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Conditions for Global Stability of n-Patches Ecological System with Dispersion *

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Abstract: In this paper, a nonautonomous predator-prey dispersion model is studied, where all parameters are time-dependent. The system, which is consisted of n-patches, the prey species can disperse among n-patches, but the predator species is confined to one patch and cannot disperse. It is proved the system is uniformly persistent under any dispersion rates effect. Furthermore, sufficient conditions are established for global stability of the system.

Key words: predator-prey system; uniform persistence; global stability.

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1. Introduction

One of the most interesting questions in mathematical biology concerns the survival of species in ecological models. Levin[1] first established this kind of model for autonomous Lotka-Volterra system, K. Kishimoto[2] and Y. Takeuchi[3] also studied this kind of models, but all the coefficients in the system they studied are constants. Song and Chen[4] extended the autonomous Lotka-Volterra system to a two species nonautonomous dispersion Lotka-Volterra system. In this paper, we consider a nonautonomous predator-prey dispersion model. The system, which is consisted of n-patches, the prey species can disperse among n-patches, but the predator species is confined to one patch and cannot disperse. Our purpose is demonstrate that the dispersion rates have no effect on uniform persistence of the solution of the system. Furthermore, we establish conditions under which the system is globally asymptotically stable.

2. Model and background concept

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In this paper, we consider the following Lotka-Volterra population model

$$
\begin{aligned}
\dot{x}_1 &= x_1(a_1(t) - b_1(t)x_1 - c(t)y) + \sum_{i=2}^{n} D_{i1}(t)(x_i - x_1), \\
\dot{x}_j &= x_j(a_j(t) - b_j(t)x_j) + \sum_{i=1}^{n} D_{ij}(t)(x_i - x_j), \ (j = 2, 3, \cdots, j \neq i) \\
\dot{y} &= y(-d(t) + e(t)x_1 - q(t)y),
\end{aligned}
\tag{2.1}
$$

where x_1 and y are population density of prey species x and predator species y in patch 1, and x_j is density of prey species x in patch j. predator species y is confined to patch 1, while the prey species x can disperse among n-patches. $D_{ij}(t)$ $(i, j = 1, 2, \cdots, n)$ are dispersion coefficients of species x .

Now we let $f^l = \inf\{f(t) : t \in R\}$ and $f^m = \sup\{f(t) : t \in R\}$, for a continuous and bounded function f(t).

In system (2.1), we always assume:

(H$_1$): $a_i(t)$, $b_i(t)$, $D_{ij}(t)$ $(i, j = 1, 2, \cdots, n)$ $c(t)$, $d(t)$, $e(t)$, $q(t)$ are continuous and strictly positive functions, which satisfy

$$\min\{a_i^l, b_i^l, D_{ij}^l, c^l, e^l, d^l, q^l\} > 0, \quad \max\{a_i^m, b_i^m, D_{ij}^m, c^m, d^m, e^m, q^m\} < \infty.$$

For ecological reasons, We always assume that

$$x \in R_{n+1}^+, \ x(0) > 0. \tag{2.2}$$

3. Uniform persistence

Definition *The system (2.1) is said to be uniformly persistent if there exists a compact region $D \subset \mathrm{Int}(R_+^{n+1})$ such that every solution, $Z(t) = (x_1(t), x_2(t), \cdots, x_n(t), y(t))$ of system (2.1) with initial condition (2.2) eventually enters and remains in the region D.*

Lemma 3.1 *Let $Z(t) = (x_1(t), x_2(t), \cdots, x_n(t), y(t))$ denote any positive solution of system (2.1) with the initial conditions (2.2). Then there exists a $T_1 > 0$ such that*

$$x_i(t) \leq M_1, \ (i = 1, 2, \cdots, n), \ y(t) \leq M_2, \ \text{for } t \geq T_1, \tag{3.1}$$

where

$$
\begin{aligned}
&M_1 > M_1^*, \qquad M_2 > M_2^*, \\
&M_1^* = \max\{\frac{a_1^m}{b_1^l}, \frac{a_2^m}{b_2^l}, \cdots, \frac{a_n^m}{b_n^l}\}, \quad M_2^* = \frac{e^m M_1^*}{q^l}.
\end{aligned}
\tag{3.2}
$$

We let $m_1^* = \min\{\frac{a_1^l - c^m M_2^*}{b_1^m}, \frac{a_2^l}{b_2^m}, \cdots, \frac{a_n^l}{b_n^m}\}$.

Theorem 3.1 *Suppose the system (2.1) satisfies*

(H_2) $a_1^l q^l/(c^m e^m) > M_1^*$,

(H_3) $e^l m_1^* > d^m$,
then system (2.1) is uniformly persistent.

Proof Suppose $z(t) = (x_1(t),\ x_2(t), \cdots, x_n(t),\ y(t))$ is a solution of system (2.1) which satisfies (2.2) .

According to the system (2.1) and Lemma 3.1 , if $t > T_1$, we can obtain

$$\begin{aligned} \dot{x}_1 &\geq x_1(a_1^l - c^m M_2 - b_1^m x_1) + \sum_{i=2}^{n} D_{i1}(t)(x_i - x_1), \\ \dot{x}_j &\geq x_j(a_j^l - b_j^m x_j) + \sum_{i=1}^{n} D_{ij}(t)(x_i - x_j) \quad (j = 2, 3, \cdots, n). \end{aligned} \tag{3.3}$$

From (3.2) and (H_3), we know $a_1^l - c^m M_2^* > 0$ holds. Also from the Lemma 3.1, we obtain that M_2 can be chosen close to M_2^* enough to make $a_1^l - c^m M_2 > 0$ holds.

Let $m_1^* = \min\{\frac{a_1^l - c^m M_2^*}{b_1^m}, \frac{a_2^l}{b_2^m}, \cdots, \frac{a_n^l}{b_n^m}\}$. We choose m_1 as: $0 < m_1 < m_1^*$, Define $V_1(t) = \min\{x_1(t), x_2(t), \cdots, x_n(t)\}$. Then calculating the lower right derivative of $V_1(t)$ along the positive solution of system (2.1), similar to the discussion of [4], we have $V_1(x_1(t), x_2(t), \cdots, x_n(t)) \geq m_1$.

From the system (2.1) and Lemma 3.1, we know that there exists $T_2 > T_1$ such that

$$\dot{y}(t) \geq y(t)[e^l m_1 - d^m - q^m y(t)].$$

From (3.2) and (H_3), the inequality $e^l m_1^* - d^m > 0$ holds, we know that m_1 can be close to m_1^* and M_2 can be close to M_2^* sufficiently to make the inequality $e^l m_1 - d^m > 0$ holds.

Suppose $y(t)$ is not oscillatory about $(e^l m_1 - d^m)/q^m > 0$,

then either

$$y(t) < (e^l m_1 - d^m)/q^m, \quad \text{for} \quad t \geq T_2, \tag{3.4}$$

or

$$y(t) > (e^l m_1 - d^m)/q^m, \quad \text{for} \quad t \geq T_2. \tag{3.5}$$

If (3.4) holds, then there exists a constant m_2, $0 < m_2 < M_3^* = (e^l m_1 - d^m)/q^m$, such that $y(t) \leq m_2$ and $e^l m_1 - d^m - m_2 q^m > 0$; thus , let $\lambda = e^l m_1 - d^m - m_2 q^m$, we obtain, $\dot{y}(t) > \lambda y(t) > 0$. This implies $y(t)$ strictly monotone increasing with speed λ. Hence there exists $T_3 > T_2$, such that $y(t) \geq m_2$, for $t \geq T_3$.

If (3.5) holds, then $y(t) > M_3^* > m_2$, for $t \geq T_2$.

Suppose now that $y(t)$ is oscillatory about M_3^*, let $y(t^*)$ $(t^* \geq T_2)$ denote an arbitrary local minimum of $y(t)$, it is easy to see from system (2.1) that

$$0 = \frac{\mathrm{d}y(t^*)}{\mathrm{d}t} \geq y(t^*)[-d^m + e^l m_1 - q^m y(t^*)],$$

and this implies $m_2 < M_3^* \leq y(t^*)$, for $t^* \geq T_2$. Since $y(t^*)$ is an arbitrary local minimum of $y(t)$, we conclude that $0 < m_2 < M_3^* \leq y(t)$ eventually .

Finally we let

$$D = \{(x_1(t), x_2(t), \cdots, x_n(t), y(t)) : m_1 \leq x_i(t) \leq M_1 \ (i = 1, 2, \cdots, n),\ m_2 \leq y(t) \leq M_2\}$$

Then D is a bounded compact region in R_+^{n+1} which has positive distance from coordinate hyperplanes let $T = T_3$, then from the proof above, we obtain that if $t \geq T$, then every positive solution of system(2.1) with the initial condition (2.2) eventually enters and remains in the region D. The proof is complete.

4. Global Asymptotic Stability

In this section , we derive sufficient conditions which guarantee that any positive solution of system (2.1) is globally asymptotically stable.

Theorem 4.1 *In addition to $(H_1)-(H_3)$, assume further that system (2.1) satisfies*

$$(H_4):\ b_1^l > e^m + \sum_{j=2}^{n} \frac{D_{1j}^m}{m_1^*},\ b_j^l > \frac{D_{j1}^m}{m_1^*} + \sum_{i=2}^{n} \frac{D_{ij}^m}{m_1^*},\ q^l > c^m, (j = 2, 3, \cdots, n)$$

then any positive solution of system (2.1) is globally asymptotically stable.

Proof For two arbitrary nontrival positive solution: $Z(t) = (x_1(t), x_2(t), \cdots, x_n(t), y(t))$ and $U(t) = (u_1(t), u_2(t), \cdots, u_n(t), v(t))$ of system (2.1), we have from uniform persistence of system (2.1) that there exist positive constants m_i and M_i, $(i = 1, 2)$ such that for all $t \geq t^*$ (t^* sufficient large),

$$0 < m_1 \leq x_i(t) \leq M_1,\ (i = 1, 2, \cdots, n),\ 0 < m_2 \leq y(t) \leq M_2,$$

$$0 < m_1 \leq u_i(t) \leq M_1,\ (i = 1, 2, \cdots, n),\ 0 < m_2 \leq v(t) \leq M_2.$$

We define

$$\tilde{x}_i(t) = \ln x_i(t),\ \tilde{y}(t) = \ln y(t);\ \tilde{u}_i(t) = \ln u_i(t),\ \tilde{v}(t) = \ln v(t)\quad (i = 1, 2, \cdots, n).$$

Consider the following Lyapunov functional

$$V_2(t) = \sum_{i=1}^{n} |\tilde{x}_i(t) - \tilde{u}_i(t)| + |\tilde{y}(t) - \tilde{v}(t)|.$$

Now we calculate and estimate the upper right derivative of $V_2(t)$ along the solutions of system (2.1), we have

$$D^+V_2(t) \leq -(b_1^l - e^m - \sum_{j=2}^{n} \frac{D_{1j}^m}{m_1})|x_1(t) - u_1(t)| - (q^l - c^m)|y(t) - v(t)| -$$
$$\sum_{j=2}^{n} (b_j^l - \frac{D_{j1}^m}{m_1} - \sum_{i=2}^{n} \frac{D_{ij}^m}{m_1})|x_j(t) - u_j(t)|.$$

From the proof of Theorem 3.1 and assumption (H_4), we can select M_2 close to M_2^* sufficiently and get m_1 close to m_1^* sufficiently too. So there exists $\alpha_1 > 0$ such that

$$D^+V_2(t) \leq -\alpha_1 (\sum_{i=1}^{n} |x_i(t) - u_i(t)| + |y(t) - v(t)|). \tag{4.1}$$

Integrating both sides of (4.1) leads to

$$V_2(t)+\alpha_1\int_{t^*}^{t}(\sum_{i=1}^{n}|x_i(s)-u_i(s)|+|y(s)-v(s)|)\mathrm{d}s\leq V_2(t^*)<+\infty \text{ for } t>t^*$$

Which leads to $\sum\limits_{i=1}^{n}|x_i(t)-u_i(t)|+|y(t)-v(t)|\in L^1(t^*,+\infty)$. From the persistence hypothesis of (2.1), the boundedness of the solutions of (2.1), we can obtain that $[x_i(t)-u_i(t)](i=1,2,\cdots,n),[y(t)-v(t)]$ and their derivatives remain bounded on $[0,\infty)$. As a consequence, $\sum\limits_{i=1}^{n}|x_i(t)-u_i(t)|+|y(t)-v(t)|$ is uniformly continuous. By Barbalat's Lemma [5, P_4, Lemma 1.2.2], it follows that

$$\lim_{t\to\infty}(\sum_{i=1}^{n}|x_i(s)-u_i(s)|+|y(s)-v(s)|)=0.$$

Hence $\lim_{t\to\infty}|x_i(t)-u_i(t)|=0,\ (i=1,2,\cdots,n),\lim_{t\to\infty}|y(t)-v(t)|=0$. The proof is complete.

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生态扩散系统全局渐近稳定的条件

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摘 要: 本文研究一类带扩散的非自治捕食系统, 该系统由 n 个斑块组成, 食饵种群可以在 n 个斑块之间扩散, 而捕食者种群限定在一个斑块不能扩散. 得到系统持续生存和全局渐近稳定的条件.