the order of $\{b_0, b_1, b_2, \ldots, b_n, b_\infty\}$ is discrete. Then L is a bounded complete domain. We give a stable map $f : L \to L$ such that $\mathrm{fix}(f)$ is not the retract of L.

Define $f : L \to L$ as follows:

$$\forall\ i \in \mathbf{N}, f(b_i) = b_i, f(a_{i+1}) = a_i, f(b_\infty) = b_\infty\ f(a_\infty) = a_\infty, f(a_0) = a_0.$$

Then $\mathrm{fix}(f) = \{a_0, b_0, b_1, b_2, \ldots, b_n, \ldots, b_\infty, a_\infty\}$. Next we show that $\mathrm{fix}(f)$ is not the retract of L. Suppose that $\mathrm{fix}(f)$ is the retract of L. Then there is a Scott continuous map $p : L \to \mathrm{fix}(f)$

and $r : \mathrm{fix}(f) \to L$ such that $p \circ r = \mathrm{id}_{\mathrm{fix}(f)}$.

We can claim $\forall\ i \in \mathbf{N}, p(a_i) = a_0$. In fact, suppose there is $i_0 \in \mathbf{N}$ such that $p(a_{i_0}) \neq a_0$. Then there is $n_0 \in \mathbf{N}$ such that $p(a_{i_0}) = b_{n_0}$ or b_∞ or a_∞. Suppose that $p(a_{i_0}) = b_{n_0}$. Then $\forall\ i \in \mathbf{N}$ and $i \geq i_0, p(b_i) \geq p(a_{i_0}) = b_{n_0}, p(a_i) \geq p(a_{i_0}) = b_{n_0}$, therefore, $p(b_i) = b_{n_0},\ p(a_i) = b_{n_0}$. It implies that $im(p)$ is finite and contradicts $p \circ r = \mathrm{id}_{\mathrm{fix}(f)}$. Similarly, suppose $p(a_{i_0}) = b_\infty$ or a_∞. Then $\mathrm{im}(p)$ is finite. This contradicts $p \circ r = \mathrm{id}_{\mathrm{fix}(f)}$. Hence $\forall i \in \mathbf{N}, p(a_i) = a_0$ and by the continuity of p, $p(a_\infty) = a_0$.

Consider the map $r : \mathrm{fix}(f) \to L$. We can claim $r(a_\infty) \neq a_\infty$. Otherwise, $p \circ r(a_\infty) = p(a_\infty) = a_0$. This contradicts $p \circ r = \mathrm{id}_{\mathrm{fix}(f)}$. Similarly, $\forall i \in \mathbf{N}, r(a_\infty) \neq a_i$, therefore, $r(a_\infty) \in \{b_0, b_1, \ldots, b_n, \ldots, b_\infty\}, r(a_\infty) \neq b_\infty$. Otherwise, $r(a_\infty) = r(b_\infty) = b_\infty$. This contradicts $p \circ r = \mathrm{id}_{\mathrm{fix}(f)}$. Similarly, $\forall\ i \in \mathbf{N}, r(a_\infty) \neq b_i$. Such r does not exist. $\mathrm{fix}(f)$ is not the retract of L. The proof is completed. $\Box$

If the stable map is strengthened to preserve arbitrary infimum, then we have the following result. The proof is basically the same as that in [2, Theorem 2.2].

Theorem 2.4 *Suppose that L is a bounded complete domain, $f : L \to L$ is Scott continuous and preserves arbitrary infimum and $\forall\ x \in L, \mathrm{fix}(f) \cap \uparrow x \neq \phi$. Then $\mathrm{fix}(f)$ is the retract of L.*

Proof Define a map $r : L \to \mathrm{fix}(f)$ as follows: $\forall\ x \in L$

$$r(x) = \wedge(\uparrow x \cap \mathrm{fix}(f))$$

in which the infimum is defined on L. Since f preserves arbitrary infimum, $\forall x \in L, \wedge(\uparrow x \cap \mathrm{fix}(f)) = f(\wedge(\uparrow x \cap \mathrm{fix}(f)) \in \mathrm{fix}(f)$. So r is well defined and preserves the order. Next, we show r is Scott continuous.

Suppose that $D \subseteq L$ is directed and $y = \vee D$. $\forall\ d \in D, d \leq \wedge(\uparrow d \cap \mathrm{fix}(f)) = r(d)$, therefore $y \leq \bigvee_{d \in D} r(d)$. Since $r(y) = \wedge(\uparrow y \cap \mathrm{fix}(f)$ and $\bigvee_{d \in D} r(d) \in \mathrm{fix}(f)$, $r(y) = r(\vee D) = \bigvee_{d \in D} r(d)$. Let $s : \mathrm{fix}(f) \to L$ be inclusion map. Then $r \circ s = \mathrm{id}_{\mathrm{fix}(f)}$. The proof is completed. $\Box$

Remark The map in Example 2.3 does not preserve arbitrary infimum.

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